Theorem \( W = \langle s_1, \ldots, s_e \mid s_e^2 = 1, (s_i s_j)^m_{ij} = 1 \ (i \neq j) \rangle \) \( m_{ij} = 2, 3, 4 \) or 6

\[ \xrightarrow{w \in \langle A \rangle} \xrightarrow{w(c)} \xrightarrow{w(A)} \]

\( \{ \text{chambers} \} \xrightarrow{\sim} \{ \text{bases} \} \)

\( C \xrightarrow{1} \xrightarrow{\Delta_c} \)

\[ \text{Coxeter Presentation} \]
Def. The length $l(w)$ of $w \in \mathfrak{w}$ if $\min \{ r \geq 0 \mid w = \overbrace{c_{i_1} \cdots c_{i_r}}^{e_i - e_i} \}$.

Lemma 7 \[ l(w) = \# \{ x \in \mathbb{R}^+ \mid w(x) \in \mathbb{R}^- \} \]
= \# \{ x \in \mathbb{R}^+ \mid c \text{ and } w(c) \text{ are on different sides of } \alpha^+ \} \]

Proof: Induction on $l(w)$. If $l(w) = 0$ then $w = 1$.

If $w \neq 1$, let $w = c_{i_1} \cdots c_{i_r}$ be some reduced expression, so $l(w) = r$.

Let $n$ be the number of positive roots sent to negative by $w$. Goal: $r = n$.

Let $w' = w c_{i_r} = c_{i_1} \cdots c_{i_{r-1}}$, must be reduced expression, so $l(w') = r - 1$.

Must have that $w'(\alpha_{i_r})$
\[ s_{i_{r-1}} \cdots s_{i_1} (\alpha_{i_r}) \in \mathbb{R}^+ \]
\[ w(\alpha_{i_r}) \in \mathbb{R}^- \]
\[ s_{i_{r-1}} \cdots s_{i_1} (\alpha_{i_r}) \]

otherwise you could apply Lemma 6 contradicting minimality of $r$. 
If we apply Lemma 5, we deduce that \( n - 1 \) positive roots to negative.

Hence, as \( L(w) < L(w) \), we obtain to get \( r - 1 = n - 1 \).

Example: The root system \( A_\ell \). Set \( n = \ell + 1 \).

Let \( E = \mathbb{R} \sum_1^\ell \oplus \cdots \oplus \mathbb{R} \sum_n \), Euclidean space so \( (\sum_i, \sum_j) = \delta_{ij} \).

Let \( E = (\sum_1 + \cdots + \sum_n)^+ = \{ \sum \sum \alpha_i \sum_i : \sum \sum \alpha_i = 0 \} \).

Then \( E \) is a Euclidean space of dimension \( \ell \).

Let \( R = \{ \sum_i - \sum_j : 1 \leq i \neq j \leq n \} \subset E \).

Consider \( \sum \sum \sum \sum_1^\ell \sum \sum : E \to \hat{E} \) reflection \( i \) \( (\sum_i, \sum_j)^+ \).

Conclude \( \sum \sum \sum \sum_1^\ell \sum \sum : E \to \hat{E} \) reflection \( i \) \( (\sum_i, \sum_j)^+ \).

Flips \( \sum_i \) and \( \sum_j \) fixes all other \( \sum_k \).
\[ w \in \mathcal{E} \quad \text{flips} \quad \xi_i \to \xi_j \]

\[ W = S_n \quad \text{and} \quad \hat{E} \quad \text{is the natural permutation representation} \]

\[ E \oplus \mathbb{R}(\xi_1 + \cdots + \xi_n) \]

\[ \Delta = \sum \chi_{\lambda} \quad \text{where} \quad \chi_\lambda = \sum_i - \xi_i + 1 \]

\[ \Xi_2 - \xi_5 = (\xi_2 - \xi_3 + \xi_3 - \xi_4 + \xi_4 - \xi_5) \]

\[ R^+ = \{ \xi_j - \xi_i \mid 1 \leq i < j \leq n \} \]

Making this choice, \( \lambda(\pi) = \# \{ 1 \leq i < j \leq n \mid \pi(i) > \pi(j) \} \)

\[ \mathcal{N} \]

\[ \Xi_n \quad \text{usual length of a permutation} \]

\[ W = S_n \]
Def: The Cartan matrix of $R$ is the matrix

$$
\begin{pmatrix}
(x_i, x_j^*)
\end{pmatrix}_{1 \leq i, j \leq l}
$$

This involves $\Delta = \{ x_1, ..., x_l \}$ but actually, as all bases are conjugate under $W$, Cartan matrix is independent of choice of base up to reordering rows/c.cols.

Cell $R$ indecomposable if you cannot partition $R = R_1 \sqcup R_2$, so $(R_1, R_2) = 0$. 
Lemma 8 TFAE:

1. \( R \) is indecomposable
2. You can't order the base so that Cartan matrix splits into blocks as:
   \[
   \begin{pmatrix}
   X & 0 \\
   0 & * \\
   \end{pmatrix}
   \]
3. \( E \) is an indecomposable \( IRW \)-module.

Proof 2 \( \Rightarrow \) 1 If \( R = R_1 \sqcup R_2 \), \( R_1, R_2 \neq \emptyset \), \( (R_1, R_2) = 0 \).

Let \( \Delta_1 = \Delta \cap R_1 \), \( \Delta_2 = \Delta \cap R_2 \) so \( \Delta = \Delta_1 \sqcup \Delta_2 \).

The Cartan matrix looks like:
\[
\begin{pmatrix}
X & 0 \\
0 & * \\
\end{pmatrix}
\]

3 \( \Rightarrow \) 2 Given a way of splitting matrix as \( \Delta \), get a partition \( \Delta = \Delta_1 \sqcup \Delta_2 \), \( \Delta_1, \Delta_2 \neq \emptyset \), \( (\Delta_1, \Delta_2) = 0 \).
Let $E_1 = \text{IR}A_1$, $E_2 = \text{IR}A_2$, so $E = E_1 \oplus E_2$.

For $x \in A_1$, so for $E_2$ part, .. leaves $E_1$ invariant.

Deduce $E = E_1 \oplus E_2$ is decomp as $\text{IR}W$-module, using also that $W = \langle s_1, .., s_e \rangle$.

$\textbf{1}\Rightarrow\textbf{3}$ Suppose $E$ is reducible, let $0 < E_1 < E$ be $\text{IR}W$-submodule. Then $E = E_1 \oplus E_2$ where $E_2 = E_1^\perp$, and this a decomp as $\text{IR}W$-module.

For $x \in R$, $s_x(E_1) = E_1 \Rightarrow$ either $x \not\in E_1$ or $x \in E_2$.

Similarly for $E_2$. This shows $R = R_1 \cup R_2$

$R_1 = R \cap E_1$, $R_2 = R \cap E_2$
Def: Let \((RCE), (R'CE')\) be two root systems. An isomorphism between them is a linear isomorphism \(f: E \rightarrow E'\) such that \(f\) takes \(R\) to \(R'\) bijectively and moreover

\[(\alpha, \beta') = (f(\alpha), f(\beta')) \quad \forall \alpha, \beta \in R\]

This implies: \(f\) induces \(W \cong W'\) (group isomorphism)

\[s \mapsto f(s)\]

\[s \mapsto f(s)\]

\[s \mapsto f(s)\]

\[s \mapsto f(s)\]

If you use this to identify \(W\) and \(W'\), then \(f: E \cong E'\) will be an isomorphism of \(1RW\)-modules. If root systems are indecomposable, you can use Schur's Lemma that \(f\) is an automorphism up-to-scalar:

\[\lambda(\alpha, \beta) = (f(\alpha), f(\beta))' \quad \forall \alpha, \beta \in E\]

for some \(\lambda \in \text{IR}_+\).
To classify root systems (up to \( \cong \)) suffices to classify the indecomposable ones.

Any root system is determined uniquely (up to \( \cong \)) by its Cartan matrix (up to reordering rows/columns).

Proof: Given Cartan matrix \( C \), WLOG it indecomposable, you get \( E \) back as \( E = IRx_1 \oplus \cdots \oplus IRx_l \). Then know \( (x_i, x_j) = \delta_{ij} \), and this determines the inner product on \( E \) up to rescaling by \( \lambda \in IR^+ \).

Then \( C \) gives all \( s_i = 5x_i \), hence, \( W = \langle s_1, \ldots, s_l \rangle \).

Finally you get \( R \) as \( R = \bigcup \omega(C) \).

In an indecomposable root system, there are at most two different root lengths with ratio long : short (when two lengths) being \( \sqrt{2} \).

Proof: Suppose there are three root lengths in \( R \).

\[ \text{short} \alpha, \beta, \gamma \text{ long} \]
As $E$ is an module $1R\!W$-module, so $W$-conjugates of $\beta$ span $E$.

You can't have $(\alpha, \omega(\beta)) = 0 \neq \omega \in W$. Replacing $\beta$ by a conjugate, reduce to $(\alpha, \beta) \neq 0$.

Similarly $(\alpha, \delta) \neq 0$.

\[ \frac{||\beta||}{||\alpha||} = \frac{\sqrt{2}}{1} = \frac{\sqrt{3}}{} \]

But then \[ \frac{||\delta||}{||\beta||} = \sqrt{3} \] ... get a contradiction by rank.

two solutions again on replacing $\beta$ by $\delta'$ with $(\beta, \delta') \neq 0$.

\[ \psi \] photon: You can encode data of indecomposable not system not by its Cartan matrix but simply by its Dynkin diagram.
Vertices: \( I \) (index set of simple roots)

Edge: \( i \rightarrow j \)

if \((\alpha_i, \alpha_j)(\alpha_j, \alpha_i) = 0\)

\[ i = 1 \]
\[ i = 2 \]
\[ i = 3 \]

Add "<" or ">" in case \( \alpha_i, \alpha_j \) are of different length.

You recover Cartan matrix angrily from Dynkin diagram, hence, root system...
Classification theorem: Dynkin diagrams classifying indecomposable root systems are exactly:

- $A_l$ \( sl(2l+1) \)
- $B_l$ \( so(2l+1) \)
- $C_l$ \( sp(2l) \)
- $D_l$ \( so(2l) \)
- $E_l$ \( l = 6, 7, 8 \)
- $F_4$
- $G_2$

- $A_1 = B_1 = C_1$
- $B_2 = C_2$

Root systems:

- $s_2(C) \cong s_3(C) \cong s_2(C)$
- $s_5(C) \cong s_4(C)$