Casimir element $z = \sum_{i=1}^{n} x_i y_i \in Z(\mathfrak{g})$ $\leftarrow$ center of $U(\mathfrak{g})$

($\mathfrak{g}$ semisimple Lie algebra so K is non-degenerate)

$x_1 \ldots x_n$ basis for $\mathfrak{g}$
$y_1 \ldots y_n$ dual basis wrt K
} $K(x_i, y_j) = \delta_{ij}$

<u>Lemma</u> $\mathfrak{g}$ simple, V be a non-trivial irreducible f.d. $\mathfrak{g}$-module.

Then z acts on V as a <u>non-zero</u> scalar.

<u>Proof</u> z acts on V by $c \cdot id_V$ by Schur's lemma for some $c \in \mathbb{C}$.

Note $tr_V(z) = c \cdot \dim V$.

Let τ be the trace form on $\mathfrak{g}$ coming from representation V $\quad \tau(x,y) = tr_V(xy)$

As $\mathfrak{g}$ is simple, $\tau = tK$ for some $t \in \mathbb{C}$. (See L6-1).

Also $t \neq 0$, as otherwise we'd get that $\mathfrak{g}$ is solvable by Cartan's criterion.

$\tau(x_i, y_j) = t\,K(x_i, y_j) = t\,\delta_{ij}$.

$\therefore tr_V(z) = \sum_{i=1}^{n} \tau(x_i, y_i) = t \cdot n = t \cdot \dim \mathfrak{g}$

$\Longrightarrow c = t \cdot \frac{\dim \mathfrak{g}}{\dim V} \neq 0$ $\quad \leftarrow \tau = tK \quad t = \frac{c \cdot \dim V}{\dim \mathfrak{g}}$

(eg) $t = c \cdot \frac{\dim V}{\dim \mathfrak{g}}$ $\quad$ V irrep. of simple $\mathfrak{g}$

$\Omega = tK$ $\quad$ c, scalar that z acts by.

For $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$

$$K = \begin{array}{c} \begin{array}{ccc} e & h & f \end{array} \\ \begin{pmatrix} & & 4 \\ & 8 & \\ 4 & & \end{pmatrix} \end{array}$$

Gram matrix of K

$\therefore z = \frac{1}{4} ef + \frac{1}{4} fe + \frac{1}{8} h^2 = \frac{1}{2} fe + \frac{1}{8} h(h+2)$

$\therefore$ On $L(n)$, z acts as $c = \frac{1}{8} n(n+2)$

$$\Omega = \begin{array}{c} \begin{array}{ccc} e & h & f \end{array} \\ \begin{pmatrix} & & 1 \\ & 2 & \\ 1 & & \end{pmatrix} \end{array}$$

$\therefore t = \frac{n(n+1)(n+2)}{24}$

$n=1$, $L(1)$ natural rep, $t = \frac{1}{4}$

<u>Theorem</u> (Weyl's theorem on complete reducibility) Let $\mathfrak{g}$ be a f.d. semisimple Lie algebra. Then any f.d. $\mathfrak{g}$-module is completely reducible.

($\mathrm{Rep}(\mathfrak{g})$ is a semisimple Abelian category)

<u>Proof</u> ① Reduce to case that $\mathfrak{g}$ is simple

Semisimple $\mathfrak{g}$ is $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ HW6-1

$U(\mathfrak{g}) = U(\mathfrak{g}_1) \otimes \cdots \otimes U(\mathfrak{g}_n)$ exercise!

+ Wedderburn's theorem

② Show every s.e.s. $0 \to V \to W \to \mathbb{C} \to 0$

f.d. irreducible trivial module

If V is trivial then image of $\mathfrak{g}$ in $\mathrm{End}_{\mathbb{C}}(W)$ looks like
$$\begin{array}{c|cc} & V & \mathbb{C} \\ \hline & \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) & \end{array}$$

Solvable!!! shows image is actually zero and W trivial 2-d rep ✓

If V is non-trivial, look at action of z.

Non-zero scalar on V by lemma, but zero on $\mathbb{C}$.
$$\begin{array}{cc} & \begin{array}{cc} V & \mathbb{C} \end{array} \\ z = & \left(\begin{array}{c|c} c \cdots 0 \\ 0 \cdots c & * \\ \hline 0 & 0 \end{array}\right) \end{array}$$

Follows that $\ker z$ gives a $\mathfrak{g}$-submodule of W

splitting the s.e.s.

③ Show every s.e.s.

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] & W \arrow[l, bend left, "f"] \arrow[r] & \mathbb{C} \arrow[r] & 0 \end{tikzcd}$$

split.

any f.d. (V) — trivial ($\mathbb{C}$)

This follows from ② by induction on dim V.

Pictures: $W = \begin{array}{|c|}\hline \mathbb{C} \\ \hline V \\ \hline \end{array}$ V irred ✓ by ② else $V = \begin{array}{|c|}\hline V' \\ \hline V'' \\ \hline \end{array}$ irred.

$$W = \begin{array}{|c|}\hline \mathbb{C} \\ \hline V' \\ \hline V'' \\ \hline \end{array} \cong \begin{array}{|c|}\hline \mathbb{C} \oplus V' \\ \hline V'' \\ \hline \end{array} \cong \mathbb{C} \oplus \begin{array}{|c|}\hline V' \\ \hline V'' \\ \hline \end{array}$$

✓

④ General case. Take s.e.s. $0 \to V \to W \to W/V \to 0$ of f.d. $\mathfrak{g}$-modules.

Consider $\mathrm{Hom}_{\mathbb{C}}(W, V) \cong W^* \otimes V$ view this as a $\mathfrak{g}$-module

$$(x \cdot f)(w) = x(f(w)) - f(xw)$$

$V \hookrightarrow W$

let $\underline{W} = \{ f \in \mathrm{Hom}_{\mathbb{C}}(W, V) \mid f|_V = c \cdot id_V$ some $c \in \mathbb{C} \}$

let $\underline{V} = \{ f \in \mathrm{Hom}_{\mathbb{C}}(W, V) \mid f|_V = 0 \}$

Note $\underline{W}$ is a $\mathfrak{g}$-submodule of $\mathrm{Hom}_{\mathbb{C}}(W, V)$

$$\underset{f}{\underline{W}} \ni \quad (x \cdot f)(v) = x(\underbrace{f(v)}_{x(cv)}) - \underbrace{f(xv)}_{cxv} = 0 \checkmark$$

So we have s.e.s.

$$0 \longrightarrow \underline{V} \longrightarrow \underline{W} \longrightarrow \mathbb{C} \longrightarrow 0$$

Splits by step (3), so $\exists f \in \underline{W}$ invariant under $\mathfrak{g}$-action

$x \cdot f = 0$ i.e. $f \in \mathrm{Hom}_{\mathfrak{g}}(W, V)$

So $\underline{W} = \underline{V} \oplus \mathbb{C} f$.

Scale f so $f|_V = id_V$ then f splits the original s.e.s. //

Applies eg $G = SL_n(\mathbb{C}), \quad Sp_{2n}(\mathbb{C}), \quad SO_n(\mathbb{C})$

These act irreducibly on natural representation V

Hence $\mathfrak{g}$ is semisimple Lie algebra

Hence Weyl's theorem shows every f.d. $\mathfrak{g}$-module is c.r.

Hence every representation of G is c.r.

Philosophy of Lie theory !!

$\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}\,\mathfrak{g}$

Remarkable rigidity of semisimple Lie algebras

Theorem Let $\mathfrak{g}$ be a f.d. semisimple Lie algebra. All derivations of $\mathfrak{g}$ are inner.

$$\underset{\text{inner derivations}}{\mathrm{ad}\,\mathfrak{g}} \trianglelefteq \underset{\text{all derivations of } \mathfrak{g}}{\mathrm{Der}(\mathfrak{g})} \leq \mathfrak{gl}(\mathfrak{g})$$

Why $\trianglelefteq$? Take $D \in \mathrm{Der}(\mathfrak{g})$, $x \in \mathfrak{g}$

$$[D, \mathrm{ad}\, x] = \mathrm{ad}\, D(x) \qquad (*)$$

Proof. As $\mathfrak{g}$ is semisimple, $\mathfrak{z}(\mathfrak{g}) = 0$, so $\mathrm{ad} : \mathfrak{g} \hookrightarrow \mathrm{Der}(\mathfrak{g})$.
Identify $\mathfrak{g}$ with its image under ad ... so $\mathfrak{g} \trianglelefteq \mathrm{Der}(\mathfrak{g})$

For $x \in \mathfrak{g}$, $D \in \mathrm{Der}(\mathfrak{g})$, $\quad [D, x] = D(x) \qquad (*)$

Let K be the Killing form on $\mathrm{Der}(\mathfrak{g})$. As $\mathfrak{g} \trianglelefteq \mathrm{Der}(\mathfrak{g})$, the restriction of K to $\mathfrak{g}$ is the Killing form on $\mathfrak{g}$, so it's non-degenerate.
Shows: $\mathfrak{g} \cap \mathfrak{g}^\perp = 0$ where $\mathfrak{g}^\perp = \{ D \in \mathrm{Der}(\mathfrak{g}) \mid K(D, x) = 0 \ \forall x \in \mathfrak{g} \}$

$\mathfrak{g} \cap \mathfrak{g}^\perp = 0 \implies$ for $D \in \mathfrak{g}^\perp$, $x \in \mathfrak{g}$ $\quad [D, x] \in \mathfrak{g} \cap \mathfrak{g}^\perp = 0$

$\mathfrak{g}, \mathfrak{g}^\perp \trianglelefteq \mathrm{Der}(\mathfrak{g})$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad D(x)$ by $(*)$ hence $D = 0$

Shows $\mathfrak{g}^\perp = 0$, and the form K is non-degenerate.

$\mathrm{Der}(\mathfrak{g}) = \mathfrak{g} \oplus \cancel{\mathfrak{g}^\perp} = \mathfrak{g}$

Take $\mathfrak{g}$ f.d. semisimple.

Could you "extend on top"

$$0 \to \mathfrak{g} \to \hat{\mathfrak{g}} \to \mathbb{C} \to 0 \quad ? \quad \underline{\underline{\text{NO}}}$$

(s.e.s. of Lie algebras / Lie algebra extension)

If you could, $D \in \hat{\mathfrak{g}}$ lift $1 \in \mathbb{C}$. Then $[D, -]$ is a derivation of $\mathfrak{g} \triangleleft \hat{\mathfrak{g}}$, hence, it's <u>inner</u> ... so $\hat{\mathfrak{g}} \cong \mathfrak{g} \oplus \mathbb{C}$

only in the split way

Could you "extend on bottom"

$$0 \to \mathbb{C} \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0 \quad ? \quad \underline{\underline{\text{NO}}}$$

(central) extensions of $\hat{\mathfrak{g}}$).

beginnings of Lie cohomology

Quite different for finite simple groups, e.g. A_n $(n \geq 5)$

$$1 \to A_n \to S_n \to C_2 \to 1$$

Can "extend on top"!

$$1 \to C_2 \to \hat{A}_n \to A_n \to 1$$

double cover of A_n

Can "extend on the bottom"!

$$\begin{tikzcd}
A_n \arrow[r, hook] \arrow[dr, phantom, "\circ"] & SO_n(\mathbb{C}) \\
\hat{A}_n \arrow[r, hook] \arrow[u, two heads] & Spin_n(\mathbb{C}) \arrow[u, two heads]
\end{tikzcd}$$