Ch. 4 Classification of semisimple Lie algebras

Let \( \mathfrak{g} \) be a f.d. semisimple Lie algebra
\[ \mathfrak{z} \] maximal toral subalgebra
\[ \mathfrak{k} = \text{Killing form} \quad (\cdot,\cdot) = \mathfrak{k}/\mathfrak{z} \quad \text{non-degenerate} \]
Using that we identified \( \mathfrak{z} = \mathfrak{t}^* \) and we
We had the
Cartan decomposition:
\[ \mathfrak{g} = \mathfrak{z} \oplus \bigoplus_{x \in \mathfrak{R}} \mathfrak{z}_x \]
\[ R \subset \mathfrak{z}^* \]
Let \( E = \mathbb{R}^R \), giving a real vector space. We showed that
\( RCE \), plus the restriction of form \( (\cdot,\cdot) \) to \( E \), gives a root system.
Lemma: $\mathfrak{g}$ is simple if and only if $R$ is indecomposable.

Proof $(\Leftarrow)$ Suppose $\mathfrak{g}$ is not simple, so $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, simple ideals. Let $\mathfrak{z}_c = \mathfrak{z} \cap \mathfrak{g}_c$ ... then $\mathfrak{z} = \mathfrak{z}_c \oplus \cdots \oplus \mathfrak{z}_n$, $\mathfrak{z}_c$ is maximal in $\mathfrak{g}_c$.

$R_c = \text{roots of } \mathfrak{g}_c \text{ cut } \mathfrak{z}_c$ so $R = R_1 \bigcup \cdots \bigcup R_n$

$E_c = \text{IrR} \mathfrak{R}_c = E \cap \mathfrak{z}_c^*$

Then $R_c \subseteq E_c$ is root system of $\mathfrak{g}_c$, and $E_c \perp E_j$ for $c \neq j$.

So root system is decomposable.

$(\Rightarrow)$ Suppose $R = R_1 \bigcup R_2$, $R_1 \bigcup R_2$.

$(R_1, R_2) = 0$

$(\alpha + \beta, \alpha) = (\alpha, \alpha) 
eq 0 \Rightarrow \alpha + \beta \not\in R_2$

$(\alpha + \beta, \beta) = (\beta, \beta) 
eq 0 \Rightarrow \alpha + \beta \not\in R_1$

So $\alpha + \beta \not\in R$, so $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$.

Consider subalgebra of $\mathfrak{g}_1$ generated by $\mathfrak{g}_2$'s $(x \in R_1)$, above shows it is an ideal, so it $0$ or $\mathfrak{g}_1$ by simplicity of $\mathfrak{g}_1$. Deduce $R = R_2$ or $R = R_1$. 

\[\text{Actually it decomposes into indecomposable components.}\]
Let me work out \( \text{so}_{2n}(\mathbb{C}) \) in detail. We did \( \text{sp}_{2n}(\mathbb{C}) \) in HW6 - Q4. Do \( \text{so}_{2n+1}(\mathbb{C}) \) similarly.

By definition, its subalgebra of \( \text{gl}_{2n}(\mathbb{C}) \) of matrix

\[
X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

so

\[
X^T \begin{pmatrix} 0 & J_n \\ J_n & 0 \end{pmatrix} + \begin{pmatrix} 0 & J_n \\ J_n & 0 \end{pmatrix} X = 0
\]

\[
J_n = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}
\]

\[
A^+ = J_n A J_n
\]

\[
D = -A^+ , \quad C = -C^+ , \quad B = -B^+ 
\]

Let \( \Sigma \) be the diagonal matrix \( W = \Sigma \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix} \) \( n \)-dimensional toral.

Let \( \Sigma \in \mathbb{T}^* \) be a map taking \( h = \text{diag}(t_1, -t_1, -t_2, \ldots, -t_r) \) \( \mapsto t^* \).
For $1 \leq i < j \leq n$, consider $h = \text{diag}(t_i, \ldots, t_n, -t_n, \ldots, -t_i) \in \mathbb{Z}$ and $h$ acts on these vectors by

$$(\varepsilon_{ij} - \varepsilon_{ji})(h) = (\varepsilon_i - \varepsilon_j)(h)$$

These lie in $\mathcal{O}_A$, in fact, together with $\mathcal{O}_A$ they give a basis for $\mathcal{O}_A$.

$$\mathcal{O}_A = \mathbb{Z} \oplus \mathcal{O}_A$$

Let $R = \{ \pm (\varepsilon_i \pm \varepsilon_j) | 1 \leq i < j \leq n \}$

$$R^+ = \{ \varepsilon_i \pm \varepsilon_j | 1 \leq i < j \leq n \}$$

Show $\mathcal{O}_A$ is maximal toral, and we've found the root system. We have $1/2$ of natural trace form ... for this induced form on $\mathbb{Z}^*$ has $\varepsilon_1, \varepsilon_2, 0, \ldots, 0$.

Base for $R$ is $\Delta = \{ \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-2} - \varepsilon_{n-1}, \varepsilon_{n-1} \pm \varepsilon_n \}$.
Now we have $R < E = RRC \mathbb{Z}^*$ root system, base $\Delta$.

So can work out Cartan matrix, here, Dynkin diagram

$$\begin{pmatrix}
 2 & -1 & & \cdots & & -1 \\
 -1 & 2 & -1 & & & \\
 & -1 & 2 & -1 & & 0 \\
 & & 0 & -1 & 2 & \\
 & & & -1 & 0 & 2
\end{pmatrix}$$

all roots have $(\alpha_i, \alpha_j) = 2$

$\chi^* = \chi$

$(\alpha_i, \alpha_j)$

$n \geq 3$ it connected, so by lemma $SO_6, SO_8, \ldots$

$D_n$ Dynkin diagram

$s\text{sl}(n)$

$n = 4$

$n = 6$

$SO_6 \subset SO_8, \ldots$

single Lie algebra

$SO_{2n} \subset \mathbb{C}$

$n = 2$
Back to general setup.

\[ \bigoplus_{x \in \mathbb{R}} a \]

Pick a base $\Delta$ for $R$, hence, $R = R^+ \oplus R^-$. 

\[ \{x_1, \ldots, x_3\} \]

We showed at end of Ch. 2 that you could pick $0 \neq e_x \in \psi_x$ (where $R^+$)

Then there's a unique $0 \neq f_x \in \psi_x$ so 

\[ [e_x, f_x] = h_x = \frac{2t}{t^2} (x, x) \]

Also $[h_x, e_x] = 2e_x$, $[h_x, f_x] = -2f_x$ so 

\[ \forall \alpha \in \mathbb{R}, [e_x, h_x, f_x] = \alpha \]

This produces a basis \[ \{e_x, f_x \mid x \in R^+ \} \cup \{h_x \mid h \in \mathbb{Z}\} \]

\[ e_i = e_{\xi_i}, f_i = f_{\xi_i}, h_i = h_{\alpha_i} \]
Lemma 4: $\mathfrak{g}$ is generated as a Lie algebra by $e_i, f_i$ (for $i \in I$).

Proof. Take $\beta \in \mathbb{R}^+$. Show $\mathfrak{e}_\beta$ is a Lie subalgebra of $\mathfrak{g}$ generated by $e_i$. Similarly, each $\mathfrak{f}_\beta$ is a Lie subalgebra generated by $f_i$. Then you get $h_i = [e_i, f_i]$. Hence, a linear basis.

Base by induction on $\text{ht}(\beta)$. Base is trivial. If $\text{ht}(\beta) > 1$,

Ch. 3 Lemma 4 $\Rightarrow f_i$. So $\beta - x_i < \mathbb{R}^+.$

Have $e_{\beta - x_i}$ by induction, suffices to show $[e_i, e_{\beta - x_i}] \neq 0$

But we saw $S_{x_i}, S_{\beta}$ associated to $x_i$, act non ad on $\mathfrak{g}_{\beta + x_i}$, and that was an irreducible $\mathfrak{g}_{x_i}$-module.
Let \( n^+ = \bigoplus_{\alpha \in \mathbb{R}^+} \mathfrak{z}_\alpha \), \( n^- = \bigoplus_{\alpha \in \mathbb{R}^-} \mathfrak{z}_\alpha \).

So Cartan decomposition gives:

\[
\mathfrak{g} = n^- \oplus \mathfrak{z} \oplus n^+
\]

Note \( n^+ \) and \( n^- \) are nilpotent Lie algebras by Engel's theorem.

Also \( \mathfrak{b} = n^+ \times \mathfrak{z} \), so \( \mathfrak{b} \) is solvable.

For \( \mathfrak{sl}_n(\mathbb{C}) \), \( \mathfrak{z} = \mathfrak{z}_{3i-3j} = \mathfrak{c} \mathfrak{e}_{ij} \) with the standard choice