Valid over any field.

Engel's Theorem Suppose $\mathfrak{g} < \mathfrak{gl}(V)$, $\dim V = n$, and all elements of $\mathfrak{g}$ are nilpotent endomorphisms of V. Then there's a basis $v_1, \dots, v_n$ for V such that $\mathfrak{g} \cdot v_i \in \langle v_1, \dots, v_{i-1} \rangle$ $\forall i$.

Corollary 1 Such a $\mathfrak{g}$ is nilpotent (as $\mathfrak{n}_n(\mathbb{C})$ is nilpotent)

Corollary 2 If $\mathfrak{g}$ is a f.d. Lie algebra and $\mathrm{ad}\, x : \mathfrak{g} \to \mathfrak{g}$ is nilpotent $\forall x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.

Lie's Theorem Suppose $\mathfrak{g} < \mathfrak{gl}(V)$, $\dim V = n$, and $\mathfrak{g}$ is solvable. Then there's a basis $v_1, \dots, v_n$ for V such that $\mathfrak{g} \cdot v_i \in \langle v_1, \dots, v_i \rangle$ $\forall i$.

Corollary 1 If $\mathfrak{g}$ is f.d. solvable Lie algebra $/\mathbb{C}$ then $\mathfrak{g}'$ is nilpotent.

Corollary 2 All f.d. irreps of f.d. solvable $\mathfrak{g} / \mathbb{C}$ are 1-D.

crucial

Requires field of char. 0 and alg. closed.

Proof of Engel $0 \neq \mathfrak{g} \leqslant \mathfrak{gl}(V)$, $\dim V = n$

All elements of $\mathfrak{g}$ are nilpotent endomorphisms of V.

Main step Show $\exists\ 0 \neq v \in V$ s.t. $\mathfrak{g} \cdot v = 0$.

Given that: Induction on n. Let $v_1 = v$. Then apply induction to the image of $\mathfrak{g}$ in $\mathfrak{gl}(V')$, $V' = V/\langle v \rangle$. Lift basis for V' to V, get $v_1, v_2, \dots, v_n$ so $\mathfrak{g} \cdot v_i \subseteq \langle v_1, \dots, v_{i-1} \rangle$ as required.

Now we must prove main step, use by induction on $\dim \mathfrak{g}$.

Base case: $\dim \mathfrak{g} = 1$. Pick $0 \neq v \in V$ to be any eigenvector for x.

$\mathfrak{g} = \langle x \rangle$ As x is nilpotent, $xv = 0$ ✓

Induction step: Let $0 \neq \mathfrak{n} < \mathfrak{g}$ maximal proper subalgebra.

Consider $\mathrm{ad} : \mathfrak{n} \longrightarrow \mathfrak{gl}(\mathfrak{g})$.

Note for $x \in \mathfrak{n}$, $\quad \mathrm{ad}\, x = (\lambda_x - \rho_x)|_{\mathfrak{g}}$

$\lambda_x \in \mathrm{End}(\mathfrak{gl}(V))$ defined by left mult. by x

$\rho_x \in \mathrm{End}(\mathfrak{gl}(V))$ " " right mult. by x

As x is nilpotent, $(\lambda_x)^n = (\rho_x)^n = 0$, also λ_x, ρ_x commute

$\therefore (\lambda_x - \rho_x)^{2n} = 0$

$\therefore$ ad x is nilpotent.

Look at induced representation $\overline{\mathrm{ad}} : \mathfrak{n} \longrightarrow \mathfrak{gl}(\mathfrak{g}/\mathfrak{n})$.

Apply induction hypothesis to image of $\mathfrak{n}$ under this homomorphism, to get a vector $y \in \mathfrak{g} - \mathfrak{n}$ so $[\mathfrak{n}, y] \subseteq \mathfrak{n}$.

Then $\mathfrak{n} \subsetneq \mathfrak{n} \oplus \mathbb{C}y \cancel{\subsetneq} \mathfrak{g} \implies \mathfrak{g} = \mathfrak{n} \oplus \mathbb{C}y.$

$=$ by maximality of $\mathfrak{n}$.

Let $W = \{ w \in V \mid \mathfrak{n} w = 0 \}$, non-zero by induction

As $[y, \mathfrak{n}] \subseteq \mathfrak{n}$, W is invariant under action of y.

$w \in W$, $x \in \mathfrak{n}$

$x(yw) = [xy]w + y(xw) = 0$ ✓

Pick $0 \neq v \in W$ that is an eigenvector for y, $yv = 0$ as y is nilpotent.

Then $\mathfrak{g} \cdot v = 0$ as $\mathfrak{g} = \mathfrak{n} \oplus \mathbb{C} y$

Proof of Lie's theorem. As before, induction on n to reduce the proof to checking

Main Step: $\exists\, 0 \neq v \in V$ s.t. $\mathfrak{g} v \in \langle v \rangle$.

To do that, $\mathfrak{g} \leq \mathfrak{gl}(V)$, $\mathfrak{g}$ is solvable. Use induction on $\dim \mathfrak{g}$.

Base case: $\dim \mathfrak{g} = 0$.

Induction step: $\dim \mathfrak{g} > 0$.

$\mathfrak{g}/\mathfrak{g}'$ is nonzero, Abelian Lie algebra. Pick codimension 1 subspace, take its pre-image in $\mathfrak{g}$, you get a codimension 1 ideal $\mathfrak{n} \triangleleft \mathfrak{g}$.

So $\mathfrak{g} = \mathfrak{n} \oplus \mathbb{C} y$, $\mathfrak{n} \triangleleft \mathfrak{g}$.

By induction $\exists\, 0 \neq w \in V$ s.t. $\mathfrak{n} w \in \langle w \rangle$.

This means for $x \in \mathfrak{n}$, $\quad xw = \lambda(x) w \quad$ for some $\lambda \in \mathfrak{n}^*$.

Let $W = \{ w \in V \mid xw = \lambda(x) w \quad \forall\, x \in \mathfrak{n} \} \neq 0$.

$\boxed{\text{Claim } yW \subseteq W.}$

need alg. closed field

Once we've proved the claim, the rest is easy... we're over $\mathbb{C}$, we can pick a $0 \neq v \in W$ that's an eigenvector for y

Then $\mathfrak{g} \cdot v \subseteq \langle v \rangle$ as $\mathfrak{g} = \mathfrak{n} \oplus \mathbb{C}y$ and we're done.

It remains to prove the claim.

Take $x \in \mathfrak{n}$, $w \in W$ $\quad (*)$

$$x(yw) = [xy]w + y(xw) = \lambda([xy])w + \lambda(x) yw$$

So we need to show $\lambda([xy]) = 0$ then done.

want $\lambda(x) yw$

Choose $n > 0$ maximal so $w, yw, y^2 w, \dots, y^{n-1}w$ are lin. independent.

Consider matrix of $x \curvearrowright \langle w, yw, y^2w, \dots, y^{n-1}w \rangle =: W'$

It is

$$
\begin{array}{ccccc}
w & yw & y^2w & \cdots & y^{n-1}w \\
\end{array}
$$

$$
\begin{pmatrix}
\lambda(x) & \lambda([x,y]) & & * & \\
0 & \lambda(x) & & & \\
\vdots & 0 & \lambda(x) & & \\
\vdots & \vdots & 0 & \ddots & \\
0 & 0 & 0 & & \lambda(x)
\end{pmatrix}
$$

upper $\triangledown$'r with $\lambda(x)$ on diagonal.

This follows like $(*)$ by induction... $x y^i w = \lambda(x) y^i w$ plus stuff in $\langle w, yw, \dots, y^{i-1}w \rangle$.

Uses char. 0.

Hence, $tr\left(x|_{W'}\right) = n\lambda(x) \quad \forall\, x \in n$

So $tr\left([x,y]|_{W'}\right) = n\,\lambda([x,y]) \Longrightarrow \lambda([x,y]) = 0$

$$\parallel$$

$0 = tr\left(x|_{W'} \circ y|_{W'} - y|_{W'} \circ x|_{W'}\right)$ $\qquad tr(AB) = tr(BA)$

Ch. 2 Semisimple Lie algebras and root systems

Working over $\mathbb{C}$.

Recall a f.d. Lie algebra $\mathfrak{g}$ is <u>semisimple</u> if it has no non-zero solvable ideals.

<u>Lemma</u> Suppose $\mathfrak{g} \leq \mathfrak{sl}(V)$ some f.d. V, and assume V is irreducible as a $\mathfrak{g}$-module. Then $\mathfrak{g}$ is semisimple.

<u>Very useful</u>!

eg $\mathfrak{sl}(V)$ is semisimple (!! $\mathfrak{sl}(V)$ is simple !!)

eg $\mathfrak{sp}(V)$ for dim V even, ≥ 2

eg $\mathfrak{so}(V)$ for dim $V \geq 3$

$Sp(V) \leq SL(V)$

$SO(V) \leq SL(V)$

G and $\mathfrak{g}$ leave same subspaces invariant $\therefore V$ is irreducible for $\mathfrak{g}$ too $\longrightarrow$ V is irreducible representation of G (Witt's theorem)

<u>Proof of lemma</u> Let $\mathfrak{n}$ be a solvable ideal, show $\mathfrak{n} = 0$.

Lie's theorem $\Longrightarrow$ $\exists\ 0 \neq v \in V$ s.t $\boxed{xv = \lambda(x) v}$ $\forall x \in \mathfrak{n}$
some $\lambda \in \mathfrak{n}^*$

For $y \in \mathfrak{g}$, $x \in \mathfrak{n}$
$$x\, y v = [xy] v + y x v \qquad (*)$$
$$= \lambda([xy]) v + \lambda(x) y v = 0$$

As V is irreducible over $\mathfrak{g}$, $V = U(\mathfrak{g})\, v$ monomials in a basis of $\mathfrak{g}$ span

$\therefore V$ has a basis consisting of $y_{i_1} \cdots y_{i_n} v$ y_{i_j}'s $\in \mathfrak{g}$.

Now like in proof of Lie's theorem, use $(*)$ and induction to see that $x \in \mathfrak{n}$ acts in an upper triangular way on this basis suitably ordered ...

$$x\big|_V = \begin{pmatrix} \lambda(x) & & * \\ & \lambda(x) & \\ 0 & & \ddots \\ & & & \lambda(x) \end{pmatrix}$$

with $\lambda(x)$ on the diagonal.

In particular, $\underset{0}{\underset{\shortparallel}{tr(x|_V)}} = (\dim V) \cdot \lambda(x)$.

$\therefore \lambda(x) = 0 \ \forall x \in \mathfrak{n}$.

Now it follows that any $x \in \mathfrak{n}$ actually has matrix zero when acting on this basis, $x \equiv 0$

$\Longrightarrow$ $\mathfrak{n} = 0$ so $\mathfrak{g}$ is semisimple