The Poincaré–Birkhoff–Witt Theorem (PBW)

Let $\mathfrak{g}$ be a Lie algebra, $U(\mathfrak{g}) = T(\mathfrak{g})$ its universal enveloping algebra.

The canonical map $i: \mathfrak{g} \to U(\mathfrak{g})$ is injective (so we can identify $\mathfrak{g}$ with a subspace of $U(\mathfrak{g})$). Moreover, if $\{x_i : i \in I\}$ is a basis for $\mathfrak{g}$ and some total order on $I$, then

$$\left\{ x_{i_1} \cdots x_{i_n} \mid n \geq 0, i_1, \ldots, i_n \in I, i_1 \leq \cdots \leq i_n \right\}$$

is a basis for $U(\mathfrak{g})$.

Usually, $\mathfrak{g}$ is f.d. with basis $x_1, \ldots, x_n$ then PBW basis is

$$\left\{ x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \mid m_1, \ldots, m_n \geq 0 \right\}.$$
Comments on proof

You probably saw the first Weyl algebra $A = \frac{\mathbb{C}[x,y]}{\langle xy - yx - 1 \rangle}$

This has basis $y^j x^i (i,j \geq 0)$.

**Usual proof**
- $y^j x^i (i,j \geq 0)$ span $A$
- Define an action of $A$ on $\mathbb{C}[y]$ so $y$ acts as mult. $x$ acts as $\frac{d}{dy}$
- Check relations: $\frac{d}{dy} (yf) = y \frac{df}{dy} + f$
- Then use this action to prove uni. independence, act on $y^k \in \mathbb{C}[y]$.

In fact, $A$ is a filtered deformation of $\mathbb{C}[x,y]$.

Note $T = \mathbb{C}[x,y]$ is a graded algebra ($x,y$ in degree 1)

But $I = \langle xy - yx - 1 \rangle$ is not homogeneous ... $A$ is merely filtered.

Let $A \leq \bigoplus_{d=0}^{n} \text{Im} (T_d)$ Then:
For a filtered algebra, its associated graded algebra \( \text{gr} A = \bigoplus_{d \geq 0} (\text{gr} A)_d \)

where \( (\text{gr} A)_d := \frac{A \leq d}{A < d} \).

For \( x \in A \leq n \), let \( \text{gr}_n x = x + A \leq n \in (\text{gr} A)_n \).

Then \( \text{gr} A \) is a graded algebra with \( (\text{gr}_n x) \cdot (\text{gr}_m y) = \text{gr}_{n+m} (xy) \).

For our \( A \), the first Weyl algebra, \((\text{gr} A)_n\) has basis \( \text{gr}_n (y^j x^i) \) \((i+j = n)\).

Also \( \text{gr}_1 x, \text{gr}_1 y \) commute in \( \text{gr} A \).

\( \text{gr}_2 (xy) = \text{gr}_2 (yx) \) So: \([x, y] \mapsto \text{gr} A \), algebra homomorphism.
Story for \( U(\mathfrak{g}) \) is similar!

To prove PBW theorem

Unpleasant step!!!

Inductive: and degree.

The ordered monomials span \( U(\mathfrak{g}) \)

- Make \( S(\mathfrak{g}) \) into a left \( U(\mathfrak{g}) \)-module so

\[ x_i \cdot x_j \cdots x_{j_n} \equiv x_i x_j \cdots x_{j_n} \quad \text{(modulo lower degree terms)} \]

and \( x_i \cdot x_j \cdots x_{j_n} = x_i x_j \cdots x_{j_n} \) if \( i \leq j \cdots j_n \)

(Unique way to do this)

- Use this action to show \( \mathfrak{g} \)-independence (action 1)

There's an axiomatization of this type of argument — Bergman's diamond key

In fact, \( U(\mathfrak{g}) \) is a filtered deformation of \( S(\mathfrak{g}) \).

Of course, \( U(\mathfrak{g}) \) is a filtered algebra as a quotient of \( T(\mathfrak{g}) \)

\[ U(\mathfrak{g}) \cong \lim_{\leftarrow n} \bigoplus_{d=0}^{\infty} T^d(\mathfrak{g}) \]
PBW \Rightarrow (\text{gr } U(g))_n \text{ has basis } \text{gr}_n(x_{c_1} \cdots x_{c_n}) \\
\text{Also } \text{gr } U(g) \text{ is commutative } (\text{gr}_1 x_{c_1}) \cdots (\text{gr}_1 x_{c_n})

[\text{gr}_1 x, \text{gr}_1 y] = \text{gr}_2 (xy - yx) = \text{gr}_2 ([x, y]) = 0

So \xymatrix{ S(g) & & \text{gr } U(g) \\ x \ar@{|->}[r] & \text{gr}_1 x \\ \text{gr} \ar@{|->}[u] & & U(g) \ar@{|->}[l]}

\text{gr} \xrightarrow{\text{U}(g)} U(g)
Last time: three basic examples of Hopf algebra $\Delta$ (consult) $\Sigma$ (consult) antipode (5)

1. $\mathfrak{g} \mathfrak{l} (G)$, $G$ a finite group
   \begin{align*}
   \Delta(g) &= g \otimes g \quad g \in G \\
   \varepsilon(g) &= 1 \quad g \in G \\
   S(g) &= g^{-1} \quad g \in G
   \end{align*}
   "group-like element"

2. $\mathfrak{g} \mathfrak{u} [G]$, $G$ algebraic group
   \begin{align*}
   \Delta &= m^* \\
   \Sigma &= e \mathfrak{u} e \\
   S &= \iota^*
   \end{align*}

3. $\mathfrak{u}(g)$, $g$ a Lie algebra
   \begin{align*}
   \Delta(x) &= x \otimes 1 + 1 \otimes x \\
   \Sigma(x) &= 0 \\
   S(x) &= -x
   \end{align*}
   Need to use uni. prop. of $\mathfrak{u}(g)$ to see that they extend appropriately.

"$\mathfrak{u}(g)$ is to $\mathfrak{g} \mathfrak{l}$ as $\mathfrak{k} \mathfrak{g} \mathfrak{l}$ is to $G$"

(Finite group $G$)

$\text{Rep}(\mathfrak{g}) = \mathfrak{u}(g) \text{- mod} \ 	ext{fd}$

$\uparrow$

f.d. representation of $\mathfrak{g}$

$\text{Rep}(G) = \text{comod} \cdot \mathfrak{k} \mathfrak{g} \mathfrak{l} \mathfrak{g} \mathfrak{l} \text{ fd}$

$\uparrow$

f.d. representation of $G$

$\text{Rep}(G) = \mathfrak{k} \mathfrak{g} \mathfrak{l} \text{- mod} \ 	ext{fd}$

$\uparrow$

All three "cats" are symmetric tensor cats.
Why is Hopf algebra structure important?

A cocommutative Hopf algebra $\Delta: A \rightarrow A \otimes A$

$\varepsilon: A \rightarrow k$

$s: A \rightarrow A$

Then $A$-mod is a symmetric tensor category

$V \otimes W = V \otimes W$ is an $A \otimes A$-module

$\text{Ik}$

$(a \otimes b)(v \otimes w) = av \otimes bw$

So it is an $A$-module via $\Delta: A \rightarrow A \otimes A$

$V^* = \text{Hom}_k (V, \text{Ik})$, linear dual

$(a \cdot f)(v) = f(S(a)v)$

$A \otimes V^*$

Trivial module $\text{Ik}$ has action defined by $\alpha \cdot 1_{\text{Ik}} = \Sigma(a) \cdot 1_{\text{Ik}}$. 
If $A$ is a Hopf algebra,

$A^*$, linear dual, is always an algebra

$$A^* \otimes A^* \rightarrow (A \otimes A)^* \rightarrow A^*$$

But $A^*$ is not necessarily a coalgebra (hence, Hopf algebra) in a natural way.

$$A^* \rightarrow (A \otimes A)^* \leftarrow A^* \otimes A^*$$

In general, the map of this map, dual or multiplication $A$, needn't be in subspace $A^* \otimes A^* \rightarrow (A \otimes A)^*$.

If $A$ is f.d., no problems, and $A^*$ is again a Hopf algebra via this construction.

$\mathbb{C} G$, $G$ finite group.

$$\text{Rep}(G) \equiv \text{Rep}(G)$$

$\mathbb{C}G$-modules $\equiv$ right $\mathbb{C}[G]$-comodules
For $G$ connected algebraic group.

Then $\mathbb{k}[G]^*$ is an algebra but "too big" to be a coalgebra / Hopf algebra

Now you consider

$$\text{Dist}(G) = \{ \Theta \in \mathbb{k}[G]^* \mid \Theta(M_e^{n+1}) = 0 \text{ for } n \gg 0 \}$$

The algebra of distributions of $G$

In a subalgebra, ever filtered with $\text{Dist}(G) \leq \mathcal{U} = (\mathbb{k}[G]/M_e^{n+1})^*$

Every a Hopf algebra with counit dual to multi on $\mathbb{k}[G]$.

In char. 0, turns out that $\text{Dist}(G) \simeq \mathcal{U}(\mathfrak{g})$ as Hopf algebras.
\[ \text{Der}(\text{Lie}(G), \text{Lie}(e)) = (\mathcal{M}_e/M_e^2)^* \subseteq \text{Der}(G) \]

Get Lie alg. hom. \[ \text{Der}(\text{Lie}(G), \text{Lie}(e)) \rightarrow \text{Der}(G) \text{ from this} \]

\[ \mathcal{U}(G) \cong \text{requins char. 0} \]