Useful lemma $\Longrightarrow$ $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{sp}_{2n}(\mathbb{C})$, $\mathfrak{so}_n(\mathbb{C})$ all are <u>semisimple</u>.

$(n \geq 2)$ $\quad (n \geq 1) \quad (n \geq 3)$

Easy: actually a simple Lie algebra

"non-Abelian simple"

$$\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{sp}_2(\mathbb{C}) \cong \mathfrak{so}_3(\mathbb{C})$$

$K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$

Another important criterion for semisimplicity.

<u>Def</u> The <u>Killing form</u> on a f.d. Lie algebra $\mathfrak{g}$ is the bilinear form K defined from $K(x,y) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}x \circ \mathrm{ad}y)$

$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$

$\mathrm{ad}x(y) = [x,y]$

- $K(x,y) = K(y,x)$ <u>symmetric</u>
- $K([x,y],z) = K(x,[y,z])$ ~~associative~~ <u>invariant</u>

$LHS = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}x \circ \mathrm{ad}y \circ \mathrm{ad}z - \mathrm{ad}y \circ \mathrm{ad}x \circ \mathrm{ad}z)$

$RHS = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}x \circ \mathrm{ad}y \circ \mathrm{ad}z - \mathrm{ad}x \circ \mathrm{ad}z \circ \mathrm{ad}y)$

These are equal ✓.

Equivalently,

$$K(\mathrm{ad}y(x), z) + K(x, \mathrm{ad}y(z)) = 0$$

Says the adjoint action of $\mathfrak{g}$ on itself leaves K invariant.

Invariance $\Longrightarrow$ $\operatorname{rad} K = \{ x \in \mathfrak{g} \mid K(x,y) = 0 \ \forall y \in \mathfrak{g} \}$

is an ideal of Lie algebra $\mathfrak{g}$.

eg $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ $\quad e, h, f$

$$\operatorname{ad} e = \begin{array}{c} \begin{matrix} e & h & f \end{matrix} \\ \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \operatorname{ad} f = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\operatorname{ad} h = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\therefore K(e,f) = \operatorname{tr} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 4$$

$$K(h,h) = \operatorname{tr} \begin{bmatrix} 4 & & 0 \\ 0 & 0 & \\ & & 4 \end{bmatrix} = 8$$

$\Longrightarrow$ Gram matrix of K

$$\begin{array}{c} \begin{matrix} e & h & f \end{matrix} \\ \begin{bmatrix} 0 & & 4 \\ & 8 & \\ 4 & & 0 \end{bmatrix} \end{array}$$

$\therefore$ non-degenerate.

There's another less painful way to get a bilinear form on $\mathfrak{g} = sl_n(\mathbb{C})$

This is the trace form $\tau : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, $\tau(x,y) = tr_V(x \circ y)$

We're using the natural representation of $\mathfrak{g}$ on $V = \mathbb{C}^n$ instead of adjoint representation.

eg $h = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & 0 & \\ & & & \ddots \\ 0 & & & & 0 \end{bmatrix}$

$\tau(h,h) = 2$

Again, τ is symmetric and invariant, so its radical is an ideal of $\mathfrak{g}$

So $\mathrm{rad}\,\tau = 0$, i.e., τ is non-degenerate on $\mathfrak{g} = sl_n(\mathbb{C})$.

Actually $K = t\tau$ some $t \in \mathbb{C}^\times$.

Proof Consider the bilinear form $t\tau - K$. Let $G_t = tG_\tau - G_K$ be its Gram matrix in some basis. $\det G_t = t^{(\dim \mathfrak{g})^2} \underbrace{\det G_\tau}_{\neq 0} + (\text{lower powers})$ $\neq 0$

Pick t to be a root, so $t\tau - K$ is degenerate, hence $\equiv 0$.

This shows $K = t\tau$ //

Theorem $\mathfrak{g}$ any f.d. Lie algebra $/\mathbb{C}$.

Then $\mathfrak{g}$ is semisimple $\iff$ K, Killing form on $\mathfrak{g}$, is non-degenerate.

Proof Depends on

Cartan's criterion for solvability. If $\mathfrak{g} \leq \mathfrak{gl}(V)$, V f.d., and $\tau(x,y) = 0$ $\forall x,y \in \mathfrak{g}$, then $\mathfrak{g}$ is solvable.

(Proof next time!)

Assuming this, let's prove current theorem.

Let $\mathfrak{n} = \operatorname{rad} K \trianglelefteq \mathfrak{g}$.

Suppose $\mathfrak{g}$ is semisimple, show $\mathfrak{n} = 0$.
$\mathfrak{z}(\mathfrak{g}) = 0$, so $\operatorname{ad}: \mathfrak{g} \hookrightarrow \mathfrak{gl}(\mathfrak{g})$. Apply Cartan's criterion to $\mathfrak{n} \leq \mathfrak{gl}(\mathfrak{g})$ to see that $\mathfrak{n}$ is solvable, hence $\mathfrak{n} = 0$ ✓

Conversely, assume $\mathfrak{n} = 0$. We must show $\mathfrak{g}$ is semisimple.

Let $\mathfrak{b}$ be a non-zero solvable ideal, get a contradiction.

Let $\mathfrak{a}$ be last non-zero term in derived series of $\mathfrak{b}$.

Then $0 \neq \mathfrak{a} \triangleleft \mathfrak{g}$ and $\mathfrak{a}$ is Abelian.

Take $x \in \mathfrak{a}, y \in \mathfrak{g}$

$$\underbrace{\mathrm{ad}x \circ \mathrm{ad}y \circ \mathrm{ad}x \circ \mathrm{ad}y}_{0} : \mathfrak{g} \xrightarrow{\mathrm{ad}y} \mathfrak{g} \xrightarrow{\mathrm{ad}x} \mathfrak{a} \xrightarrow{\mathrm{ad}y} \mathfrak{a} \xrightarrow{\mathrm{ad}x} 0$$

hence $(\mathrm{ad}x \circ \mathrm{ad}y)^2 = 0$

$\therefore$ $\mathrm{ad}x \circ \mathrm{ad}y$ is nilpotent, $\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}x \circ \mathrm{ad}y) = 0$

$K(x, y) = 0$

Shows $\mathfrak{a} \subseteq \mathfrak{n} = 0$ $\#$

Digression Jordan decomposition in linear algebra.

any alg. closed field ok

Theorem Let V be a f.d. vector space over $\mathbb{C}$, $x \in \mathrm{End}_{\mathbb{C}}(V)$.

Then $\exists!\ x_s, x_n \in \mathrm{End}_{\mathbb{C}}(V)$ s.t.

(1) x_s is semisimple ($\equiv$ diagonalizable $\equiv$ min. poly. of x_s has distinct linear factors)

(2) x_n is nilpotent

(3) $x = x_s + x_n$

(4) x_s and x_n commute

x_s = the semisimple part of x

x_n = the nilpotent part of x

Moreover, there are polynomials $p(t), q(t) \in \mathbb{C}[t]$ with $p(0) = q(0) = 0$ such that $x_s = p(x)$ and $x_n = q(x)$.

Proof. Uniqueness Suppose x_s, x_n are as in Theorem, let x_s', x_n' be more endomorphisms satisfying (1) – (4).

$$x = x_s + x_n = x_s' + x_n'$$

$$\therefore \quad x_s - x_s' = x_n' - x_n$$

As $x_s = p(x)$, x_s' commutes with x_s. Similarly x_n' commutes with x_n. So $x_s - x_s'$ is semisimple and $x_n' - x_n$ is nilpotent.

Hence, $x_s - x_s' = x_n' - x_n = 0$

Existence Pick a basis so x is in J.N.F.

$$x = \begin{bmatrix} \begin{matrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda \end{matrix} & & 0 \\ & \begin{matrix} \mu & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \mu \end{matrix} & \\ 0 & & \ddots \end{bmatrix}$$

Take x_s = "diagonal part" $= \begin{bmatrix} \lambda & & & & & 0 \\ & \ddots & & & & \\ & & \lambda & & & \\ & & & \mu & & \\ & & & & \ddots & \\ 0 & & & & & \mu \\ & & & & & & \ddots \end{bmatrix}$

x_n = "upper triangular parts" $= \begin{bmatrix} \begin{matrix} 0 & 1 & 0 \\ & \ddots & 1 \\ & & \end{matrix} & & 0 \\ & \begin{matrix} 0 & 1 & 0 \\ & \ddots & 1 \\ & & \end{matrix} & \\ 0 & & \ddots \end{bmatrix}$

Clearly commute, $x = x_s + x_n$, x_s is diagonal, x_n is nilpotent

But need final part... $x_s = p(x)$, $x_n = q(x)$

Let $\chi(t) = \prod_{i=1}^{k} (t-\lambda_i)^{n_i}$ be the characteristic polynomial of x

distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.

Let $V_i = \ker (x - \lambda_i \cdot id)^{n_i}$ generalised λ_i-eigenspace of x

$V = V_1 \oplus \cdots \oplus V_k$.

C.R.T. $\Rightarrow$ $\exists\, p(t)$ so $p(t) \equiv \lambda_i \pmod{(t-\lambda_i)^{n_i}}$ $\forall i$

and $p(t) \equiv 0 \pmod{t}$

$(x-\lambda_i \cdot id)^{n_i}$ is zero on V_i

so constant term of p is zero.

$p(x)\big|_{V_i} = \lambda_i \cdot id_{V_i}$

So $p(x) = x_s$

Let $q(t) = t - p(t)$ so $x_n = q(x)$

Discussion

If you have $\mathfrak{g} \leq \mathfrak{gl}(V)$, V f.d.

$\overset{\cup}{x}$ ... have $x_s, x_n \in \mathfrak{gl}(V)$

Do x_s, x_n Lie in the subspace $\mathfrak{g}$? <u>**NOT**</u> IN GENERAL

We'll show it <u>IS</u> true if $\mathfrak{g}$ is a semisimple Lie algebra.

eg $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) \leq \mathfrak{gl}_n(\mathbb{C})$

$\overset{\cup}{x} = x_s + x_n \qquad tr(x) = 0, \ tr(x_n) = 0 \Rightarrow tr(x_s) = 0$

so $x_n, x_s \in \mathfrak{sl}_n(\mathbb{C})$.

Cheap argument in this case!

Given that they do, what happens if you have some other representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V')$, V' f.d. ?

Is it true that $\rho(x_s) = \rho(x)_s$, $\rho(x_n) = \rho(x)_n$?

$\mathfrak{g}$

Again we'll prove this (for semisimple $\mathfrak{g}$).

ρ(semisimple x) is semisimple, ρ(nilpotent x) is nilpotent.

eg $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, $\rho = \mathrm{ad}$.

$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$

If $x \in \mathfrak{g}$ is nilpotent, we already know $\mathrm{ad}\, x$ is nilpotent too ✓

$\lambda_x - \rho_x$

If $x \in \mathfrak{g}$ is semisimple, also true that $\mathrm{ad}\, x$ is semisimple.

WLOG $x = \mathrm{diag}(t_1, \ldots, t_n)$.

Then $(\mathrm{ad}\, x) e_{ij} = (t_i - t_j) e_{ij}$. $\therefore$ $\mathrm{ad}\, x$ is diagonal in basis of matrix units.