$\mathfrak{g}$ semisimple Lie algebra, $\mathfrak{t}$ maximal toral subalgebra — could be $\mathfrak{k}$.

$(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ some non-degenerate symmetric invariant bilinear form.

<u>Cartan decomposition</u> $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$

$\mathfrak{t} = \mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{t})$

$(\cdot,\cdot)|_{\mathfrak{t}\times\mathfrak{t}}$ is non-degenerate

$R = \{0 \neq \alpha \in \mathfrak{t}^* \mid \mathfrak{g}_\alpha \neq 0\}$

$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x \ \ \forall h \in \mathfrak{t}\}$

<u>Example</u> $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. We've shown this is semisimple (even simple).

Let $\mathfrak{t}$ = subalgebra of trace zero diagonal matrices. Toral ✓

Basis for $\mathfrak{t}$: $h_1, h_2, \dots, h_{n-1}$ $\quad h_i = \mathrm{diag}(0,\dots,0,\underset{i\text{th}}{1},\underset{(i+1)}{-1},0,\dots,0)$

Let $\varepsilon_i \in \mathfrak{t}^*$, $\varepsilon_i(\mathrm{diag}(t_1,\dots,t_n)) \mapsto t_i$. These span $\mathfrak{t}^*$

subject to one linear relation $\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n = 0$.

Then $\mathfrak{g} = \mathfrak{z} \oplus \bigoplus_{i \neq j} \mathbb{C} e_{ij}$ Cartan decomposition of $\mathfrak{g}$ wrt $\mathfrak{z}$

matrix mult.

If $h = \mathrm{diag}(t_1, \dots, t_n) \in \mathfrak{z}$ then $h e_{ij} = t_i e_{ij}$, $e_{ij} h = t_j e_{ij}$

$$\Longrightarrow [h, e_{ij}] = (t_i - t_j) e_{ij} = (\varepsilon_i - \varepsilon_j)(h)\, e_{ij}$$

Shows:

- $\mathfrak{g}_0 = \mathfrak{z}$, hence, $\mathfrak{z}$ is <u>maximal</u> toral subalgebra
- $R = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i,j \leq n,\ i \neq j \}$
- each $\mathfrak{g}_\alpha$ for $\alpha \in R$ is 1-D (spanned by e_{ij} if $\alpha = \varepsilon_i - \varepsilon_j$)
- Say $(\cdot,\cdot) = \tau$, trace form. Then:

$$\begin{array}{cc} e_{ij} & e_{ji} \\ \end{array}$$
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

non-degenerate ✓

Gram matrix of $(\cdot,\cdot)$

$$\begin{array}{cccc} h_1 & h_2 & \cdots & h_{n-1} \end{array}$$
$$\begin{bmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 2 \end{bmatrix}$$

determinant $n \neq 0$

non-degenerate ✓

Back to general case! $\mathfrak{g}$, $\mathfrak{h}$, $(\cdot,\cdot)$

Properties of the root system

$(\cdot,\cdot) : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ is non-degenerate

Use it to identify $\mathfrak{h}$ with $\mathfrak{h}^*$ ... so $\lambda \in \mathfrak{h}^*$ is identified with $t_\lambda \in \mathfrak{h}$

where $(t_\lambda, h) = \lambda(h)$ $\forall h \in \mathfrak{h}$.

Note R spans $\mathfrak{h}^*$.

[Proof Else, you could find $0 \neq h \in \mathfrak{h}$ so $\alpha(h) = 0$ $\forall \alpha \in R$.

Then $[h, \mathfrak{g}_\alpha] = 0$ $\forall \alpha \in R$, so $h \in \mathfrak{z}(\mathfrak{g}) = 0$ #]

Take $\alpha \in R$. $(\cdot,\cdot)|_{\mathfrak{g}_\alpha} = 0$, $(\cdot,\cdot)|_{\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}}$ is non-degenerate,

hence, $-\alpha \in R$, and ...

Pick $0 \neq e_\alpha \in \mathfrak{g}_\alpha$, $\exists f_\alpha \in \mathfrak{g}_{-\alpha}$ s.t. $(e_\alpha, f_\alpha) \neq 0$.

(Choice here!!!)

Lemma | ① $[e_\alpha, f_\alpha] = (e_\alpha, f_\alpha)\, t_\alpha \neq 0$

for $\alpha \in R$ ...

② $(t_\alpha, t_\alpha) \neq 0$, hence, after rescaling f_α if necessary, we can assume $(e_\alpha, f_\alpha) = \dfrac{2}{(t_\alpha, t_\alpha)}$.

③ Let $h_\alpha = \dfrac{2t_\alpha}{(t_\alpha, t_\alpha)}$. Then

$$[e_\alpha, f_\alpha] = h_\alpha, \quad [h_\alpha, e_\alpha] = 2e_\alpha, \quad [h_\alpha, f_\alpha] = -2f_\alpha.$$

$\Longrightarrow (e_\alpha, h_\alpha, f_\alpha)$ span a subalgebra of $\mathfrak{g}$ $\cong sl_2(\mathbb{C})$.

eg $\mathfrak{g} = sl_n(\mathbb{C})$ as above, $\alpha = \varepsilon_i - \varepsilon_j$

Take $e_\alpha = e_{ij}$, $f_\alpha = e_{ji}$, $h_\alpha = \mathrm{diag}(0 \cdots \underset{i\text{th}}{1} \cdots \underset{j\text{th}}{-1} \cdots 0)$

Proof ① Know $(e_\alpha, f_\alpha)\, t_\alpha \neq 0$.

Take $h \in \mathfrak{t}$. Then $(h, [e_\alpha, f_\alpha]) = ([h, e_\alpha], f_\alpha)$

$$= \alpha(h)\,(e_\alpha, f_\alpha)$$

$$= (t_\alpha, h)(e_\alpha, f_\alpha)$$

$$= ((e_\alpha, f_\alpha)\, t_\alpha, h)$$

$\forall h \in \mathfrak{t}$

hence $[e_\alpha, f_\alpha] = (e_\alpha, f_\alpha)\, t_\alpha$

using non-degeneracy of form on $\mathfrak{t}$

② Suppose $(t_\alpha, t_\alpha) = 0$. Then $[t_\alpha, e_\alpha] = 0 = [t_\alpha, f_\alpha]$

So $\mathfrak{g}_\alpha' = \mathbb{C} e_\alpha + \mathbb{C} t_\alpha + \mathbb{C} f_\alpha$ is a solvable Lie subalg. of $\mathfrak{g}$.

Lie's theorem $\Longrightarrow \exists$ a basis for $\mathfrak{g}$ wrt $\mathfrak{g}_\alpha' \leq \begin{pmatrix} & * \\ 0 & \end{pmatrix}$

But then $t_\alpha \in \mathfrak{g}_\alpha'$ which consists of nilpotent matrices in adjoint representation

Shows $\operatorname{ad} t_\alpha$ is nilpotent, also semisimple, hence, $\operatorname{ad} t_\alpha = 0$

So $t_\alpha = 0$ $\#$.

③ $[e_\alpha, f_\alpha] = h_\alpha$ ✓ $\qquad h_\alpha : \dfrac{2t_\alpha}{(t_\alpha, t_\alpha)}$

$$[h_\alpha, e_\alpha] = \frac{2}{(t_\alpha, t_\alpha)}[t_\alpha, e_\alpha] = \frac{2\alpha(t_\alpha)}{(t_\alpha, t_\alpha)} e_\alpha = 2e_\alpha$$

$[h_\alpha, f_\alpha] = -2f_\alpha$ similar

<u>Lemma 2</u> For $\alpha \in R$, only multiples of α that belong to R are $\pm\alpha$. Moreover, $\dim \mathfrak{g}_\alpha = 1$, so $\dim \mathfrak{g} = \dim \mathfrak{h} + |R|$.

<u>Proof</u> Let $e_\alpha, h_\alpha, f_\alpha$ be as in Lemma 1, so $\mathfrak{g}_\alpha = \mathbb{C}e_\alpha \oplus \mathbb{C}h_\alpha \oplus \mathbb{C}f_\alpha$
$\cong sl_2(\mathbb{C})$.

Let $M = \bigoplus_{c \in \mathbb{C}} \mathfrak{g}_{c\alpha}$

Note $\mathfrak{g}_\alpha \curvearrowright M$ via ad, h_α acts on $\mathfrak{g}_{c\alpha}$ as

Rep. theory of $sl_2(\mathbb{C}) \Longrightarrow$ $\qquad c\alpha(h_\alpha) = \dfrac{2c\cancel{\alpha(t_\alpha)}}{\cancel{(t_\alpha, t_\alpha)}} = 2c.$

$\mathfrak{g}_{c\alpha} = 0$ unless $2c \in \mathbb{Z}$. (eigenvalues of h_α).

$\mathfrak{g}_\alpha$ acts trivially on $\ker(\alpha : \mathfrak{t} \to \mathbb{C})$ ← codim 1 subspace of $\mathfrak{t}$.

So $M = \underbrace{\boxed{0 \; - - \; - - \; 0}}_{\dim \mathfrak{t} - 1} \oplus \underbrace{\boxed{\begin{matrix} 2 \\ 0 \\ -2 \end{matrix}}}_{\mathfrak{g}_\alpha \text{ itself}} \oplus \cancel{\mathfrak{g}_{c\alpha}\text{'s}}$ for $2c$ being odd.

$$\begin{matrix} 2c \\ 2c-2 \\ \vdots \\ 2-2c \\ -2c \end{matrix}$$

no 0 wt space in remaining $\mathfrak{sl}_2$-strings

Shows: If $\alpha \in R$, then $-\alpha \in R$, $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ are 1-D

and $2\alpha \notin R$

We can't have any $\mathfrak{g}_{c\alpha} \neq 0$ for $2c$ odd

As if one did, we'd have 1-eigenspace for h_α,

so $c = \frac{1}{2}$ ... $\mathfrak{g}_{\frac{1}{2}\alpha} \neq 0$ as if $\frac{1}{2}\alpha \in R$ then $\alpha \notin R$ #

So now we've shown:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \underset{\text{1-D}}{\mathfrak{g}_\alpha}$$

e_α basis, $\exists\, f_\alpha \in \mathfrak{g}_{-\alpha}$

$h_\alpha = [e_\alpha, f_\alpha]$

so $e_\alpha, h_\alpha, f_\alpha$ $\mathfrak{sl}_2$-triple

$$h_\alpha = \frac{2 t_\alpha}{(t_\alpha, t_\alpha)}.$$

h_α's $(\alpha \in R)$ span $\mathfrak{t}$

e_α's $(\alpha \in R)$ give a basis for the rest

$$\dim \mathfrak{g} = \underbrace{\dim \mathfrak{t}}_{\operatorname{rank}(\mathfrak{g})} + |R|$$

Lemma 3 $\alpha, \beta \in R$, $\beta \neq \pm\alpha$

Let $r, q \in \mathbb{N}$ be maximal so $\beta - r\alpha$ and $\beta + q\alpha$ lie in R.

Then all $\beta + i\alpha$ $(-r \leq i \leq q)$ belong to R.

and $\beta(h_\alpha) = r - q$

Hence, $\beta(h_\alpha) \in \mathbb{Z}$ and $\beta - \beta(h_\alpha)\alpha \in R$.

Proof Let $\mathfrak{g}_\alpha$ be as above. $\mathfrak{g}_\alpha \hookrightarrow M$ via ad.

1-D or 0 by lemma 2

Let $M = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\beta + i\alpha}$

h_α acts on $\mathfrak{g}_{\beta+i\alpha}$ as $(\beta + i\alpha)(h_\alpha) = \beta(h_\alpha) + i \dfrac{2\cancel{\alpha(t_\alpha)}}{\cancel{(t_\alpha, t_\alpha)}}$

$= \beta(h_\alpha) + 2i$

$\mathbb{Z}_{\psi}$

If $\beta(h_\alpha)$ even ... all are even, 0 wt space is 1-D
If $\beta(h_\alpha)$ odd ... all are odd, 1 wt space is 1-D
$\Longrightarrow$ M is an irreducible $\mathfrak{g}_\alpha$-module

So M is irred, highest h_α-weight is $\beta(h_\alpha)+2q$

lowest " " $\beta(h_\alpha)-2r$

We have every $\beta(h_\alpha)+2i$, $-r \leq i \leq q$ (just one sl_2-string)

Finally $\beta(h_\alpha)-2r = -(\beta(h_\alpha)+2q)$

$\therefore 2\beta(h_\alpha) = 2r-2q$

$\therefore \beta(h_\alpha) = r-q$