The Poincaré-Birkhoff-Witt Theorem (PBW)

graded $\bigoplus_{d \geq 0} T^d(\mathfrak{g})$

Let $\mathfrak{g}$ be a Lie algebra, $U(\mathfrak{g}) = T(\mathfrak{g}) \big/ \langle \underbrace{x \otimes y - y \otimes x}_{\text{in deg. 2}} - \underbrace{[x,y]}_{\text{in deg. 1}} \mid x, y \in \mathfrak{g} \rangle$ its universal enveloping algebra.

The canonical map $i : \mathfrak{g} \to U(\mathfrak{g})$ is injective (so we can identify $\mathfrak{g}$ with a subspace of $U(\mathfrak{g})$). Moreover, if x_i $(i \in I)$ is a basis for $\mathfrak{g}$ and some total order on I, then

$$\{ x_{i_1} \cdots x_{i_n} \mid n \geq 0,\ i_1, \ldots, i_n \in I,\ i_1 \leq \cdots \leq i_n \}$$

Ordered monomials / PBW monomials

is a basis for $U(\mathfrak{g})$.

Usually, $\mathfrak{g}$ is f.d. with basis $x_1, \ldots, x_n$ then PBW basis is

$$\{ x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \mid m_1, \ldots, m_n \geq 0 \}.$$

Comments on proof

You probably saw the first Weyl algebra $A = \mathbb{C}\langle x,y\rangle / \langle \underbrace{xy-yx}_{\deg 2} - \underbrace{1}_{\deg 0}\rangle$

This has basis $y^j x^i$ $(i,j \geq 0)$.

Usual proof

- $y^j x^i$ $(i,j \geq 0)$ span A
- define an action of A on $\mathbb{C}[y]$ so y acts as mult.
  x acts as $\frac{d}{dy}$

  check relations! $\frac{d}{dy}(yf) = y\frac{df}{dy} + f$
- Then use this action to prove lin. independence, act on $y^k \in \mathbb{C}[y]$.

In fact, A is a *filtered deformation* of $\mathbb{C}[x,y]$

Note $T = \mathbb{C}\langle x,y\rangle$ is a graded algebra (x,y in degree 1)

But $I = \langle xy-yx-1\rangle$ is not homogeneous ... A is merely <u>filtered</u>.

Let $A_{\leq n} = \operatorname{Im}\left(\bigoplus_{d=0}^{n} T_d\right)$. Then:

- $\mathbb{k}1 = A_{\leq 0} \subseteq A_{\leq 1} \subseteq A_{\leq 2} \subseteq \cdots$ subspaces
- $A_{\leq m} \cdot A_{\leq n} \subseteq A_{\leq (m+n)}$ — $(*)$

For a filtered algebra, its <u>associated graded algebra</u> $\operatorname{gr} A = \bigoplus_{d \geq 0} (\operatorname{gr} A)_d$

where $(\operatorname{gr} A)_d := A_{\leq d} / A_{< d}$.

For $x \in A_{\leq n}$, let $\operatorname{gr}_n x = x + A_{<n} \in (\operatorname{gr} A)_n$.

Then $\operatorname{gr} A$ is a graded algebra with $(\operatorname{gr}_n x) \cdot (\operatorname{gr}_m y) = \operatorname{gr}_{n+m}(xy)$.

Check well-defined using $(*)$

For our A, the first Weyl algebra, $(\operatorname{gr} A)_n$ has basis $\underbrace{\operatorname{gr}_n (y^j x^i)}_{(\operatorname{gr}_1 y)^j (\operatorname{gr}_1 x)^i}$ $(i+j=n)$.

Also $\operatorname{gr}_1 x$, $\operatorname{gr}_1 y$ commute in $\operatorname{gr} A$.

$\operatorname{gr}_2(xy) = \operatorname{gr}_2(yx)$

So: $\mathbb{C}[x,y] \hookrightarrow\!\!\!\rightarrow \operatorname{gr} A$, algebra ~~homomorphism~~ isomorphism.

$x \longmapsto \operatorname{gr}_1 x$

$y \longmapsto \operatorname{gr}_1 y$

Story for $U(\mathfrak{g})$ is similar!

To prove PBW theorem

- The ordered monomials span $U(\mathfrak{g})$
- Make $S(\mathfrak{g})$ into a left ~~$U(\mathfrak{g})$~~-module so

$$\underset{\substack{\cap\\ \mathfrak{g}}}{x_i} \cdot \underset{\substack{\cap\\ S(\mathfrak{g})}}{x_{j_1} \cdots x_{j_n}} \equiv x_i x_{j_1} \cdots x_{j_n} \quad \text{(modulo lower degree terms)}$$

Unpleasant step!!! Induction on degree.

and $x_i \cdot x_{j_1} \cdots x_{j_n} = x_i x_{j_1} \cdots x_{j_n}$ if $i \leq j_1, \dots, j_n$

(unique way to do this)

- Use this action to show lin. independence (act on 1)

[There's an axiomatization of this type of argument — Bergman's diamond lemma]

In fact, $U(\mathfrak{g})$ is a filtered deformation of $S(\mathfrak{g})$.

Of course $U(\mathfrak{g})$ is a filtered algebra as a quotient of $T(\mathfrak{g})$

$$U(\mathfrak{g})_{\leq n} = \operatorname{Im}\left(\bigoplus_{d=0}^{n} T^d(\mathfrak{g}) \right)$$

PBW $\Longrightarrow$ $(\mathrm{gr}\, U(\mathfrak{g}))_n$ has basis $\mathrm{gr}_n(x_{i_1} \cdots x_{i_n})$ $\quad i_1 \le \cdots \le i_n$

$$\mathrm{gr}_n(x_{i_1} \cdots x_{i_n}) = (\mathrm{gr}_1 x_{i_1}) \cdots (\mathrm{gr}_1 x_{i_n})$$

Also $\mathrm{gr}\, U(\mathfrak{g})$ is commutative

$$[\mathrm{gr}_1 x, \mathrm{gr}_1 y] = \mathrm{gr}_2(xy - yx) = \mathrm{gr}_2([x,y]) = 0$$

So

$$S(\mathfrak{g}) \xrightarrow{\sim} \mathrm{gr}\, U(\mathfrak{g})$$

$$x \longmapsto \mathrm{gr}_1 x$$

$x \in \mathfrak{g}$

$$\mathfrak{g} \hookrightarrow U(\mathfrak{g})$$

Last time: three basic examples of Hopf algebra Δ (comult.) ε (counit) antipode (S)

① $\Bbbk G$, G a finite group

$\Delta(g) = g \otimes g$ $\quad g \in G$

"group-like elements"

$\varepsilon(g) = 1$ $\quad g \in G$

$S(g) = g^{-1}$ $\quad g \in G$

② $\Bbbk[G]$, G algebraic group

$\Delta = m^*$

$\varepsilon = ev_e$

$S = i^*$

③ $U(\mathfrak{g})$, $\mathfrak{g}$ a Lie algebra

$\Delta(x) = x \otimes 1 + 1 \otimes x$

$\varepsilon(x) = 0$ $\quad x \in \mathfrak{g}$

$S(x) = -x$

Need to use univ. prop. of $U(\mathfrak{g})$ to see that these extend appropriately.

"$U(\mathfrak{g})$ is to $\mathfrak{g}$ as $\Bbbk G$ is to G"

(finite group G)

$\mathrm{Rep}(\mathfrak{g}) = U(\mathfrak{g})\text{-mod}_{fd}$

f.d. representations of $\mathfrak{g}$

$\mathrm{Rep}(G) = \Bbbk G\text{-mod}_{fd}$

$\mathrm{Rep}(G) = \mathrm{comod}_{fd}\text{-}\Bbbk[G]$

algebraic group

All three cats. are symmetric tensor cats.

Why is Hopf algebra structure important?

$$A \xrightarrow{\Delta} A \otimes A$$
$$\Delta \downarrow \quad \circ \quad \nearrow \text{flip}$$
$$A \otimes A$$

A cocommutative Hopf algebra $\Delta : A \to A \otimes A$ $\quad \varepsilon : A \to \mathbb{k}$ $\quad S : A \to A$

Then $A\text{-mod}_{fd}$ is a symmetric tensor category

$V \otimes W \xrightarrow{\sim} W \otimes V$
$v \otimes w \mapsto w \otimes v$
is an A-module isomorphism

- Abelian
- There's a $\otimes$
- There's a duality $*$
- There's a unit object, trivial module $\mathbb{k}$

V, W

$V \otimes W = V \otimes_{\mathbb{k}} W$ is an $A \otimes A$-module

$(a \otimes b)(v \otimes w) = av \otimes bw$

So it an A-module via $\Delta : A \to A \otimes A$

$V^* = \mathrm{Hom}_{\mathbb{k}}(V, \mathbb{k})$, linear dual $\qquad (a f)(v) = f(S(a)v)$, $a \in A$, $f \in V^*$

Trivial module $\mathbb{k}$ has action defined by $\quad a \cdot 1_{\mathbb{k}} = \varepsilon(a) \cdot 1_{\mathbb{k}}$.

If A is a Hopf algebra,

A^*, linear dual, is always an algebra $A^* \otimes A^* \hookrightarrow (A \otimes A)^* \longrightarrow A^*$

dual map to comult. Δ on A

But A^* is <u>NOT</u> necessarily a coalgebra (hence, Hopf algebra) in a natural way. $A^* \longrightarrow (A \otimes A)^* \hookleftarrow A^* \otimes A^*$

In general the image of this map, dual of mult. on A, needn't be in subspace $A^* \otimes A^* \hookrightarrow (A \otimes A)^*$

If A is f.d., no problems, and A^* is again a Hopf algebra via this construction

eg $\Bbbk G$, G finite group. $(\Bbbk G)^* \equiv \Bbbk[G]$ as Hopf algebras

$$\mathrm{Rep}(G) \equiv \mathrm{Rep}(G)$$

left $\Bbbk G$-modules $\equiv$ right $\Bbbk[G]$-comodules

For G connected algebraic group.

Then $\mathbb{k}[G]^*$ is an algebra but "too big" to be a coalgebra / Hopf algebra

$\mathfrak{m}_e = \ker ev_e \triangleleft \mathbb{k}[G]$

Now you consider

$$\mathrm{Dist}(G) = \{ \theta \in \mathbb{k}[G]^* \mid \theta(\mathfrak{m}_e^{n+1}) = 0 \text{ for } n \gg 0 \}$$

$$= \bigcup_{n \geq 0} \left(\mathbb{k}[G] / \mathfrak{m}_e^{n+1} \right)^*$$

algebra of distributions of G

It's a subalgebra, even filtered with $\mathrm{Dist}(G)_{\leq n} = \left(\mathbb{k}[G] / \mathfrak{m}_e^{n+1} \right)^*$.

Even a Hopf algebra with comult. dual to mult. on $\mathbb{k}[G]$

In char. 0, turns out that $\mathrm{Dist}(G) \cong U(\mathfrak{g})$ as Hopf algs.

$$\mathfrak{g} = \mathrm{Der}(\mathbb{k}[G], \mathbb{k}_e) \cong \left(\mathfrak{m}_e / \mathfrak{m}_e^2\right)^* \subseteq \mathrm{Dist}(G)$$

Get Lie alg. hom. $\mathfrak{g} \longrightarrow \mathrm{Dist}(G)$ from this

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[d, hook] & \mathrm{Dist}(G) \\ U(\mathfrak{g}) \arrow[ur, "\sim"] & \end{tikzcd}$$

$\cong$ requires char. 0.