Action of algebraic group $G \curvearrowright V \otimes W$ ⟵ $g(v \otimes w) = gv \otimes gw$

and of its Lie algebra $\mathfrak{g} \curvearrowright V \otimes W$ ⟵ $x(v \otimes w) = xv \otimes w + v \otimes xw$

are related.

$U(\mathfrak{g})$

$\lVert$ char. zero.

Use point of view from L4-3

$$\mathfrak{g} = \left(\frac{m_e}{m_e^2}\right)^* \hookrightarrow \underset{\text{subalgebra}}{\mathrm{Dist}(G)} \subset \underbrace{\Bbbk[G]^*}_{\text{an algebra}}.$$

The comult. on $\mathrm{Dist}(G)$ is dual of multiplication on $\Bbbk[G]$

Take $x \in \mathfrak{g}$, $f \in \Bbbk[G]$. Then $f = f(e) + f'$ for $f' \in m_e$

$x(f) = x(f')$ and x is zero on m_e^2.

m_e^2

For $f, g \in \Bbbk[G]$. Then $fg = (f(e) + f')(g(e) + g') = f(e)g(e) + f'g(e) + f(e)g' + f'g'$

So $x(fg) = x(f)g(e) + f(e)x(g) = (x \otimes 1 + 1 \otimes x)(f \otimes g)$

Shows $\Delta x = x \otimes 1 + 1 \otimes x$ matches dual of multiplication on $\Bbbk[G]$.

Note this multiplication is also comorphism of diagonal map $G \to G \times G$, $g \mapsto (g, g)$

Last time $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ $\quad \mathrm{Rep}(\mathfrak{g}) \equiv U(\mathfrak{g})\text{-mod}_{fd}$ is a semisimple Abelian category with irreps$/\cong \longleftrightarrow \mathbb{N}$

$L(n) \longleftrightarrow n$ irreducible highest weight module of highest weight n

h-eigenvalue

v_1^n	n
$v_1^{n-1}v_2$	$n-2$
$\vdots$	$\vdots$
$v_1 v_2^{n-1}$	$2-n$
v_2^n	$-n$

$$L(n) \cong S^n V, \quad V = \mathbb{C}^2 \text{ natural representation, basis } \underset{1}{v_1}, \underset{-1}{v_2}$$

Using that weights add when you take tensor product.

$$h(v \otimes w) = hv \otimes w + v \otimes hw$$

weight λ weight μ

$$= \lambda v \otimes w + \mu v \otimes w$$

$$= (\lambda + \mu) v \otimes w$$

Consider $G = SL_2(\mathbb{C})$.

Of course V lifts to a representation of G

$\mathfrak{g}$ action is derived from G action

Hence, so does $S^n V \cong L(n)$.

Hence, we see the action of $\mathfrak{g}$ on any f.d. representation lifts to G.

This shows that our functor

$$\mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(\mathfrak{g})$$

is an isomorphism of categories. False for $\mathbb{G}_m, \mathbb{G}_a$.

(Knew: full and faithful)

Special! Need G semisimple & simply connected

Example

$$S^2V \otimes S^3V$$
$$\cong$$
$$L(5)\oplus L(3)\oplus L(1)$$

$L(2)\backslash L(3)$	3	1	−1	−3
2	5	3	1	−1
0	3	1	−1	−3
−2	1	−1	−3	−5

Knowledge of weights + multiplicities determines structure up to isomorphism.

5		
3	3	
1	1	1
−1	−1	−1
−3	−3	
−5		

$v_1^2 \otimes v_1^3$ h/w vector in $L(5)$

$v_1v_2 \otimes v_1^3 - v_1^2 \otimes v_1^2 v_2$ h/w in $L(3)$

$$\underbrace{v_2^2 \otimes v_1^3 \quad v_1v_2 \otimes v_1^2 v_2 \quad v_1^2 \otimes v_1 v_2^2}_{\text{some lin. comb. to get h/w vector in } L(1)}$$

Final topic in this chapter   More about nilpotent and solvable Lie algebras.

In Ch. 0, we saw that "Every algebraic group is linear".

What about f.d. Lie algebras? YES

The proof of Ado's theorem is tricky!

Any Lie algebra has

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$
$$x \mapsto \mathrm{ad}\, x \quad \text{where } (\mathrm{ad}\, x)(y) = [x, y]$$

$\ker \mathrm{ad} = \mathfrak{z}(\mathfrak{g})$, so any f.d. centerless Lie algebra is linear via adjoint representation.

All algebraic Lie algebras are linear, already known too...

$\boxed{\text{Ado's Theorem}}$ (1938)

Any f.d. Lie algebra over any field is isomorphic to a subalgebra of $\mathfrak{gl}(V)$ for some f.d. V.

Proof: Terence Tao's blog.

$\boxed{\text{Nilpotent Ado}}$

If $\mathfrak{g}$ is a f.d. nilpotent Lie algebra then it is isomorphic to a subalgebra of $\mathfrak{gl}(V)$ consisting of nilpotent endomorphisms of V, some f.d. V.

Do want to prove two much more classical results:

Proofs next lecture! Valid over any field.

Engel's Theorem Suppose $\mathfrak{g} < \mathfrak{gl}(V)$, $\dim V = n$, and all elements of $\mathfrak{g}$ are nilpotent endomorphisms of V. Then there's a basis $v_1, \dots, v_n$ for V such that $\mathfrak{g} \cdot v_i \in \langle v_1, \dots, v_{i-1} \rangle \ \forall i$.

__Corollary 1__ Such a $\mathfrak{g}$ is nilpotent (as $\mathfrak{n}_n(\mathbb{C})$ is nilpotent)

__Corollary 2__ If $\mathfrak{g}$ is a f.d. Lie algebra and $\mathrm{ad}\, x : \mathfrak{g} \to \mathfrak{g}$ is nilpotent $\forall x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.

$\mathfrak{sl}_2(\mathbb{k})$, char $\mathbb{k} = 2$ nilpotent but $\mathbb{k}^2$ is irred.

Lie's Theorem Suppose $\mathfrak{g} < \mathfrak{gl}(V)$, $\dim V = n$, and $\mathfrak{g}$ is solvable. Then there's a basis $v_1, \dots, v_n$ for V such that $\mathfrak{g} \cdot v_i \in \langle v_1, \dots, v_i \rangle \ \forall i$.

__Corollary 1__ If $\mathfrak{g}$ is f.d. solvable Lie algebra $/\mathbb{C}$ then $\mathfrak{g}'$ is nilpotent.

__Corollary 2__ All f.d. irreps of f.d. solvable $\mathfrak{g} / \mathbb{C}$ are 1-D.

Requires field of char. 0 and alg. closed.

Discussion of Engel & Lie.

A basis $v_1, \dots, v_n$ for V identifies $\mathfrak{gl}(V) \equiv \mathfrak{gl}_n(\mathbb{C})$

$$\begin{pmatrix} 0 & * & * & & \\ \vdots & 0 & * & & * \\ \vdots & \vdots & 0 & & \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Engel says $\mathfrak{g} \hookrightarrow \mathfrak{n}_n(\mathbb{C})$, subalgebra of strictly upper ∇'r matrices

Lie says $\mathfrak{g} \hookrightarrow \mathfrak{b}_n(\mathbb{C})$, subalgebra of weakly upper ∇'r matrices

$$\begin{pmatrix} * & * & & \\ 0 & * & & * \\ \vdots & \vdots & & \\ 0 & 0 & 0 & \end{pmatrix}$$

Note $\mathfrak{n}_n(\mathbb{C})$ IS a nilpotent Lie algebra

$\mathfrak{b}_n(\mathbb{C})$ IS a solvable Lie algebra with $\mathfrak{b}_n(\mathbb{C})' = \mathfrak{n}_n(\mathbb{C})$.

Combine Engel/Lie with nilpotent Ado / Ado, deduce:

$\mathfrak{g}$ f.d. nilpotent $\iff$ $\mathfrak{g} \hookrightarrow \mathfrak{n}_n(\mathbb{C})$ some n

$\mathfrak{g}$ f.d. solvable $\iff$ $\mathfrak{g} \hookrightarrow \mathfrak{b}_n(\mathbb{C})$ some n