Ch. 3 Classification of root systems

E Euclidean space, inner product $(\cdot,\cdot)$

$\alpha^\vee = \dfrac{2\alpha}{(\alpha,\alpha)}$ $\qquad s_\alpha : E \to E$ reflection in hyperplane $\alpha^\perp$

$s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\,\alpha$

Root system: $(R \subseteq E)$

① R is finite and spans E

② $\alpha \in R$, $c\alpha \in R \iff c = \pm 1$ (so: $0 \notin R$)

③ $s_\alpha(R) = R$

④ $(\beta, \alpha^\vee) \in \mathbb{Z}$ $\forall\, \alpha, \beta \in R$.

$\operatorname{rank}(R) = \dim E$.

$(\alpha, \beta) = \|\alpha\| \, \|\beta\| \cos\theta$

Basic properties of $(R \subseteq E)$

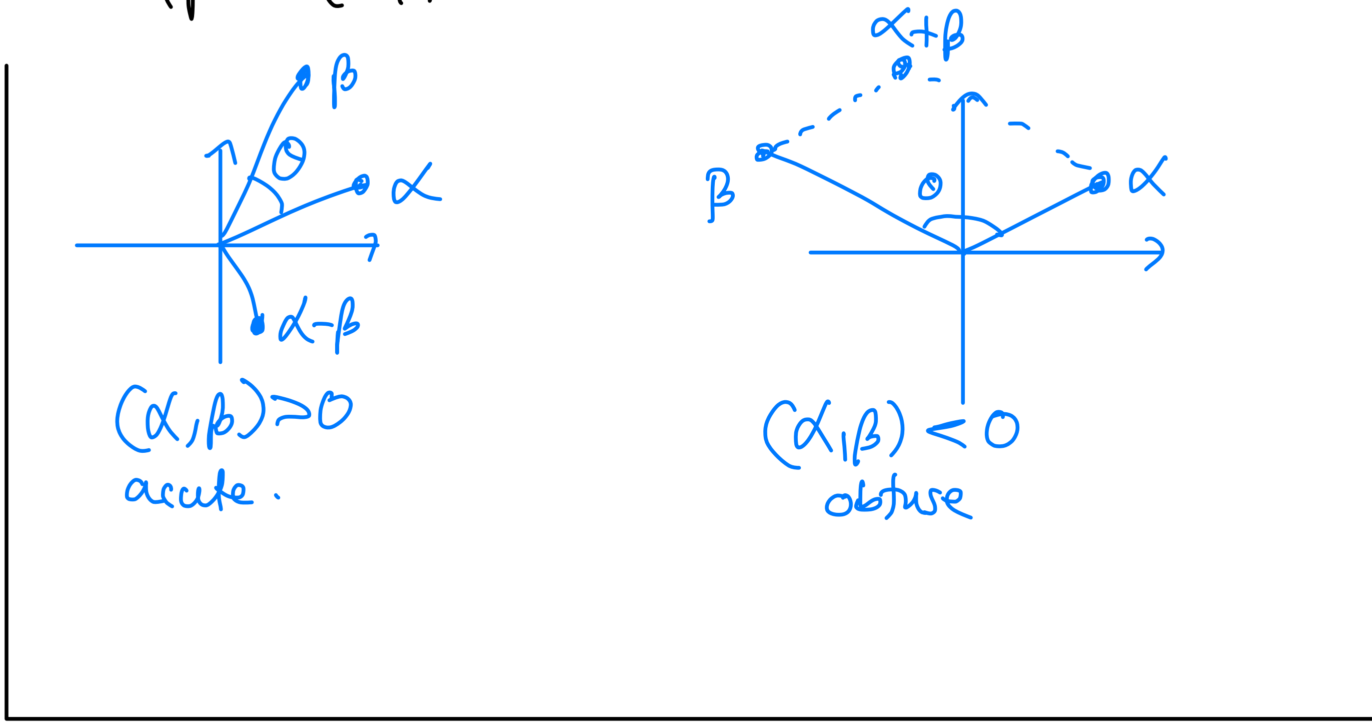
Lemma 1 $\alpha, \beta \in R$, $\alpha \neq \pm\beta$. If $(\alpha, \beta) > 0$ then $\alpha - \beta \in R$

If $(\alpha, \beta) < 0$ then $\alpha + \beta \in R$

Proof $(\alpha, \beta) > 0$.

$$(\beta, \alpha^\vee) = \frac{2\|\beta\|}{\|\alpha\|} \cos\theta$$

$$\therefore \underbrace{(\beta, \alpha^\vee)}_{\in \mathbb{Z}} \underbrace{(\alpha, \beta^\vee)}_{\in \mathbb{Z}} = \underbrace{4\cos^2\theta}_{< 4} \in \mathbb{Z}$$



It's 1, 2 or 3.

WLOG $(\alpha, \beta^\vee) = 1$

$(\beta, \alpha^\vee) = 1, 2$ or 3.

$$S_\beta(\alpha) = \alpha - (\alpha, \beta^\vee)\beta = \alpha - \beta \in R \text{ by } ③$$

Useful table $\alpha, \beta \in R$ linearly independent

$(\alpha, \beta^\vee)$	$(\beta, \alpha^\vee)$	θ	$\|\beta\| / \|\alpha\|$
0	0	$\pi/2$	?
1	1	$\pi/3$	1
1	2	$\pi/4$	$\sqrt{2}$
1	3	$\pi/6$	$\sqrt{3}$
-1	-1	$2\pi/3$	1
-1	-2	$3\pi/4$	$\sqrt{2}$
-1	-3	$5\pi/6$	$\sqrt{3}$

or flip columns and $\frac{1}{\cdot}$ the last column.

Lemma 2 Take $\alpha, \beta \in R$, $\alpha \neq \pm\beta$. Let $r, q \in \mathbb{N}$ be maximal so $\beta - r\alpha$ and $\beta + q\alpha$ are both roots.

Then $\beta + i\alpha$ is a root $\forall$ $-r \leq i \leq q$

Moreover, $(\beta, \alpha^\vee) = r - q \in \{0, \pm 1, \pm 2, \pm 3\}$

We saw this in Lemma 3 at end of last chapter.

α-string through β

$\underbrace{\beta - r\alpha, \ldots, \beta + i\alpha, \ldots, \beta + q\alpha}_{\text{all are roots}}$

Proof Suppose some $\beta + i\alpha$ is <u>not</u> a root.

Pick i minimal, $-r < i < q$, so $\beta + i\alpha \notin R$.

Pick j maximal, $-r < j < q$, so $\beta + j\alpha \notin R$.

Then $i \leq j$

$\left.\begin{array}{l}\beta + i\alpha \notin R \\ \beta + (i-1)\alpha \in R\end{array}\right\}$ Lemma 1 $\Longrightarrow (\beta + (i-1)\alpha, \alpha) \geq 0$

$\left.\begin{array}{l}\beta + j\alpha \notin R \\ \beta + (j+1)\alpha \in R\end{array}\right\}$ Lemma 1 $\Longrightarrow (\beta + (j+1)\alpha, \alpha) \leq 0$

Subtract:

$\underbrace{(i - j - 2)}_{<0} \underbrace{(\alpha, \alpha)}_{>0} \geq 0$

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$$s_\alpha(\beta + i\alpha) = \beta + i\alpha - (\beta + i\alpha, \alpha^\vee)\alpha = \beta - (\beta, \alpha^\vee)\alpha - i\alpha$$

also in α-string through β

This must just "reverse" the string.

$$s_\alpha(\beta + q\alpha) = \beta - r\alpha = \beta - (\beta, \alpha^\vee)\alpha - q\alpha$$

$$\therefore\ r = (\beta, \alpha^\vee) + q$$

$$\therefore\ (\beta, \alpha^\vee) = r - q \quad \in \{0, \pm 1, \pm 2, \pm 3\}$$

how far β is from bottom of α-string

how far β is from top of string

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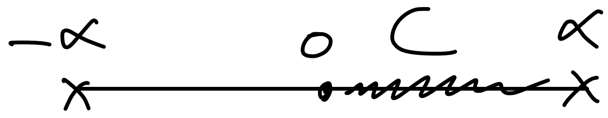
Examples

$\boxed{A_1}$

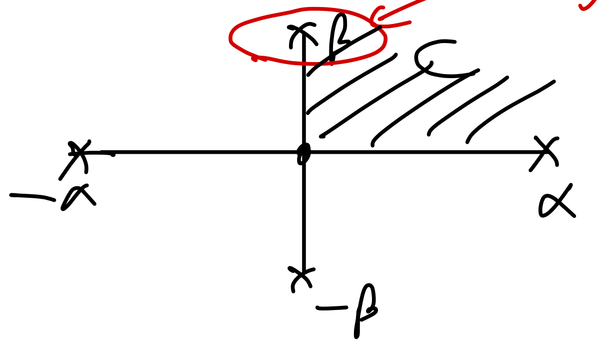


$\Delta_C = \{\alpha\}$

$R = \{\pm\alpha\}$

$\boxed{A_1 \times A_1}$

α-string through β, length 1

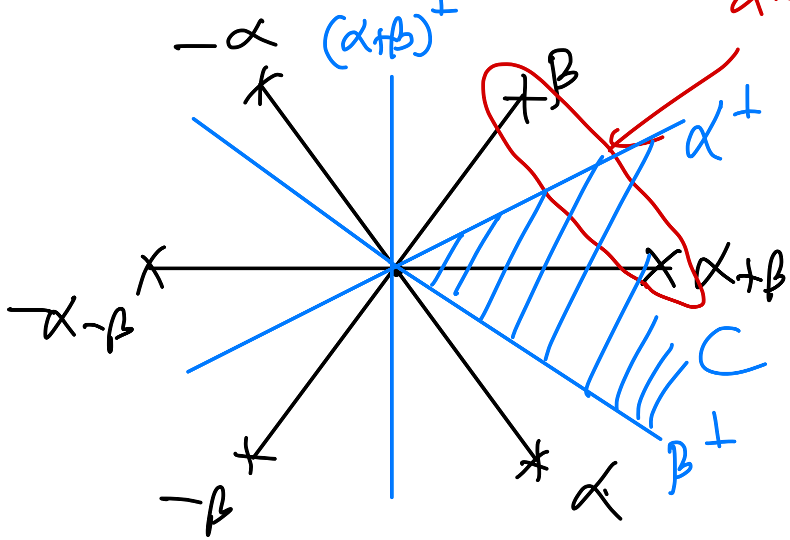


$R = \{\pm\alpha, \pm\beta\}$

$\Delta_C = \{\alpha, \beta\}$

$\boxed{A_2}$

α-string through β, length 2



$|R| = 6$

$\Delta_C = \{\alpha, \beta\}$

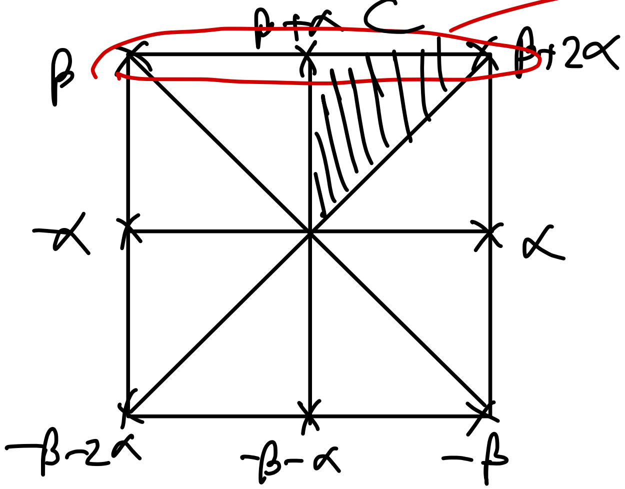
$(\alpha, \beta^\vee) = -1, \quad (\beta, \alpha^\vee) = -1$

$\theta = \frac{2\pi}{3}, \quad \|\alpha\| = \|\beta\|$

$\boxed{B_2}$

$(\alpha, \beta^\vee) = -1, \ (\beta, \alpha^\vee) = -2$

$\theta = \frac{3\pi}{4}, \ \|\beta\| = \sqrt{2}\|\alpha\|$



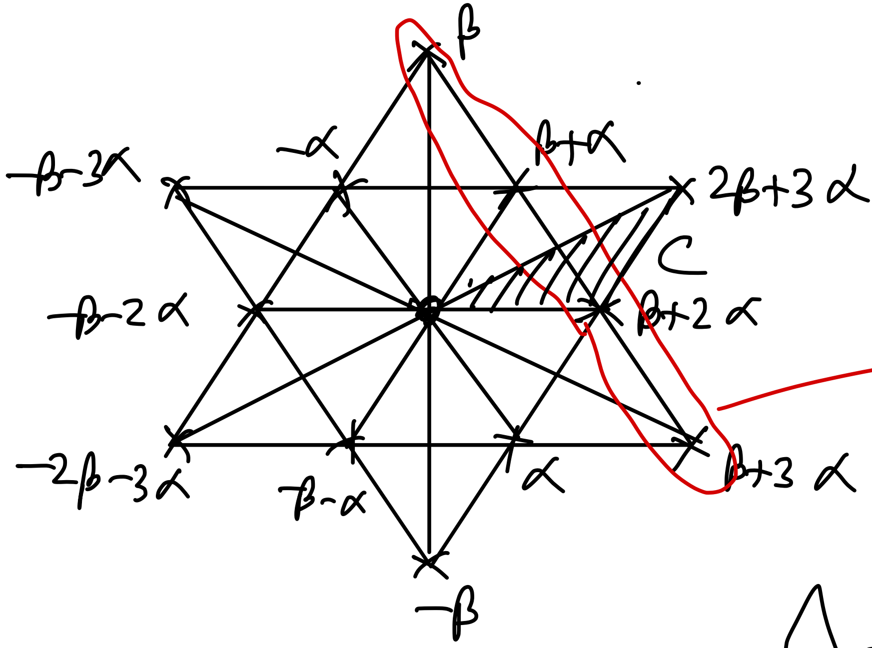
α-string through β, length 3

$\Delta_C = \{\alpha, \beta\}$

$\boxed{G_2}$

$(\alpha, \beta^\vee) = -1, \ (\beta, \alpha^\vee) = -3$

$\theta = \frac{5\pi}{6}, \ \|\beta\| = \sqrt{3}\|\alpha\|$



α-string through β, length 4

$\Delta_C = \{\alpha, \beta\}$

Bases A *base* for a root system ($R \subseteq E$) is a subset Δ of R such that Δ is a basis for E, and if $\beta \in R$ written as $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$ then *either* all $c_\alpha \geq 0$ *or* all $c_\alpha \leq 0$.

Given a base $\Delta = \{ \overbrace{\alpha_1, \alpha_2, \ldots \alpha_\ell}^{\text{at obtuse angles to each other}} \}$, ($\ell = \operatorname{rank}(R) = \dim E$)

we'll call the $\alpha_i \in \Delta$ the *simple roots*. Any $\beta \in R$ is either a positive sum of α_i's or a negative sum. We'll call β *positive* or *negative* accordingly. So

$$R = \underbrace{R^+}_{\text{positive roots}} \sqcup \underbrace{R^-}_{\text{negative roots}}$$

Note also for $\alpha_i \neq \alpha_j$ in Δ, $(\alpha_i, \alpha_j) \leq 0$ by Lemma)
(as $\alpha_i - \alpha_j \notin R$)

Theorem Bases exist for any root system.

See Humphreys 10.1.

Show you the construction of bases

Consider $E - \bigcup_{\alpha \in R} \alpha^\perp$. This has a bunch of connected components called chambers.

Given a chamber C, define

$$\Delta_C = \{\alpha \in R \mid \alpha^\perp \text{ is bounding hyperplane of } C, \text{ and } \alpha \text{ is at an acute angle to every vector in } C\}$$

Then proof of theorem shows Δ_C is a base, and all bases arise from chambers in this way.