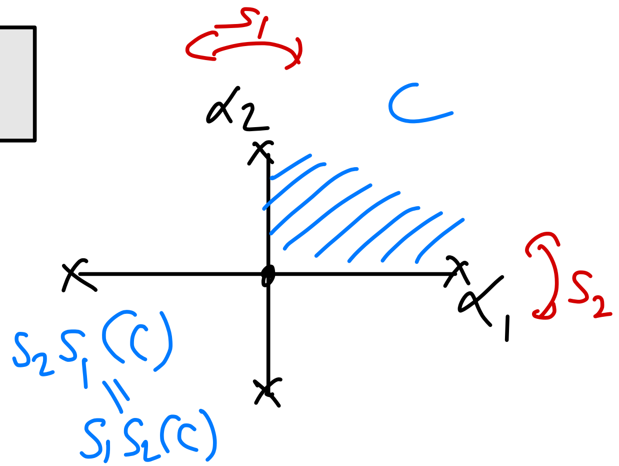
Theorem $W = \langle s_1, \dots, s_\ell \mid s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1 \quad (i \neq j) \rangle$ $\quad m_{ij} = 2, 3, 4$ or 6

COXETER PRESENTATION

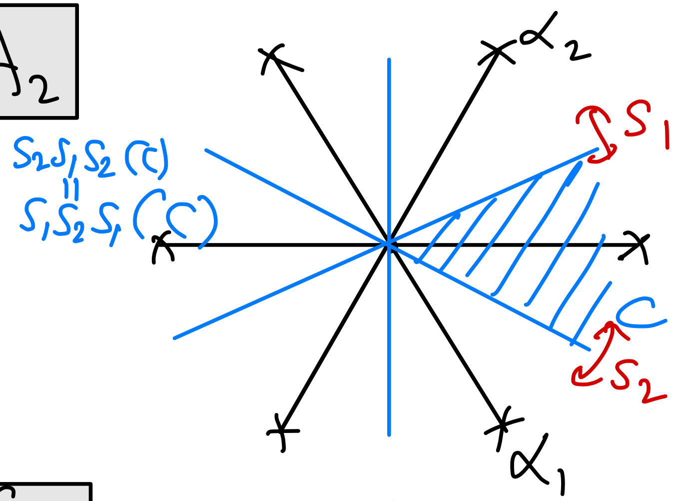
$$\begin{tikzcd} & W \arrow[dl, "w(C)"'] \arrow[dr, "w(\Delta)"] & \\ \{\text{chambers}\} \arrow[rr, "\sim"] & & \{\text{bases}\} \end{tikzcd}$$

$$C \longmapsto \Delta_C$$

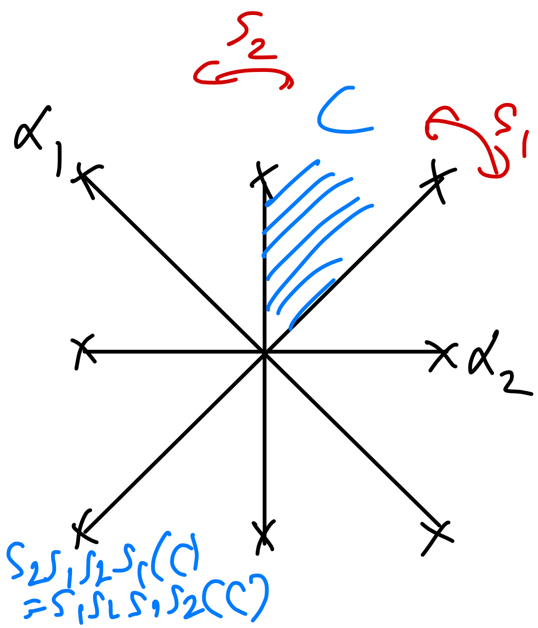
$A_1 \times A_1$



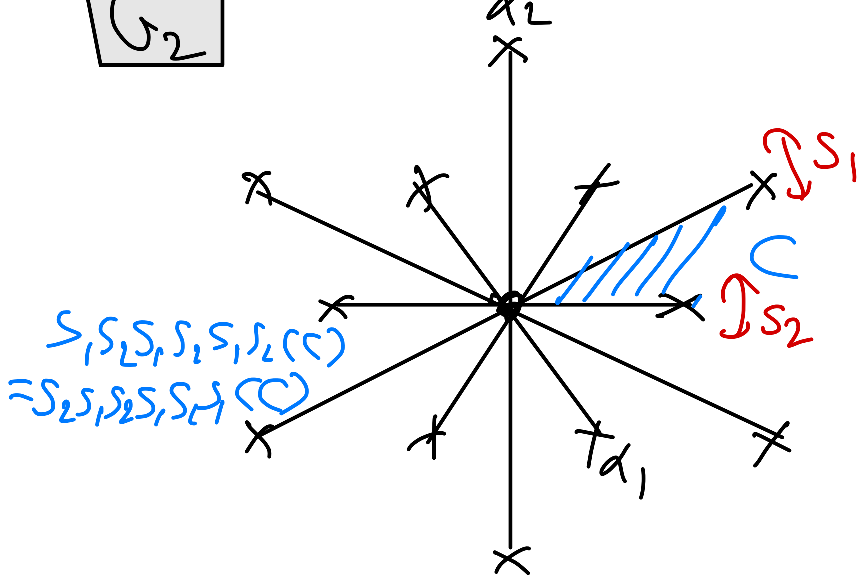
A_2



B_2



G_2



Def. The **length** $\ell(w)$ of $w \in W$ is $\min\{r \geq 0 \mid w = s_{i_1} \cdots s_{i_r},\ i_1, \ldots, i_r \in I\}$

reduced expression for w if $r = \ell(w)$

Lemma 7 $\ell(w) = \#\{\alpha \in R^+ \mid w(\alpha) \in R^-\}$

$= \#\{\alpha \in R^+ \mid C \text{ and } w(C) \text{ are on different sides of } \alpha^\perp\}$

Proof Induction on $\ell(w)$. If $\ell(w) = 0$ then $w = 1$

If $w \neq 1$, let $w = s_{i_1} \cdots s_{i_r}$ be some reduced expression, so $\ell(w) = r$.

Let n be the number of positive root sent to negative by w. Goal: $r = n$.

Let $w' = w s_{i_r} = s_{i_1} \cdots s_{i_{r-1}}$, must be reduced expression, so $\ell(w') = r-1$.

Must have that $w'(\alpha_{i_r})$

$$\overset{\shortparallel}{s_{i_1} \cdots s_{i_{r-1}}}(\alpha_{i_r}) \in R^+$$

otherwise you could apply Lemma 6 contradicting minimality of r.

$$\therefore \quad w(\alpha_{i_r}) \in R^-$$

$$\overset{\shortparallel}{s_{i_1} \cdots s_{i_{r-1}} s_{i_r}(\alpha_{i_r})}$$

If we apply Lemma 5, we deduce that ω' sends $n-1$ positive roots to negative.

Hence, as $\ell(\omega') < \ell(\omega)$, we induction to get $r-1 = n-1$

$\therefore r = n$ //

<u>Example</u> The root system A_ℓ. Set $n = \ell+1$.

Let $\hat{E} = \mathbb{R}\varepsilon_1 \oplus \cdots \oplus \mathbb{R}\varepsilon_n$ Euclidean space so $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$

Let $E = (\varepsilon_1 + \cdots + \varepsilon_n)^\perp = \left\{ \sum_{i=1}^n a_i \varepsilon_i \;\middle|\; \sum_{i=1}^n a_i = 0 \right\}$ "trace zero"

Then E is a Euclidean space of dimension ℓ.

Let $R = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i,j \leq n,\ i \neq j \} \subset E$.

(1)–(4) Axioms of root system all hold.

$R \subseteq E$ ✓

Consider $\hat{s}_{\varepsilon_i - \varepsilon_j} : \hat{E} \to \hat{E}$ reflection in $(\varepsilon_i - \varepsilon_j)^\perp$

flips ε_i and ε_j

fixes all other ε_k.

$$s_{\varepsilon_i - \varepsilon_j} = \hat{s}_{\varepsilon_i - \varepsilon_j}\Big|_E : E \to E$$

W ? $\quad W \hookrightarrow \hat{E}$ $\quad \hat{s}_{\varepsilon_i - \varepsilon_j}$ flips $\varepsilon_i \leftrightarrow \varepsilon_j$

$W \equiv S_n$ and $\hat{E}$ is its natural permutation representation.

$$\hat{E} = E \oplus \mathbb{R}(\varepsilon_1 + \dots + \varepsilon_n)$$

$\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ base $\quad \alpha_i = \varepsilon_i - \varepsilon_{i+1}$

$$\varepsilon_2 - \varepsilon_5 = (\varepsilon_2 - \varepsilon_3) + (\varepsilon_3 - \varepsilon_4) + (\varepsilon_4 - \varepsilon_5)$$

$R^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}$

Lemma 7

Making this choice, $\quad \ell(w) = \# \{1 \leq i < j \leq n \mid w(i) > w(j)\}$

$w \in W = S_n$

Usual length of a permutation in S_n.

Def The Cartan matrix of R is the matrix

$$\left((\alpha_i, \alpha_j^\vee) \right)_{1 \le i,j \le \ell}$$

This involves $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ but actually as all bases are conjugate under W, Cartan matrix is independent of choice of base up to reordering rows/cols.

Call R indecomposable if you cannot partition $R = R_1 \sqcup R_2$, $R_1, R_2 \neq \emptyset$

so $(R_1, R_2) = 0$.

Lemma 8 TFAE.

(1) R is indecomposable

(2) You can't order the base so that Cartan matrix splits into blocks as

$$\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$$

(3) E is an irreducible $\mathbb{R}W$-module.

Proof (2) $\Rightarrow$ (1) If $R = R_1 \sqcup R_2$, $R_1, R_2 \neq \emptyset$, $(R_1, R_2) = 0$.

Let $\Delta_1 = \Delta \cap R_1$, $\Delta_2 = \Delta \cap R_2$ so $\Delta = \Delta_1 \sqcup \Delta_2$.

Then Cartan matrix looks like

$$\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$$

(3) $\Rightarrow$ (2) Given a way of splitting matrix as, get a partition

$\Delta = \Delta_1 \sqcup \Delta_2$, $\Delta_1, \Delta_2 \neq \emptyset$, $(\Delta_1, \Delta_2) = 0$.

Let $E_1 = \mathbb{R}\Delta_1$, $E_2 = \mathbb{R}\Delta_2$; so $E = E_1 \oplus E_2$.

For $\alpha \in \Delta_1$, s_α fixes E_2 pointwise ... leaves E_1 invariant.

Deduce $E = E_1 \oplus E_2$ is decomposition as $\mathbb{R}W$-modules,

using also that $W = \langle s_1, \dots, s_\ell \rangle$

(1) $\Rightarrow$ (3) Suppose E is reducible, let $0 < E_1 < E$ be $\mathbb{R}W$-submodule. Then $E = E_1 \oplus E_2$ where $E_2 = E_1^\perp$, and this a decomposition as $\mathbb{R}W$-modules.

For $\alpha \in R$, $s_\alpha(E_1) = E_1$ $\Longrightarrow$ either $\alpha \in E_1$ or $\alpha \in E_2$

Similarly for E_2. This shows $R = R_1 \sqcup R_2$

$R_1 = R \cap E_1$, $R_2 = R \cap E_2$

Def Let $(R \subset E)$, $(R' \subset E')$ be two root systems. An isomorphism between them is a linear isomorphism $f : E \to E'$ such that f takes R to R' bijectively and moreover

$$(\alpha, \beta^\vee) = (f(\alpha), f(\beta)^\vee)' \qquad \forall\, \alpha, \beta \in R$$

This implies: f induces $W \xrightarrow{\sim} W'$ (group isomorphism)

$$s_\alpha \mapsto s'_{f(\alpha)}$$
$$w \mapsto f \circ w \circ f^{-1}$$

If you use this to identify W and W', then $f : E \xrightarrow{\sim} E'$ will be an isomorphism of $\mathbb{R}W$-modules. If root systems are indecomposable, you can use Schur's Lemma that f is an isometry-up-to-scalar:

$$\lambda(\alpha, \beta) = (f(\alpha), f(\beta))' \qquad \forall\, \alpha, \beta \in E$$

for some $\lambda \in \mathbb{R}_+$.

- To classify root systems (up to $\cong$) sufficies to classify the indecomposable ones.
- Any root system is determined uniquely (up to $\cong$) by its Cartan matrix (up to reordering row/cols).

<u>Proof</u> Given Cartan matrix $\underset{l \times l}{C}$, WLOG its indecomposable, you get E back as $E = \mathbb{R}\alpha_1 \oplus \cdots \oplus \mathbb{R}\alpha_l$. Then know $(\alpha_i, \alpha_j^\vee)$ $\forall i,j$, and this determines the inner product on E up to rescaling by $\lambda \in \mathbb{R}_+$. Then C gives all $s_i = s_{\alpha_i}$, hence, $W = \langle s_1, \dots, s_l \rangle$. Finally you get R as $R = \bigcup_{w \in W} w(\Delta)$ //

- In an indecomposable root system, there are at most two different root lengths with ratio long : short (when two lengths) being $\left.\begin{matrix}\sqrt{2} \\ \sqrt{3}\end{matrix}\right\}$.

<u>Proof</u> Suppose there are <u>three</u> root lengths in R. short α, β, γ long

Know this by rank two analysis

As E is an irreducible $\mathbb{R}W$-module, so W-conjugates of β span E.
You can't have $(\alpha, \omega(\beta)) = 0 \ \forall \omega \in W$. Replacing β by a conjugate, reduce to $(\alpha, \beta) \neq 0$

Similarly $(\alpha, \gamma) \neq 0$.

$$\|\beta\| / \|\alpha\| = \sqrt{2} \qquad \|\gamma\| / \|\alpha\| = \sqrt{3}$$

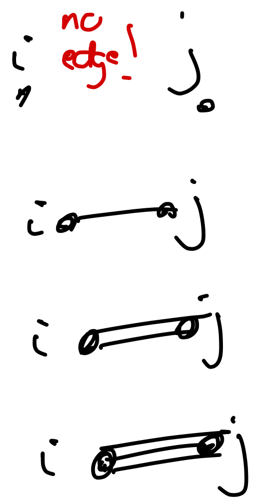
But then $\|\gamma\| / \|\beta\| = \sqrt{3/2}$ ... get a contradiction to ank two situation again on replacing γ by γ' with $(\beta, \gamma') \neq 0$.

$\#$

<u>Upshot</u>: You can encode data of indecomposable root system not by its Cartan mx but simply by its <u>Dynkin diagram</u>

Vertices : I (index set of simple roots)

Edge



if $(\alpha_i, \alpha_j^\vee)(\alpha_j, \alpha_i^\vee) = 0$

" $= 1$

" $= 2$

" $= 3$

Add "<" or ">" in case α_i, α_j are of different lengths,

if $\|\alpha_i\| < \|\alpha_j\|$

You recover Cartan matrix uniquely from Dynkin diagram, hence, root system...

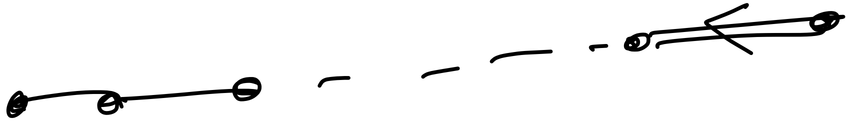
__Classification theorem__ Dynkin diagrams classifying indecomposable root systems $/\cong$ are exactly:

$\ell = \text{rank} = \#\text{vertices}$

A_ℓ $\mathfrak{sl}_{\ell+1}(\mathbb{C})$ 

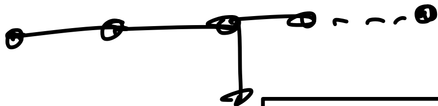
$$\begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & -1 & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

B_ℓ $\mathfrak{so}_{2\ell+1}(\mathbb{C})$ 

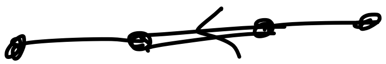
C_ℓ $\mathfrak{sp}_{2\ell}(\mathbb{C})$ 

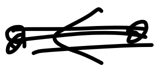
$$B_2 \quad \begin{array}{cc} \alpha_1 & \alpha_2 \\ \hline \end{array}\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$

D_ℓ $\mathfrak{so}_{2\ell}(\mathbb{C})$ 

E_ℓ $(\ell = 6, 7, 8)$ 

$$G_2 \quad \begin{array}{cc} \alpha_1 & \alpha_2 \\ \hline \end{array}\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

F_4 

G_2 

$A_1 = B_1 = C_1$!  $\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{so}_3(\mathbb{C}) \cong \mathfrak{sp}_2(\mathbb{C})$

$B_2 = C_2$ $\mathfrak{so}_5(\mathbb{C}) \cong \mathfrak{sp}_4(\mathbb{C})$