Ch. 4 Classification of semisimple Lie algebras

Let $\mathfrak{g}$ be a f.d. semisimple Lie algebra

$\mathfrak{t}$ maximal toral subalgebra

K = Killing form $\qquad (\cdot,\cdot) = K|_{\mathfrak{t}}$ non-degenerate

Using that we identified $\mathfrak{t} \cong \mathfrak{t}^*$, $t_\lambda \leftrightarrow \lambda$, and we transported form on $\mathfrak{t}$ to get $(\cdot,\cdot)$ on $\mathfrak{t}^*$.

We had the Cartan decomposition:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \qquad R \subset \mathfrak{t}^*$$

Let $E = \mathbb{R}R$, giving a real vector space. We showed that $R \subset E$, plus the restriction of form $(\cdot,\cdot)$ to E, gives a root system.

Lemma $\mathfrak{g}$ is simple if and only if R is indecomposable.

Proof ($\Leftarrow$) Suppose $\mathfrak{g}$ is not simple, so $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \mathfrak{g}_n$ simple ideal.

Let $\mathfrak{t}_i = \mathfrak{t} \cap \mathfrak{g}$ ... then $\mathfrak{t} = \mathfrak{t}_1 \oplus \cdots \oplus \mathfrak{t}_n$, $\mathfrak{t}_i$ is max. toral in $\mathfrak{g}_i$.

R_i = roots of $\mathfrak{g}_i$ wrt $\mathfrak{t}_i$ so $R = R_1 \sqcup \cdots \sqcup R_n$

Actually it's decomposition into indecomposable components.

$E_i = \mathbb{R} R_i = E \cap \mathfrak{t}_i^*$

Then $R_i \subset E_i$ is root system of $\mathfrak{g}_i$, and $E_i \perp E_j$ $i \neq j$

so root system is decomposable.

($\Rightarrow$) Suppose $R = \underset{\alpha}{\underset{\cup}{R_1}} \sqcup \underset{\beta}{\underset{\cup}{R_2}}$

$(R_1, R_2) = 0$

$(\alpha+\beta, \alpha) = (\alpha, \alpha) \neq 0 \Rightarrow \alpha+\beta \notin R_2$

$(\alpha+\beta, \beta) = (\beta, \beta) \neq 0 \Rightarrow \alpha+\beta \notin R_1$

So $\alpha+\beta \notin R$, so $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$.

Consider subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_\alpha$'s ($\alpha \in R_1$), above shows its an ideal, so its 0 or $\mathfrak{g}$ by simplicity of $\mathfrak{g}$. Deduce $R = R_2$ or $R = R_1$

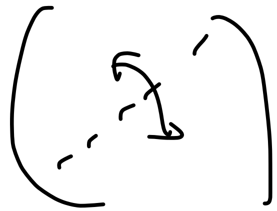
Let me work out $so_{2n}(\mathbb{C})$ in detail. $\longleftarrow$ We did $sp_{2n}(\mathbb{C})$ in HWG-Q4. Do $so_{2n+1}(\mathbb{C})$ similarly.

$$\begin{array}{c} so_{2n}(\mathbb{C}) \\ \| \\ \mathfrak{g} \end{array}$$

By definition, its subalgebra of $\mathfrak{gl}_{2n}(\mathbb{C})$ of matrices

$$X = \begin{array}{c} 1 \to n \;\; -n \to -1 \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ n \qquad n \end{array} \quad \text{so} \quad X^T \begin{pmatrix} 0 & J_n \\ J_n & 0 \end{pmatrix} + \begin{pmatrix} 0 & J_n \\ J_n & 0 \end{pmatrix} X = 0$$

$$J_n = \underbrace{\begin{pmatrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{pmatrix}}_{n \times n} \qquad A^\dagger = J_n A J_n$$



flip

$$\Longrightarrow \cancel{A^T J_n + J_n D = 0}$$

$$\boxed{D = -A^\dagger, \quad C = -C^\dagger, \quad B = -B^\dagger}$$

Let $\mathfrak{t}$ be the diagonal matrices in $\mathfrak{g} = \left\{ \begin{pmatrix} \begin{matrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{matrix} & 0 \\ 0 & \begin{matrix} -t_n & & 0 \\ & \ddots & \\ 0 & & -t_1 \end{matrix} \end{pmatrix} \right\}$ n-dimensional torus

Let $\varepsilon_i \in \mathfrak{t}^*$ be map taking $h = \mathrm{diag}(t_1, \dots, t_n, -t_n, \dots, -t_1) \mapsto t_i$.

For $1 \leq i < j \leq n$, consider

$h = \mathrm{diag}(t_1, \dots, t_n, -t_n, \dots, -t_1) \in \mathfrak{t}$

ad h acts on these vectors by

$e_{ij} - e_{-j,-i}$ $\quad (\varepsilon_i - \varepsilon_j)(h)$ $\qquad e_{ji} - e_{-i,-j}$ $\quad -(\varepsilon_i - \varepsilon_j)(h)$

upper ∇'r $\qquad$ lower Δ'r

$e_{i,-j} - e_{j,-i}$ $\quad (\varepsilon_i + \varepsilon_j)(h)$ $\qquad e_{-j,i} - e_{-i,j}$ $\quad -(\varepsilon_i + \varepsilon_j)(h)$

upper ∇'r $\qquad$ lower Δ'r

These lie in $\mathfrak{g}$, in fact, together with $\mathfrak{t}$ they give a basis for $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \qquad R = \{ \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i < j \leq n \}$$

$$R^+ = \{ (\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i < j \leq n \}$$

Shows $\mathfrak{t}$ is maximal toral, and we've found the root system. We $\frac{1}{2}$ of natural trace form ... for this induced form on $\mathfrak{t}^*$ has $\varepsilon_1, \dots, \varepsilon_n$ o.n.

Base for R is $\Delta = \{ \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_{n-2} - \varepsilon_{n-1}, \varepsilon_{n-1} \pm \varepsilon_n \}$

Now we have $R \subset E = \mathbb{R}R \subset \mathfrak{z}^*$ root system, base Δ,

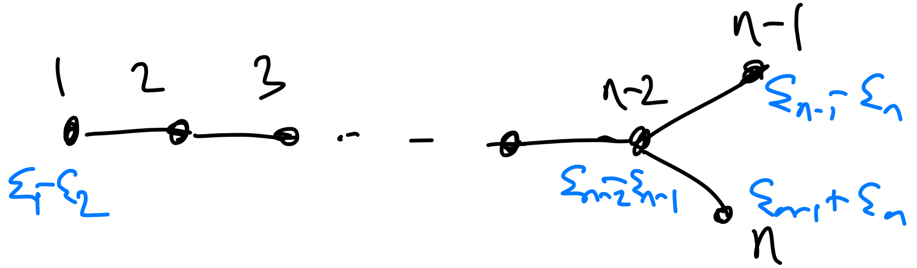
so can work out Cartan matrix, here, Dynkin diagram

$$\begin{array}{c}
\begin{array}{cccc} \varepsilon_1-\varepsilon_2 \ \cdots & \varepsilon_{n-2}-\varepsilon_{n-1} & \varepsilon_{n-1}-\varepsilon_n & \varepsilon_{n-1}+\varepsilon_n \end{array} \\
\begin{pmatrix}
2 & -1 & & & & \\
-1 & \ddots & \ddots & & 0 & \\
 & \ddots & \ddots & -1 & & \\
 & & -1 & 2 & -1 & -1 \\
 & 0 & & -1 & 2 & 0 \\
 & & & -1 & 0 & 2
\end{pmatrix}
\end{array}$$

all roots have $(\alpha, \alpha) = 2$

$\alpha^\vee = \alpha$

(α_i, α_j)

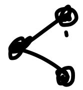


$\ell = n$ $\quad D_n$ Dynkin diagram

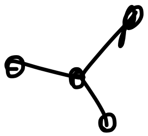
$\updownarrow$

$\mathfrak{so}_{2n}(\mathbb{C})$

$\mathfrak{sl}_4(\mathbb{C})$



$n \geq 3$ it's connected, so by Lemma $\mathfrak{so}_6, \mathfrak{so}_8, \ldots$ simple Lie algebras

$n=4$ 

$n=2$  $\mathfrak{so}_4(\mathbb{C})$ $\quad \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$

Back to general setup.

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$$

Pick a base $\underset{\{\alpha_1,\dots,\alpha_\ell\}}{\overset{}{\Delta}}$ for R, hence, $R = R^+ \cup R^-$.

We showed at end of Ch-2 that you could pick $0 \neq e_\alpha \in \mathfrak{g}_\alpha$ $(\alpha\in R^+)$

Then there's a unique $0 \neq f_\alpha \in \mathfrak{g}_{-\alpha}$ so $[e_\alpha, f_\alpha] = h_\alpha = t_{\alpha^\vee} = \dfrac{2t_\alpha}{(t_\alpha, t_\alpha)}$

Also $[h_\alpha, e_\alpha] = 2e_\alpha$, $[h_\alpha, f_\alpha] = -2f_\alpha$ so $s_\alpha = \langle e_\alpha, h_\alpha, f_\alpha\rangle \cong \mathfrak{sl}_2(\mathbb{C})$

This produces a <u>basis</u> $\{e_\alpha, f_\alpha \mid \alpha\in R^+\} \cup \{h_1, \dots, h_\ell\}$.

$e_i = e_{\alpha_i}$, $f_i = f_{\alpha_i}$, $h_i = h_{\alpha_i}$

Lemma $\mathfrak{g}$ is generated as a Lie algebra by $\underbrace{e_i, f_i}_{2\ell \text{ elements}}$ $(i \in I)$.

Proof Take $\beta \in R^+$, show e_β lies in Lie subalgebra of $\mathfrak{g}$ generated by $e_1, \dots, e_\ell$. Similarly each f_β lies in subalgebra generated by $f_1, \dots, f_\ell$, then you get $h_i = [e_i, f_i]$, hence, a linear basis.

Goes by induction on $ht(\beta)$. Base is trivial. If $ht(\beta) > 1$,

Ch. 3 Lemma 4 $\Rightarrow \exists i$ so $\beta - \alpha_i \in R^+$.

Have $e_{\beta-\alpha_i}$ by induction, suffices to show $[e_i, e_{\beta-\alpha_i}] \neq 0$

But we saw S_{α_i}, sl_2 associated to α_i, acts via ad on

$\bigoplus_{r \in \mathbb{Z}} \mathfrak{g}_{\beta + r\alpha_i}$, and that was an irreducible S_{α_i}-module. //

Let $\mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$, $\quad \mathfrak{n}^- = \bigoplus_{\alpha \in R^-} \mathfrak{g}_\alpha$.

So Cartan decomposition gives:

$$\mathfrak{g} = \mathfrak{n}^- \oplus \underbrace{\mathfrak{t} \oplus \mathfrak{n}^+}_{\mathfrak{b}}$$

triangular decomposition of $\mathfrak{g}$. Choice of $\mathfrak{t}$ plus $R^+ \sqcup R^-$, i.e., fundamental chamber / base.

($\mathfrak{b}$ is for Borel subalgebra).

Note $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent Lie algebras by Engel's theorem.

Also $\mathfrak{b} = \mathfrak{n}^+ \rtimes \mathfrak{t}$, so $\mathfrak{b}$ is solvable.

eg $\mathfrak{sl}_n(\mathbb{C})$ $\quad \mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C} e_{ij}$ with the standard choice

$R^+ = \{ (\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq n \}$ $\quad \mathfrak{n}^+ = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$, $\mathfrak{n}^- = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$