Root system R, base $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$, $c_{ij} = (\alpha_i, \alpha_j^\vee)$
Cartan integers

Serre's Theorem: Let $\mathfrak{g}$ be the Lie algebra with gens e_i, h_i, f_i $(i = 1, \ldots, \ell)$ and relations below. Then $\mathfrak{g}$ is a f.d. semisimple Lie algebra with maximal toral subalgebra $\mathfrak{t} = \langle h_1, \ldots, h_\ell \rangle$ and corresponding root system R.

(S1) $[h_i, h_j] = 0$

(S2) $[e_i, f_j] = \delta_{ij} h_i$

(S3) $[h_i, e_j] = c_{ji} e_j$ $[h_i, f_j] = -c_{ji} f_j$

$$\left.\begin{array}{ll} (S_{ij}^+) & (\mathrm{ad}\, e_i)^{1-c_{ji}}(e_j) = 0 \\ (S_{ij}^-) & (\mathrm{ad}\, f_i)^{1-c_{ji}}(f_j) = 0 \end{array}\right\} i \neq j$$

Proof part I: Consequences of (S1) – (S3)

Let $\hat{\mathfrak{g}}$ be generated by $\hat{e}_i, \hat{h}_i, \hat{f}_i$ $(i = 1, \ldots, \ell)$ subject to relations (S1) – (S3).

Let $V = \mathbb{C}\langle x_1, \ldots, x_\ell \rangle$ tensor algebra

Make $\hat{\mathfrak{g}} \curvearrowright V \ldots$

Note V is spanned by monomials $x_{i_1} \cdots x_{i_r}$.

$$\hat{h}_j \cdot x_{i_1} \cdots x_{i_r} = -(c_{i_1 j} + \cdots + c_{i_r j})\, x_{i_1} \cdots x_{i_r}$$

sum is zero if $r=0$

$$\hat{f}_j \cdot x_{i_1} \cdots x_{i_r} = x_j x_{i_1} \cdots x_{i_r}$$

sum is zero if $r=1$

$$\hat{e}_j \cdot x_{i_1} \cdots x_{i_r} = x_{i_1}(\hat{e}_j \cdot x_{i_2} \cdots x_{i_r}) - \delta_{i_1, j}(c_{i_2 j} + \cdots + c_{i_r j})\, x_{i_2} \cdots x_{i_r}$$

defined recursively!

$\hat{e}_j \cdot 1 = 0$

Check (S1) – (S3) hold so this action is well-defined. Induction exercise.

eg $\hat{e}_j \hat{f}_j - \hat{f}_j \hat{e}_j = \hat{h}_j$?

$$(\hat{e}_j \hat{f}_j - \hat{f}_j \hat{e}_j) \cdot x_{i_1} \cdots x_{i_r} = \hat{e}_j(x_j x_{i_1} \cdots x_{i_r}) - x_j(\hat{e}_j \cdot x_{i_1} \cdots x_{i_r})$$

$$= -(c_{i_1 j} + \cdots + c_{i_r j})\, x_{i_1} \cdots x_{i_r} = \hat{h}_j \cdot x_{i_1} \cdots x_{i_r}.$$

Other rels ...

Now use action $\hat{\mathfrak{g}} \curvearrowright V$ to prove ..

Lemma) $\hat{h}_1, \dots, \hat{h}_\ell$ span an ℓ-dimensional Abelian subalgebra $\hat{\mathfrak{z}}$ of $\hat{\mathfrak{g}}$,

and
$$\hat{\mathfrak{g}} = \underbrace{\hat{\mathfrak{n}}^-}_{\text{subalg. gen'd by } \hat{f}_1, \dots, \hat{f}_\ell} \oplus \hat{\mathfrak{z}} \oplus \underbrace{\hat{\mathfrak{n}}^+}_{\text{subalg. gen'd by } \hat{e}_1, \dots, \hat{e}_\ell}$$
as a vector space.

Proof Let $\varphi : \hat{\mathfrak{g}} \to \mathfrak{gl}(V)$ be representation above.

Observe $\varphi|_{\hat{\mathfrak{z}}}$ is injective. If $\sum a_j \hat{h}_j \in \ker \varphi$, act

Abelian subalg. spanned by $\hat{h}_1, \dots, \hat{h}_\ell$

on x_i to deduce that

$$\sum a_j c_{ij} = 0 \quad \forall i$$

Cartan matrix is invertible by non-degeneracy of form

$\therefore\ a_j = 0 \ \forall j$ ✓

By relation, $\hat{e}_i, \hat{h}_i, \hat{f}_i$ span a quotient of $sl_2(\mathbb{C})$. Since $\hat{h}_i \neq 0$, it's not zero quotient. As $sl_2(\mathbb{C})$ is simple, it follows that

$$\langle \hat{e}_i, \hat{h}_i, \hat{f}_i \rangle \cong sl_2(\mathbb{C}).$$

Notation
$[a\, b\, c\, d] = [a[b[cd]]]$

Now show $\hat{g} = \hat{n}^- \oplus \hat{z} \oplus \hat{n}^+$.

Use induction on r, (S1) - (S3) and Jacobi to show ...

$$[\hat{h}_j\, [\hat{e}_{i_1} \cdots \hat{e}_{i_r}]] = (\varsigma_{j i_1} + \cdots + \varsigma_{j i_r})\, [\hat{e}_{i_1} \cdots \hat{e}_{i_r}]$$

$$[\hat{h}_j\, [\hat{f}_{i_1} \cdots \hat{f}_{i_r}]] = -(\varsigma_{j i_1} + \cdots + \varsigma_{j i_r})\, [\hat{f}_{i_1} \cdots \hat{f}_{i_r}]$$

$$[\hat{f}_j\, [\hat{e}_{i_1} \cdots \hat{e}_{i_r}]] = \hat{n}^+ \quad \text{if } r \geq 2$$

$$[\hat{e}_j\, [\hat{f}_{i_1} \cdots \hat{f}_{i_r}]] \in \hat{n}^- \quad \text{if } r \geq 2.$$

It follows that $\hat{n}^- + \hat{z} + \hat{n}^+$ is actually a Lie subalgebra

Finally need to show $+$ is actually $\oplus$, but that follows by considering the decomposition into ad $\hat{z}$-weight spaces.

Now $\hat{e}_{ij} = (\mathrm{ad}\, \hat{e}_i)^{1-c_{ji}}(\hat{e}_j)$, $\hat{f}_{ij} = (\mathrm{ad}\hat{f}_i)^{1-c_{ji}}(\hat{f}_j)$ $i \neq j$.

Elements of $\hat{\mathfrak{J}}$.

<u>Lemma 2</u> $[\hat{e}_k, \hat{f}_{ij}] = [\hat{f}_k, \hat{e}_{ij}] = 0$ $\forall\, k,\ i \neq j$.

<u>Proof</u> Do first one.

$k \neq i$ $[\hat{e}_k, \hat{f}_i] = 0$ so $\mathrm{ad}\,\hat{e}_k$ and $\mathrm{ad}\,\hat{f}_i$ commute.

$\therefore$ $\mathrm{ad}\,\hat{e}_k(\hat{f}_{ij}) = (\mathrm{ad}\,\hat{f}_i)^{1-c_{ji}}([\hat{e}_k\ \hat{f}_j]) = 0$ if $k \neq j$

and if $k = j$: $= (\mathrm{ad}\hat{f}_i)^{1-c_{ji}}(\hat{h}_j) = (\mathrm{ad}\hat{f}_i)^{-c_{ji}}(\hat{f}_i)\cdot(-c_{ji})$

If $c_{ji} = 0$ it's zero 'cos of the scalar, if $c_{ji} < 0$ it's obviously zero.

$k = i$ Let $\hat{S}_i = \langle \hat{e}_i, \hat{h}_i, \hat{f}_i \rangle \cong sl_2(\mathbb{C})$. $\mathrm{ad}\hat{e}_i(\hat{f}_j) = 0$ as $i \neq j$.

Shows $\hat{f}_j$ is highest weight vector for $\hat{S}_i$ of weight $\lambda = -c_{ji}$.

$(\mathrm{ad}\hat{e}_i)(\mathrm{ad}\hat{f}_i)^m(\hat{f}_j) = m(\lambda - m + 1)(\mathrm{ad}\hat{f}_i)^{m-1}(\hat{f}_j)$

So it's zero when $m = 1 - c_{ji}$

Proof part II : Consequences of $S_{ij}^{\pm}$.

Consider $\mathfrak{g} = \hat{\mathfrak{g}}/K$ where K is ideal gen'd by $\hat{e}_{ij}, \hat{f}_{ij}$ $(i \neq j)$

meant to be f.d.

$\hat{\mathfrak{g}}$ is usually ∞ dimensional here

Lemma 3 $K = I^- + I^+$ where I^- ideal of $\hat{n}^-$ gen'd by $\hat{f}_{ij}$
I^+ " " $\hat{n}^+$ " " $\hat{e}_{ij}$.

Proof

Show $I^{\pm}$ is actually an ideal of $\hat{\mathfrak{g}}$ by checking its invariance under bracketing with all generators of $\hat{\mathfrak{g}}$. Uses Lemma 2.

Then you get that $K = I^- + I^+$ as the RHS contains all the generators of K ~~/~~.

Since $\hat{\mathfrak{g}} = \hat{n}^- \oplus \mathfrak{h} \oplus \hat{n}^+$, deduce $\quad \mathfrak{g} = \hat{\mathfrak{g}}/I = \hat{n}^-/I^- \oplus \hat{\mathfrak{h}} \oplus \hat{n}^+/I^+$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n^- \oplus \mathfrak{h} \oplus n^+$

Moreover $\mathfrak{h}$ is l-dimensional.

Also the $\mathfrak{sl}_2(\mathbb{C})$ argument made earlier shows $\mathfrak{s}_i = \langle e_i, h_i, f_i \rangle \cong \mathfrak{sl}_2(\mathbb{C})$

Now for $\lambda \in \mathfrak{z}^*$, let $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid [h,x] = \lambda(h)x \ \forall h \in \mathfrak{z}\}$.

Identify $\alpha \in R$ with $\alpha \in \mathfrak{z}^*$ by $\alpha(h_i) = (\alpha, \alpha_i^\vee)$ as usual.

$R \subset E$ This is the form on E.

<u>Lemma 4</u> For $0 \neq \lambda \in \mathfrak{z}^*$, $\mathfrak{g}_\lambda$ is non-zero if and only if $\lambda \in R$, in which case $\mathfrak{g}_\lambda$ is 1-D.

(so we "right" Cartan decomposition, $\dim \mathfrak{g} = |R| + |\Delta| < \infty$)

<u>Proof</u> THE MOST INTERESTING PLACE!

Consider $\operatorname{ad} e_i : \mathfrak{g} \to \mathfrak{g}$.

Claim: This is locally nilpotent.

Let $M = \bigcup_{n \geq 1} \ker(\operatorname{ad} e_i)^n$. Need to show $M = \mathfrak{g}$.

Check that M is a subalgebra of $\mathfrak{g}$. Then note all generators of $\mathfrak{g}$ lie in M, hence, $M = \mathfrak{g}$ ✓.

$\left\{ \begin{array}{l} e_k \in M \text{ by } (S_{ij}^+) \\ f_k \in M \text{ by } (S3) \\ h_k \in M \text{ by } (S2) \end{array} \right.$

Hence, it makes sense to define a linear map

$$\exp(\operatorname{ad} e_i) = 1 + \operatorname{ad} e_i + \frac{(\operatorname{ad} e_i)^2}{2!} + \frac{(\operatorname{ad} e_i)^3}{3!} + \cdots : \mathfrak{g} \to \mathfrak{g}$$

In fact, its an automorphism, with inverse $\exp(-\operatorname{ad} e_i)$.

So we can define

$$\sigma_i := \exp(\operatorname{ad} e_i) \exp(\operatorname{ad}(-f_i)) \exp(\operatorname{ad} e_i) : \mathfrak{g} \to \mathfrak{g}.$$

Now you show $\sigma_i(\mathfrak{g}_\lambda) = \mathfrak{g}_{s_i(\lambda)}$

$s_i \in W$ simple reflection

$s_i(\lambda) = \lambda - (\lambda, \alpha_i^\vee)\alpha_i$

So $\dim \mathfrak{g}_\lambda = \dim \mathfrak{g}_{s_i(\lambda)}$ $\forall\, i \in I$

$\dim \mathfrak{g}_\lambda = \dim \mathfrak{g}_{w(\lambda)}$ $\forall\, w \in W$

Now $\mathfrak{g}_{\alpha_i}$ is 1-D spanned by e_i — even true in $\hat{\mathfrak{g}}$.

Since every root is conjugate under W to a simple root, it follows that $\dim \mathfrak{g}_\alpha = 1$ $\forall\, \alpha \in R$.

Similarly you see that $\dim \mathfrak{g}_{c\alpha} = 0$ unless $c = \pm 1$, for $\alpha \in R$.

Finally we need to show $\dim \mathfrak{g}_\lambda = 0$ for $\lambda \in \mathfrak{t}^*$ not a multiple of any root.

If $\mathfrak{g}_\lambda \neq 0$ then $\lambda = \sum_{i=1}^{n} c_i \alpha_i$ all $c_i \in \mathbb{Z}_{\geq 0}$ or all $c_i \in \mathbb{Z}_{\leq 0}$

$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}^+$.

Now appeal to:

<u>HW8-Q6</u> If $\lambda = \sum_{i=1}^{l} c_i \alpha_i$ all $c_i \in \mathbb{N}$, and not a multiple of any root, then $\exists\, w \in W$ s.t. $w(\lambda)$ has both positive and negative coefficients when written in terms of simple roots.

This completes the proof //

Finally we can finish the proof of Serre's Theorem ...

Let's show $\mathfrak{g}$ is semisimple. It suffices to show any Abelian ideal $A \triangleleft \mathfrak{g}$ is zero.

Consider

$$A = (A \cap \mathfrak{h}) \oplus \bigoplus_{\alpha \in R} (A \cap \mathfrak{g}_\alpha).$$

If some $A \cap \mathfrak{g}_\alpha \neq 0$, then $\mathfrak{g}_\alpha \subseteq A$. But A is invariant under those σ_i's ... so some $\mathfrak{g}_{\alpha_i} \subseteq A$.

$\therefore e_i \in A$ Then bracket with f_i to get $e_i, h_i, f_i \in A$,

but contradicts A being Abelian

So $A \subseteq \mathfrak{t}$. $[A, \mathfrak{g}_\alpha] = 0$.

So $A \subseteq \bigcap_{\alpha \in R} \ker \alpha = 0$. ✓

Final step. Identify form on $\mathfrak{t}^*$ coming from Killing form with original form coming from E.

Killing form has $$K(h_i, h_j) = \sum_{\alpha \in R} \alpha(h_i)\alpha(h_j)$$

$$K(\alpha_i^\vee, \alpha_j^\vee) = \sum_{\alpha \in R} (\alpha, \alpha_i^\vee)(\alpha, \alpha_j^\vee)$$

K gives us a W-invariant inner product on E

$$K(\lambda,\mu) = \sum_{\alpha\in R} (\alpha,\lambda)(\alpha,\mu)$$

But such a form is unique up to rescaling by positive reals on each indecomposable component (due irreducibility of E as $\mathbb{R}W$-module when root system is indecomposable).

$\blacksquare$