\[ \text{Rep}(sl_2) = U(\mathfrak{g})-\text{mod}_{f.d.} \]

Today \( \mathfrak{g} = sl_2(\mathbb{C}) \) . \( e = (0, 1) \) \( h = (0, 1) \) \( f = (1, 0) \)

\[ [e, f] = h \quad [h, e] = 2e \quad [h, f] = -2f \]

If \( v \) is a non-zero vector in some \( \mathfrak{g} \)-module \( V \) such that \( hv = \lambda v \) \( (\lambda \in \mathbb{C}) \)

then
\[ h(ev) = (\lambda + 2)ev \]
\[ h(fv) = (\lambda - 2)fv \]

\[ [h, e] v = h(ev) - e(hv) \]

\[ 2e v = h(ev) - \lambda ev \]

\[ h(ev) = (\lambda + 2)ev \]
If \( 0 \neq V \) is any f.d. \( \mathfrak{g} \)-module, you can always find a (non-zero) weight vector \( v \in V \) of weight \( \lambda \in \mathbb{C} \).

\[ v, \ e v, \ e^2 v, \ldots \quad \text{Can't go forever as } V \text{ is f.d.} \]

\[ \lambda, \ \lambda + 2, \ \lambda + 4 \]

Eventually, you get to a weight vector killed by \( e \)

\[ \begin{align*}
O \neq v \in V & \quad \{ \text{highest weight vector } \} \\
\forall \mu = \lambda v & \quad e v = 0
\end{align*} \]

We're going to show any f.d. irreducible representation is determined uniquely up to isomorphism by its highest weight — the weight of a highest weight vector.

This weight is always \( \lambda = n \in \mathbb{N} \)

\[ L(n) \quad (n \in \mathbb{N}) \quad \text{"highest weight } n" \]

\[ \dim L(n) = n + 1. \]
Examples

Highest weight 0

L(0)

Highest weight 1

L(1)

Highest weight 1

L(n), dimension (n+1)

\( h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)

\( S^n V \)

\( n^{th} \) symmetric power

Proof that said. Any weight vector generates
Lemma  If \( v \in V \) is any vector in a \( g \)-module with \( e_v = 0 \), then

\[
e^{f^{n+1}} v = (n+1) f^n (h-n) v \quad (n \in \mathbb{N})
\]

Proof  \( e^{f^{n+1}} v = [e_f, f^{n+1}] v = \sum_{c=0}^{n} f^c [e_f] f^{n-c} v \)

\[
= \sum_{c=0}^{n} f^c f^{n-c} v = \sum_{c=0}^{n} f^c f^{n-c} (h-2n+2c) v
\]

\[
= f^n \cdot (n+1) (h-n) v
\]

Corollary  If \( v \) is a vector killed by \( e \) then

\[
e^{f^{n+1}} v = (n+1)! h (h-1) \ldots (h-n) v
\]

Proof  \( e^n (e^{f^{n+1}} v) = (n+1) e^n f^n (h-n) v \) then induct
Now take $\lambda \in \mathbb{C}$. "Borel"

Let $\mathfrak{g} = \mathfrak{ch} \oplus \mathfrak{ce} < \mathfrak{g}$

This is a solvable subalgebra, $\mathfrak{g} \rightarrow \mathfrak{ch}$.

Let $\mathfrak{c}_\lambda$ be l-d. mod $\mathfrak{b}$-module on which $\mathfrak{h}$ acts as $\lambda$ and $\mathfrak{e}$ acts as $0$.

As $\mathfrak{g} < \mathfrak{g}$, $U(\mathfrak{b}) < U(\mathfrak{g})$... form the Verma module

$$M(\lambda) = U(\mathfrak{g}) \otimes \mathfrak{c}_\lambda$$

As $U(\mathfrak{g})$ has basis $f_i h^j e^k (ij, k \geq 0)$ by PBW Theorem, it is free as a right $U(\mathfrak{b})$-module with basis $f_i^j$ ($i \geq 0$). So $M(\lambda)$ has basis $f_i^j e_i (i \geq 0)$... denote these by $f_i^j v_i$.
Note also that \( f^i v_+ \) is a weight vector of weight \( \lambda - 2i \).

**Theorem** For \( \lambda \in \mathbb{C} - \mathbb{N} \), \( M(\lambda) \) is an irreducible \( \mathfrak{g} \)-module.

If \( \lambda = n \in \mathbb{N} \), \( M(n) \) has a unique submodule isomorphic to \( M(-n-2) \), and unique quotient \( L(n) := M(n) / M(-n-2) \), which is a f.d. irreducible representation of highest weight \( n \), denoted \( \Pi^+ n \).

Every f.d. irreducible representation of \( \mathfrak{g} \) is isomorphic to \( L(n) \) for \( \lambda \in \mathbb{N} \).

**Proof** Take \( \lambda \in \mathbb{C} - \mathbb{N} \).

Take \( 0 \neq v \in M(\lambda) \), \( v = \sum_{c \in \mathbb{N}} c_i \ f^i v_+ \).

Choose biggest \( i \) so \( c_i \neq 0 \). Compute \( e^c v \).
\[
e f^{n+1} v_+ = (n+1) (\lambda - n) f^n v_+ \quad \text{by lemma} \quad \neq 0
\]

It follows that \( e^2 v \) is a non-zero multiple of \( v_+ \), cyclic vector.
Hence \( v \) generates and \( M(\lambda) \) is irreducible.

But if \( \lambda = n \in \mathbb{N} \), then \( f^{n+1} v_+ \) is killed by \( e \).
So \( \langle f^{n+1} v_+, f^{n+2} v_+, \ldots \rangle \) span a submodule of \( M(\lambda) \).

\[
\text{weight } n-2(n+1) = -n-2 \ldots \text{ copy of } M(-n-2) \text{, irreducible.}
\]

The quotient of \( M(\lambda) \) by this submodule has basis
\[
\overline{v_+}, \overline{f v_+}, \ldots, \overline{f^n v_+} \quad \text{dim } n+1 \ldots \text{ irreducible.}
\]
Now take my f.d. indeg. $g$-module $L$

As above, find $O \neq V$ highest weight vector of weight $\lambda \in \mathbb{C}$.

$$[h V = \lambda V, \ ev = 0]$$

$$\Hom\left(\mathbb{C} \xrightarrow{c} \text{res}_{U(b)}^{U(g)} L\right) \neq 0$$

$$\text{SL} \downarrow$$

$$\Hom\left(\left.\left.\left(\mathbb{U(g) \otimes \mathbb{C}_x \xrightarrow{\text{U(b)}} \mathbb{M(x)}\right)\right)\right) \neq 0$$

\begin{align*}
&\text{Shows there's non-zero} \quad M(\lambda) \rightarrow L \\
&\text{So } L \text{ is a f.d. quotient of } M(\lambda) \xrightarrow{f^i} V, \ x^i \in \text{f.d. hom.} \end{align*}
Theorem 2 Any f.d. rep. of $g$ is completely reducible.

Proof Suffice to show $\text{Ext}^1_g(L(a), L(b))$ for $a, b \in \mathbb{N}$.

Case one $a > b$

The generalized $a$-eigenspace of $h$ on $V$ is $V^V$.

So you can find $v \in V$ of weight $a$ mapping to $v_+ \in L(a)$ under $\pi$.

By universal property of $M(a)$, get $g$-module hom.

(c $v = 0$). This gives splitting of $\pi$.

Case two $a < b$ $\text{Ext}^1_g(L(a), L(b)) \cong \text{Ext}^1_g(L(b)^*, L(a)^*)$.

Done.
Case three  \( a = b \)

\[ 0 \to L(a) \to V \to L(a) \to 0 \]

Highest weight \( a \)-generalized \( a \)-eigenvector for \( h \in V \) is \( 2D \).

The argument in case one works fine if \( h \) is diagonalizable on this eigenspace. We don't know this -- why not a Jordan block \((a \mid 1)\) ?

Take any \( 0 \not= v \) in this generalized eigenspace.

\[ e^v = 0 = f^a v \]

\[ \therefore e^{a+1} f^{a+1} v = (a+1)! h(h-1) \ldots (h-a) v \text{ by Corollary.} \]

Show: min. poly. of \( h \) acting on this eigenspace divides \( x(x-1) \ldots (x-a) \).

This is a product of distinct linear factors, hence, \( h \) is indeed diagonalizable.