

Action of algebraic group $G \hookrightarrow V \otimes W \longleftarrow g (v \otimes w) = gv \otimes gw$

and of its Lie algebra $\mathfrak{g} \hookrightarrow V \otimes W \longleftarrow x (v \otimes w) = xv \otimes w + v \otimes xw$

are related.

$U(\mathfrak{g})$

|| char. zero.

Use part of result from L4-3

$\mathfrak{g} = (M_e / M_e^2)^* \hookrightarrow \text{Dut}(G) \subset \underbrace{[k[G]]^*}_{\text{an algebra}}$. The counit on $\text{Dut}(G)$ is dual of multiplication on $[k[G]]$

Subalgebra

Take $x \in \mathfrak{g}$, $f \in [k[G]]$. Then $f = f(e) + f'$ for $f' \in M_e$.
 $x(f) = x(f')$ and x is zero on M_e^2 . M_e^2
 \sim

For $f, g \in [k[G]]$. Then $fg = (f(e) + f')(g(e) + g') = f(e)g(e) + f'g(e) + f(e)g' + f'g'$

So $x(fg) = x(f)g(e) + f(e)x(g) = (x \otimes 1 + 1 \otimes x)(f \otimes g)$

Shows $\Delta x = x \otimes 1 + 1 \otimes x$ matches dual of multiplication on $[k[G]]$.

Note this multiplication is also comorphism of diagonal map $G \rightarrow G \times G, g \mapsto (g, g)$

Last time $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ $\text{Rep}(\mathfrak{g}) \cong U(\mathfrak{g})\text{-mod}_{\text{fd}}$ is a semisimple Abelian

category with irreps $\cong \mathbb{N}$ Irreducible highest weight module v_1^n | (n)
 $L(n) \cong n$ of highest weight n $v_1^{n-1}v_2$ | $n-2$
 \vdots | \vdots
 $v_1 v_2^{n-1}$ | $2-n$
 v_2^n | $-n$
↑ h-eigenvalue

III
 $S^n V, V = \mathbb{C}^2$ natural representation, basis v_1, v_2
 $\begin{matrix} 1 & -1 \\ & \end{matrix}$

Consider $G = \text{SL}_2(\mathbb{C})$. *g action is derived from G action*

Of course V lifts to a representation of G

Hence, so does $S^n V \cong L(n)$.

Hence, we see the action of \mathfrak{g} on any f.d. representation lifts to G .

This shows that our functor

$\text{Rep}(G) \rightarrow \text{Rep}(\mathfrak{g})$ is an isomorphism of categories. *False for G_m, G_a .*
Special! Need G semisimple & simply connected
 (knew: full and faithful)

Using that weights add when you take tensor product.
 $h(v \otimes w) = hv \otimes w + v \otimes hw$
 weight λ \uparrow \uparrow weight μ
 $= \lambda v \otimes w + \mu v \otimes w = (\lambda + \mu) v \otimes w$

Example

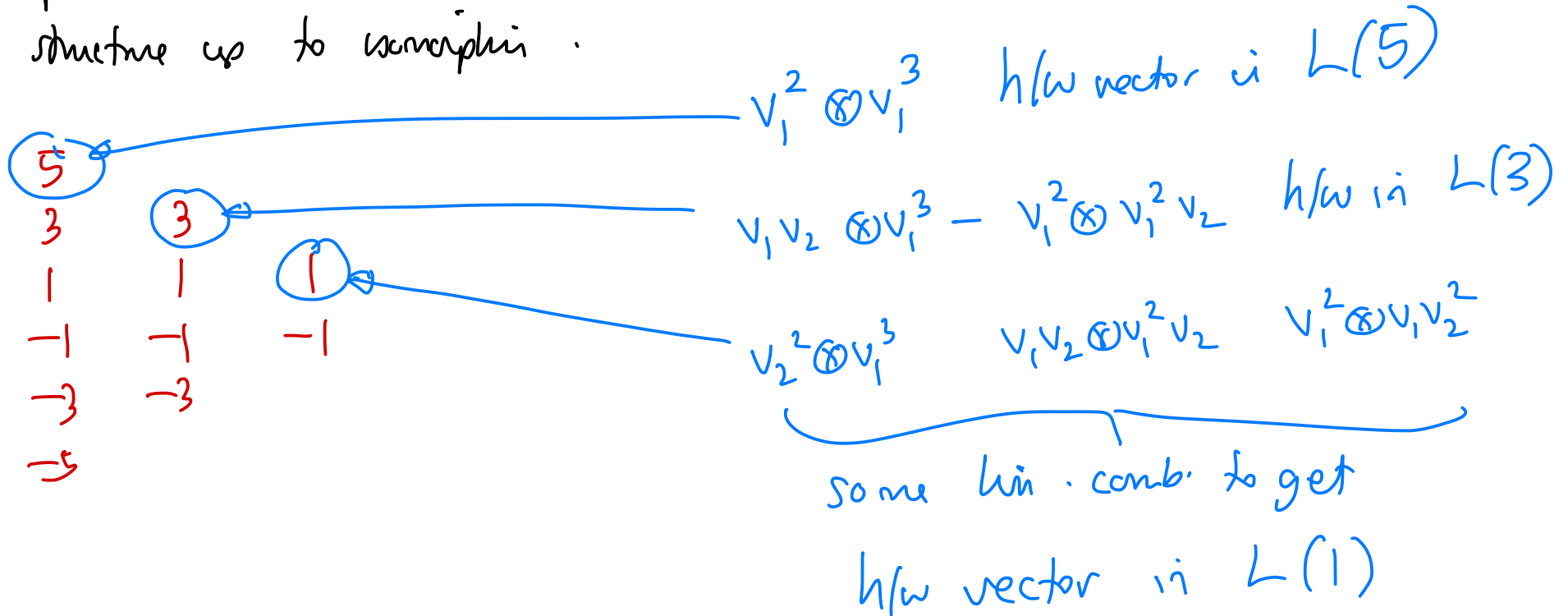
$$S^2 V \otimes S^3 V$$

$S //$

$$L(5) \oplus L(3) \oplus L(1)$$

$L(2) \setminus L(3)$	3	1	-1	-3
2	5	3	1	-1
0	3	1	-1	-3
-2	1	-1	-3	-5

Knowledge of weights + multiplicities
determines structure up to isomorphism.



Final topic in this chapter

More about nilpotent and solvable Lie algebras.

In Ch. 0, we saw that "Every algebraic group is linear."

What about f.d. Lie algebras? YES

The proof of Ado's theorem is tricky!

Any Lie algebra has

$$\begin{aligned} \text{ad}: \mathfrak{g} &\rightarrow \mathfrak{gl}(\mathfrak{g}) \\ x &\mapsto \text{ad } x \text{ where } (\text{ad } x)(y) = [x, y] \end{aligned}$$

$\ker \text{ad} = \mathfrak{z}(\mathfrak{g})$, so any f.d. centerless Lie algebra is linear via adjoint representation.

All algebraic Lie algebras are linear, already known too...

Ado's Theorem (1938)

Any f.d. Lie algebra over any field is isomorphic to a subalgebra of $\mathfrak{gl}(V)$ for some f.d. V .

Proof: Terence Tao's blog.

Nilpotent Ado

If \mathfrak{g} is a f.d. nilpotent Lie algebra then it is isomorphic to a subalgebra of $\mathfrak{gl}(V)$ consisting of nilpotent endomorphisms of V , some f.d. V .

Do want to prove two much more classical results:

Proofs next lecture!
Valid over any field.

Engel's Theorem Suppose $\mathfrak{g} \leq \mathfrak{gl}(V)$, $\dim V = n$, and all elements of \mathfrak{g} are nilpotent endomorphisms of V . Then there's a basis v_1, \dots, v_n for V such that $\mathfrak{g} \cdot v_i \in \langle v_1, \dots, v_{i-1} \rangle \forall i$.

Corollary 1 Such a \mathfrak{g} is nilpotent (as $\mathfrak{n}_n(\mathbb{C})$ is nilpotent)

Corollary 2 If \mathfrak{g} is a f.d. Lie algebra and $\text{ad}x: \mathfrak{g} \rightarrow \mathfrak{g}$ is nilpotent $\forall x \in \mathfrak{g}$, then \mathfrak{g} is nilpotent.

$\rightarrow \mathfrak{sl}_2(\mathbb{k}), \text{char } \mathbb{k} = 2$
 \rightarrow nilpotent but \mathbb{k}^2 is irred.

Lie's Theorem Suppose $\mathfrak{g} \leq \mathfrak{gl}(V)$, $\dim V = n$, and \mathfrak{g} is solvable.

Then there's a basis v_1, \dots, v_n for V such that $\mathfrak{g} \cdot v_i \in \langle v_1, \dots, v_i \rangle \forall i$.

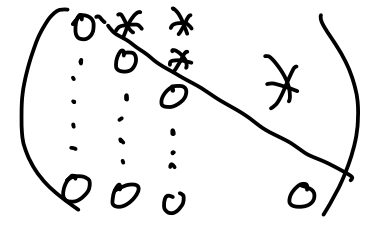
Corollary 1 If \mathfrak{g} is f.d. solvable Lie algebra / \mathbb{C} then \mathfrak{g}^1 is nilpotent.

Corollary 2 All f.d. irreps of f.d. solvable $\mathfrak{g} / \mathbb{C}$ are 1-D.

\rightarrow Requires field of char. 0 and alg. closed.

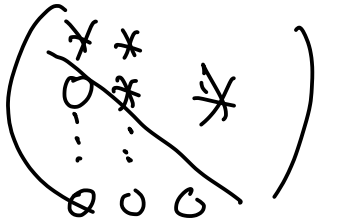
Discussion of Engel & Lie

A basis v_1, \dots, v_n for V identifies $\mathfrak{gl}(V) \cong \mathfrak{gl}_n(\mathbb{C})$



Engel says $\mathfrak{g} \hookrightarrow \mathfrak{n}_n(\mathbb{C})$, subalgebra of strictly upper \mathbb{C} 'r matrices

Lie says $\mathfrak{g} \hookrightarrow \mathfrak{b}_n(\mathbb{C})$, subalgebra of weakly upper \mathbb{C} 'r matrices



Note $\mathfrak{n}_n(\mathbb{C})$ IS a nilpotent Lie algebra

$\mathfrak{b}_n(\mathbb{C})$ IS a solvable Lie algebra with $\mathfrak{b}_n(\mathbb{C})' = \mathfrak{n}_n(\mathbb{C})$.

Combine Engel/Lie with nilpotent Ado / Ado, deduce:

- \mathfrak{g} f.d. nilpotent $\iff \mathfrak{g} \hookrightarrow \mathfrak{n}_n(\mathbb{C})$ some n
- \mathfrak{g} f.d. solvable $\iff \mathfrak{g} \hookrightarrow \mathfrak{b}_n(\mathbb{C})$ some n