BRAUER AND IQUANTUM CATEGORIFICATION

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Abstract. We show that locally finite Abelian module categories over the affine Brauer category are the same thing as locally finite Abelian 2-representations of a certain 2-iquantum group satisfying some additional nilpotency and finiteness properties. Our results are valid over algebraically closed fields of characteristic other than 2 and for arbitrary spectra. This extends recent work of Gao, Rui and Song, who treated characteristic 0 with half-integral spectrum. The relevant Satake diagrams are of quasi-split type AIII and of diagonal type A, including affine types in positive characteristic. The key step in our approach identifies the affine Brauer category with a topologically dense monoidal subcategory of a completion of the 2-iquantum group. As an application, we show that deformed cyclotomic quotients on the two sides are equivalent.

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1. Introduction

The notion of a Kac–Moody categorification was introduced by Rouquier in [Rou08], building on his earlier work with Chuang [CR08] treating the $\mathfrak{sl}_2$ case. It provides an elegant axiomatic framework encapsulating ideas of higher representation theory which emerged in the 1990s, extending the approach to representation theory of symmetric and general linear groups which can be traced back to work of Frobenius and Schur at the beginning of the 20th century and, later, Macdonald, Bernstein and Zelevinsky. A key ingredient in Rouquier's setup comes from the *quiver Hecke algebras* (a.k.a. Khovanov–Lauda–Rouquier algebras) discovered in [Rou08, KL09], which are half of the *Kac–Moody 2-categories* $\mathfrak{U}(\mathfrak{g})$ introduced in [Rou08, KL10]. We also refer to Kac–Moody 2-categories as *2-quantum groups* because $\mathfrak{U}(\mathfrak{g})$ categorifies Lusztig's modified integral form for the Drinfeld–Jimbo quantum group associated to the Kac–Moody algebra $\mathfrak{g}$. This is a theorem of Khovanov and Lauda [KL10, Ths. 1.1–1.2], modulo the non-degeneracy of 2-quantum groups which was proved in full generality in [Web26]; see also [BWW25, Th. 5.6].

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We work from now on over a fixed algebraically closed ground field $\Bbbk$. Let $\mathfrak{g}$ be a Kac-Moody algebra with weight lattice X. A *Kac–Moody categorification* is a locally finite Abelian category

$$\mathbf{R} = \bigoplus_{\lambda \in X} \mathbf{R}_\lambda$$

equipped with a categorical action of a 2-quantum group $\mathfrak{U}(\mathfrak{g})$, i.e., it is a 2-representation of $\mathfrak{U}(\mathfrak{g})$, such that for each $\lambda \in X$ and $V \in \mathbf{R}_\lambda$ the set $\{i \in I \mid E_i V \neq 0 \text{ or } F_i V \neq 0\}$ is finite. We assume in addition that $\mathbf{R}$ is *nilpotent*, which means that the natural transformations defined by dots act nilpotently on any object. It follows that the Grothendieck group $K_0(\mathbf{R}) = \bigoplus_{\lambda \in X} K_0(\mathbf{R}_\lambda)$ is an integrable representation of the Kostant $\mathbb{Z}$-form of the universal enveloping algebra $U(\mathfrak{g})$. Moreover, the isomorphism classes of irreducible objects of $\mathbf{R}$ equip $K_0(\mathbf{R})$ with a *perfect basis* in the sense of [BK07, Def. 5.49]. Aspects of the structure of the category $\mathbf{R}$ can then be understood via the rich combinatorics of integrable representations, e.g., the action of the underlying braid group permuting weight spaces, and the theory of canonical/crystal bases of Lusztig and Kashiwara, both of which play an important role.

There are many naturally occurring examples of Kac–Moody categorifications arising in type A representation theory; see the first part of [BSW26] for a recent survey. In all of these examples, the underlying Kac–Moody algebra $\mathfrak{g}$ has simple roots indexed by the set $I := \Bbbk$, and its Cartan matrix $(a_{i,j})_{i,j \in I}$ is defined by $a_{i,j} := 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}$; the connected components are all of type A_∞ if $p = 0$ or $\mathrm{A}^{(1)}_{p-1}$ if $p > 0$, where p is the characteristic of $\Bbbk$. To prove the existence of the categorical action of $\mathfrak{U}(\mathfrak{g})$ in the first place, one needs to construct explicit endofunctors and natural transformations satisfying the defining relations of $\mathfrak{U}(\mathfrak{g})$. The verification of Rouquier's *inversion relation* is particularly difficult. In [Rou08, Th. 5.27], Rouquier developed a general approach to this which he called "control by K_0". In our work [BSW20], we explained another approach which is more convenient in practice. Both approaches depend ultimately on the isomorphism between cyclotomic quotients of degenerate affine Hecke algebras of type A and corresponding cyclotomic quiver Hecke algebras from [BK09] or, equivalently, the isomorphism between completions of these algebras from [Rou08].

By a *Heisenberg categorification* we mean a locally finite Abelian category $\mathbf{R}$ equipped with an action of a Heisenberg category $\mathbf{Heis}_\kappa$ for some central charge $\kappa \in \mathbb{Z}$, i.e., $\mathbf{R}$ is a strict left $\mathbf{Heis}_\kappa$-module category. By *Heisenberg category* here, we mean the strict $\Bbbk$-linear monoidal category defined originally by Khovanov for central charge one in [Kho14] and generalized to all central charges in [BCNR17, MS18, Bru18]. The main idea in [BSW20] is that many of the Abelian categories of classical importance in type A representation theory are Heisenberg categorifications. Moreover, the action of $\mathbf{Heis}_\kappa$ is usually easy to construct. Then we showed that any Heisenberg categorification is a nilpotent Kac–Moody categorification, and vice versa (assuming that the Cartan type of $\mathfrak{g}$ is as described above and the defining parameters of the 2-quantum group $\mathfrak{U}(\mathfrak{g})$ are chosen correctly). Roughly speaking, we did this by upgrading the algebra isomorphisms between appropriate completions of degenerate affine Hecke algebras and quiver Hecke algebras arising from [Rou08, BK09] to Heisenberg categories and 2-quantum groups, although we did not use this language explicitly. In the setup of [BSW20], the rightward cup and cap generators of $\mathbf{Heis}_\kappa$ map in an obvious way to the rightward cup and cap generators in the 2-quantum group. However, it is much harder to understand the relationship between the "second adjunctions" defined by the leftward cup and cap generators in these categories. We finally worked out how to do that explicitly in the second part of [BSW26].

For later reference, we should also mention an application of the theory developed in [BSW20]: it implies the existence of explicit isomorphisms between *generalized cyclotomic quotients* (GCQs) of the Heisenberg category and GCQs of the corresponding 2-quantum group. These isomorphisms are a substantial generalization of the isomorphisms between ordinary cyclotomic quotients from

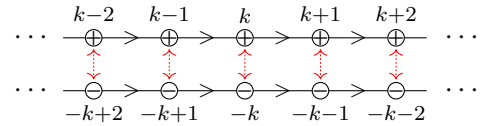
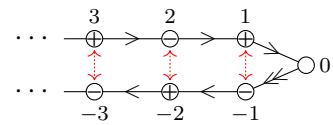
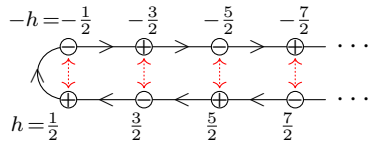
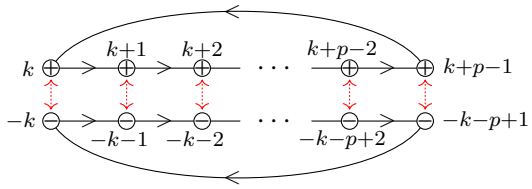
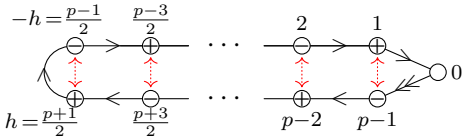
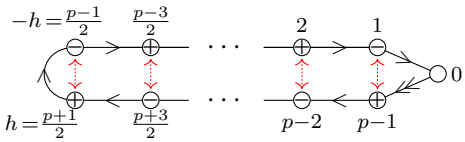
A_∞	$p=0,\ k\not\sim 0,h$	
$\mathrm{AIII}_{2\infty+1}$	$p=0,\ k\sim 0$	
$\mathrm{AIII}_{2\infty}$	$p=0,\ k\sim h$	
$\mathrm{A}^{(1)}_{p-1}$	$p>2,\ k\not\sim 0$	
$\mathrm{AIII}^{(1)}_{p-1}$	$p\equiv 1 \pmod 4,\ k\sim 0$	
$\mathrm{AIII}^{(1)}_{p-1}$	$p\equiv 3 \pmod 4,\ k\sim 0$	

TABLE 1. Satake diagrams

[BK09]. For example, taking the ground field $\Bbbk$ to be $\mathbb{C}$, the Deligne category $\underline{\mathrm{Rep}}(GL_\delta)$, which by definition is the additive Karoubi envelope of the oriented Brauer category $\mathbf{OB}_\delta$, is a generalized cyclotomic quotient of level one. This is a useful result: for example, in [BM27], we use it to show for $\delta\in\mathbb{Z}$ that $\underline{\mathrm{Rep}}(GL_\delta)$ is Morita equivalent to a certain limiting version of Khovanov's arc algebra, as was originally conjectured in [BS12].

In the last decade, there has been a great deal of work related to the Kolb–Letzter theory of quantum symmetric pairs (QSPs) [Let99, Kol14]. We adopt here the terminology *iquantum group* introduced by Bao and Wang to refer to the coideal subalgebra in a QSP. The influential papers [BW18b, ES18] revealed that certain naturally occurring categories in type B/C/D representation theory are related from a combinatorial perspective to certain iquantum groups. These categories include blocks of the BGG category $\mathcal{O}$ for orthogonal and symplectic Lie algebras (indeed, for orthosymplectic Lie superalgebras), the category of finite-dimensional representations of the orthosymplectic supergroup $OSp_{m|2n}(\mathbb{C})$, and representations of the classical Brauer category. The Satake diagrams of the relevant iquantum groups, displayed in Table 1, are of diagonal type A and of *quasi-split type AIII*. Wang refers to the latter as "the new type A" as there are analogs of many results, e.g., ones involving Schur–Weyl duality, which we used to think were peculiar to type A.

The existence of *icanonical bases* for iquantum groups established in [BW18a] suggested that there might be a categorification of iquantum groups in the spirit of the Khovanov–Lauda–Rouquier categorification of quantum groups. For the Satake type $\mathrm{AIII}_{2\infty}$, the appropriate 2-category was

constructed in [BSWW18]. Building on this, categorifications of all quasi-split iquantum groups whose underlying Cartan matrix is symmetric (and, conjecturally, most non-symmetric types too) were constructed in [BWW25]. These are denoted $\mathfrak{U}^\imath(\mathfrak{g})$, depending on an underlying Kac–Moody algebra together with an additional involution encoded combinatorially by the Satake diagram. For the ones in Table 1, this involution is indicated by the red arrows. Other decorations on the Satake diagrams there, such as the $+/-$ labels on the nodes, and the arrows on the edges, will be explained in section 3—they are needed in order to specify the rather delicate parameter choices we use to define $\mathfrak{U}^\imath(\mathfrak{g})$. We call $\mathfrak{U}^\imath(\mathfrak{g})$ a *2-iquantum group*. This terminology is justified by the main result of [BWW25], which shows that the graded Grothendieck ring of $\mathfrak{U}^\imath(\mathfrak{g})$ is isomorphic to the Bao–Wang modified integral form $\dot{\mathrm{U}}^\imath_{\mathbb{Z}}(\mathfrak{g})$ for $\mathrm{U}^\imath(\mathfrak{g})$. Under the isomorphism, the class $[B_i\mathbb{1}_\lambda]$ of the generating 1-morphism $B_i\mathbb{1}_\lambda$ of the 2-iquantum group corresponds to the generator $b_i 1_\lambda$ of $\dot{\mathrm{U}}^\imath_{\mathbb{Z}}(\mathfrak{g})$.

In [RS19], Rui and Song introduced a strict monoidal category **AB** called the *affine Brauer category*, motivated by the observation that **AB** acts naturally on many Abelian categories studied classically in type B/C/D representation theory. We review the definition of **AB** in section 2. It is similar in spirit to the definition of $\mathbf{Heis}_\kappa$; actually, it is a little easier as there is only one generating object B—there is no orientation on the string diagrams used to represent morphisms in **AB**—and there is no central charge κ involved. By a *Brauer categorification*, we mean a locally finite Abelian category **R** which is a strict left **AB**-module category. By an *iquantum categorification*, we mean a locally finite Abelian category

$$\mathbf{R} = \bigoplus_{\lambda \in X^\imath} \mathbf{R}_\lambda$$

which is a 2-representation of a 2-iquantum group $\mathfrak{U}^\imath(\mathfrak{g})$, where $X^\imath$ is the associated iweight lattice. We also require for each $\lambda \in X^\imath$ and $V \in \mathbf{R}_\lambda$ that the set $\{i \in I \mid B_i V \neq 0\}$ is finite. We say that **R** is *even* if $\mathbf{R}_\lambda$ is trivial for all iweights of odd parity, and *nilpotent* if the natural transformations defined by dots act nilpotently on any object. The works [BW18b, ES18] strongly suggested that there should be a relationship similar to the one in [BSW20] connecting Brauer categorifications to iquantum categorifications for the appropriate Satake diagrams.

In recent work [GRS26, Th. A], Gao, Rui and Song have constructed exactly such a bridge assuming that the ground field $\Bbbk$ is of characteristic 0 and the spectrum of the Brauer categorification is $\frac{1}{2} + \mathbb{Z}$. This is the case that corresponds to the iquantum group of quasi-split type $\mathrm{AIII}_{2\infty}$ in Table 1, exactly the one which was categorified in [BSWW18]. At the time that [GRS26] appeared on the `arXiv` (early September 2026), we had also been working on this problem for some time, and had independently developed a similar theory with no restrictions on the ground field (except char $\Bbbk \neq 2$) or on the spectrum. The relevant iquantum groups are all of the Satake types in Table 1. However, at that time, we had not yet checked all of the relations, so our proofs were incomplete. In fact, the relations interacting with the 0 node in these Satake diagrams turn out to be substantially more complicated.

In this article, we present our approach, which gives a complete solution to the problem in the general case. A key difference compared to Gao, Rui and Song's paper is that we have formulated the results using the unified general setup of [BWW25], whereas Gao, Rui and Song used the older setup from [BSWW18] which only applies in type $\mathrm{AIII}_{2\infty}$. This is essential in order to include all possible spectra in characteristic 0 and any case in positive characteristic. The principal difficulty in the Brauer/iquantum case compared to the Heisenberg/Kac–Moody case is that there is no counterpart of the degenerate affine Hecke algebra or quiver Hecke algebra ("half"), so it is simply not possible to avoid the problem of relating (unoriented) cups and caps in the affine Brauer category to the ones in the 2-iquantum group. Our formulae which do this explicitly involve natural transformations which we call *internal bubbles*; see section 7. They are closely related to the internal bubbles used to describe the relationship between leftward cups and caps in Heisenberg/Kac–Moody categorification which we worked out in [BSW26]. In the type $\mathrm{AIII}_{2\infty}$ case, Gao, Rui and Song discovered the same

natural transformations, although their setup is somewhat different. Internal bubbles on strings labelled by 0 of the Satake diagrams in Table 1 are more difficult to understand.

Our first main theorem, which is Corollary 8.3 in the main body of the text, is as follows:

Theorem A. *Assuming that $\mathfrak{g}$ is of the Cartan type described above and that the parameter choices for the 2-iquantum group $\mathfrak{U}^\imath(\mathfrak{g})$ are as specified in section 3, any even nilpotent iquantum categorification $\mathbf{R} = \bigoplus_{\lambda \in X^\imath} \mathbf{R}_\lambda$ can be made into a Brauer categorification in a canonical way.*

We have not developed enough of the setup to be able to state Theorem A more precisely in this introduction, but of course that is done later in the paper. In particular, we define **AB** in section 2, the relevant 2-iquantum group $\mathfrak{U}^\imath(\mathfrak{g})$ in section 3, and develop some of the basic notions of Brauer and iquantum categorifications in sections 4 and 5, respectively; these results are also discussed in [GRS26]. The endofunctor B on the Brauer side is related to the functors B_i ($i \in I$) on the iquantum side by $B = \bigoplus_{i \in I} B_i$; specifically, B_i is the generalized i-eigenspace of the endomorphism of B defined by the dot generator of the affine Brauer category. The explicit formulae relating the action of the generating morphisms of **AB** to the action of the generating 2-morphisms of $\mathfrak{U}^\imath(\mathfrak{g})$ are recorded in (8.3) to (8.6). The generating function for dotted bubbles in **AB** acts via

$$\bigcirc(u) \;=\; \prod_{i \in I} \bar{\imath}\bigcirc(u - i)\ .$$

We expected that this would be the case from the outset, and this expectation led us to a key intermediate result, Theorem 7.2 below, the proof of which is similar to the proof of [BSW26, Th. 7.1]. This is how we discovered the appropriate definition of internal bubbles in this setting.

Our second main theorem, and the one which is most important from the point of view of applications to type B/C/D representation theory, shows that the construction in Theorem A is invertible, so that the notions of Brauer categorification and even nilpotent iquantum categorification are equivalent. This is Theorem 9.9 in the main text.

Theorem B. *Any Brauer categorification $\mathbf{R}$ can be made into an even nilpotent iquantum categorification in a canonical way. Moreover, this construction and the construction in Theorem A are inverse to each other.*

Theorem B is deduced from a stronger result, Theorem 9.5, which identifies **AB** with a topologically dense monoidal subcategory of a topological strict monoidal category denoted $\widehat{\mathbf{AB}}$. The latter is constructed by first passing from $\mathfrak{U}^\imath$ to a completion $\widehat{\mathfrak{U}}^\imath$, then collapsing $\widehat{\mathfrak{U}}^\imath$ to a monoidal category by replacing its morphism categories with products over certain pairs of objects. This construction is explained in section 6. Our approach to Theorem A also involves a stronger result, Theorem 8.2, which establishes the existence of a strict $\Bbbk$-linear monoidal functor $\Psi\colon \mathbf{AB} \to \widehat{\mathbf{AB}}$. Then, to prove Theorem 9.5, we show that Ψ is faithful and topologically dense. Finally, the proof of Theorem B is completed by showing that the action of **AB** on any Brauer categorification extends by continuity to an action of $\widehat{\mathbf{AB}}$, hence, to $\mathfrak{U}^\imath$ from which $\widehat{\mathbf{AB}}$ was constructed in the first place. This general strategy cuts down considerably on the number of relations that need to be checked explicitly compared to the approach taken in [GRS26]: we only need to make such calculations in one direction in order to prove Theorem 8.2.

In the main body of the article, we refer to the categorifications discussed above as *Abelian* Brauer and iquantum categorifications. There are also analogous notions of *Karoubian* Brauer and iquantum categorifications, in which the underlying category is merely required to be a *locally finite Karoubian category*, that is, an additive idempotent-complete $\Bbbk$-linear category whose morphism spaces are finite-dimensional. On both the Brauer side and on the iquantum side, there are universal families of examples of Karoubian categorifications: for a monic polynomial $f(u) \in \Bbbk[u]$ of degree $\beta \geq 0$, there is a corresponding *cyclotomic quotient* of **AB** denoted $\mathbf{cB}(f)$ (Definition 10.1); for monomials $f_i(u) := u^{\beta_i}$ for $\beta_i \geq 0$ with $\beta_i = 0$ for all but finitely many i, there is a corresponding

cyclotomic quotient of $\mathfrak{U}^\imath(\mathfrak{g})$ denoted $\mathbf{cU}^\imath(f_i \mid i \in I)$ (Definition 10.5). There are analogs of Theorems A and B in the Karoubian case. In the last section of the article, we apply these to prove that $\mathbf{cB}(f)$ and $\mathbf{cU}^\imath(f_i \mid i \in I)$ are equivalent categories when $f(u) = \prod_{i \in I} f_i(u-i)$, i.e., β_i is the multiplicity of i as a root of $f(u)$. This result is the Brauer/iquantum counterpart of the isomorphism theorem between GCQs in [BSW20]; see Theorem 10.8. Gao, Rui and Song also prove such a theorem in the $\mathrm{AIII}_{2\infty}$ case in [GRS26, Th. B].

For example, this result implies the existence of an equivalence between the most important cyclotomic quotient of the affine Brauer category, the Deligne category $\underline{\mathrm{Rep}}\, O_\delta$ at parameter $\delta \in \mathbb{C}$, and an iquantum cyclotomic quotient. Indeed, $\underline{\mathrm{Rep}}\, O_\delta$ is the additive Karoubi envelope of the classical Brauer category $\mathbf{B}_\delta$ over $\mathbb{C}$ with $\bigcirc = \delta\, \mathrm{id}_{\mathbb{1}}$. By Rui and Song's theory, the additive Karoubi envelope of $\mathbf{B}_\delta$ is isomorphic to the level one cyclotomic quotient $\mathbf{cB}\big(u - (1-\delta)/2\big)$. It follows for any $\delta \in \mathbb{C}$ that $\underline{\mathrm{Rep}}\, O_\delta$ is equivalent to $\mathbf{cU}^\imath\big(u^{\delta_{i,(1-\delta)/2}} \,\big|\, i \in I\big)$, with the result for $\delta \in 2\mathbb{Z}$ following already from [GRS26]. These results solve a problem which has been asked by several different groups since the time of [BK09], e.g., see [Li14]. The interesting $\mathbb{Z}$-grading on the Brauer category revealed by the equivalence was constructed in [ES16] by a different approach.

We hope there will be further applications to the general theory of cyclotomic quotients of **AB** which has been developed in other work of Gao, Rui and Song, e.g., see [GRS23b, GRS23a]. The explicit connection to iquantum groups and their combinatorics which this paper and [GRS26] establish should allow for a more conceptual interpretation of these earlier results, and it provides further impetus for the development of the theory of iquantum groups and their categorifications.

Preliminaries. Throughout the paper we work over an algebraically closed field $\Bbbk$ of characteristic $p \neq 2$. We fix a choice of a square root of -1, denoting it by $\sqrt{-1}$. As usual, "$\Bbbk$-linear" indicates enrichment in the monoidal category of vector spaces. We use the word "graded" to indicate enrichment in the monoidal category of $\mathbb{Z}$-graded vector spaces.

We let $I := \Bbbk$. This set will also be the labelling set for simple roots in a certain root system as in Table 1; this will be explained at the start of section 3. We reserve the letter h for $\frac{1}{2} \in \Bbbk$. In string diagrams, we will often label strings by elements $i, j, k, \dots \in I$, and will use $\overline{\imath}, \overline{\jmath}, \overline{k}, \dots$ instead of $-i, -j, -k, \dots$ for clarity in diagrams.

We will often work with formal Laurent series $f(u) = \sum_{n \in \mathbb{Z}} a_n u^n \in A(\!(u^{-1})\!)$ with coefficients in some algebra A. We use the notation $[f(u)]_{u:n}$ to denote the u^n-coefficient a_n, $[f(u)]_{u:\geq 0}$ for the polynomial part $\sum_{n \geq 0} a_n u^n$, and so on. The formal power series

$$\tfrac{1}{u-x} = u^{-1} + xu^{-2} + x^2u^{-3} + \cdots \in A[x](\!(u^{-1})\!)$$

will be particularly important. Some basic tricks: for any polynomial $f(x) \in A[x]$, we have that

$$\frac{f(x)}{u-x} = \left[\frac{f(u)}{u-x}\right]_{u:<0}, \qquad f(x) = \left[\frac{f(u)}{u-x}\right]_{u:-1}. \tag{1.1}$$

Let $\partial_{x,y} \colon A[x,y] \to A[x,y]$ be the *Demazure operator*, the linear map defined by

$$\partial_{x,y} f(x,y) := \frac{f(x,y) - f(y,x)}{x-y}. \tag{1.2}$$

We also use the notation ∂_x to denote the one-variable Demazure operator defined by

$$\partial_x f(x) := \frac{f(x) - f(-x)}{2x} \tag{1.3}$$

for $f(x) \in A[x]$. Note that $\partial_{x,y} = -\partial_{y,x}$. We extend $\partial_{x,y}$ to formal series in an auxiliary variable u in the obvious way. Using the second identity in (1.1) together with partial fractions, it follows that

$$\partial_{x,y}\Big(\frac{f(x)}{u-x}\Big) = \left[\frac{f(u)}{(u-x)(u-y)}\right]_{u:<0}, \qquad \partial_{x,y} f(x) = \left[\frac{f(u)}{(u-x)(u-y)}\right]_{u:-1} \tag{1.4}$$

in $A[x,y]((u^{-1}))$. Later, we will also apply Demazure operators to formal power series; the extended versions are defined in the same way.

We will eventually need the following version of Hensel's lemma:

Lemma 1.1 ([BSW20, Cor. 2.4])**.** *Let A be a finite-dimensional commutative local $\Bbbk$-algebra with unique maximal ideal $\mathfrak{m}$, so that $A/\mathfrak{m} = \Bbbk$. Let $f(u) \in A[u]$ be a monic polynomial and $\bar{f}(u) \in \Bbbk[u]$ be its reduction modulo $\mathfrak{m}$. Let β_i be the multiplicity of i as a root of $\bar{f}(u)$. Then there are unique monic $f_i(u) \in A[u]$, all but finitely many of which are 1, such that $\bar{f}_i(u) = u^{\beta_i}$ and $f(u) = \prod_{i\in I} f_i(u-i)$.*

We call the polynomial $f_i(u)$ defined by Lemma 1.1 the *local factor* of $f(u)$ at $u = i$. Similarly, we call the rational function $R_i(u)$ appearing in the following corollary the *local factor* of $R(u)$ at $u = i$.

Corollary 1.2. *For A and $\mathfrak{m}$ as in the preceding lemma, let $R(u) = \frac{f(u)}{g(u)} \in A(u)$ be a rational function with $f(u), g(u) \in A[u]$ monic, and $\bar{R}(u) \in \Bbbk(u)$ be its reduction modulo $\mathfrak{m}$. Letting $\mathrm{ord}_i(R) \in \mathbb{Z}$ be the order of $\bar{R}(u)$ at $u = i$ (the multiplicity of i as a zero minus its multiplicity as a pole), there are unique $R_i(u) \in u^{\mathrm{ord}_i(R)} + u^{\mathrm{ord}_i(R)-1}\mathfrak{m}[u^{-1}]$ such that $R_i(u) = 1$ for all but finitely many i and $R(u) = \prod_{i\in I} R_i(u-i)$.*

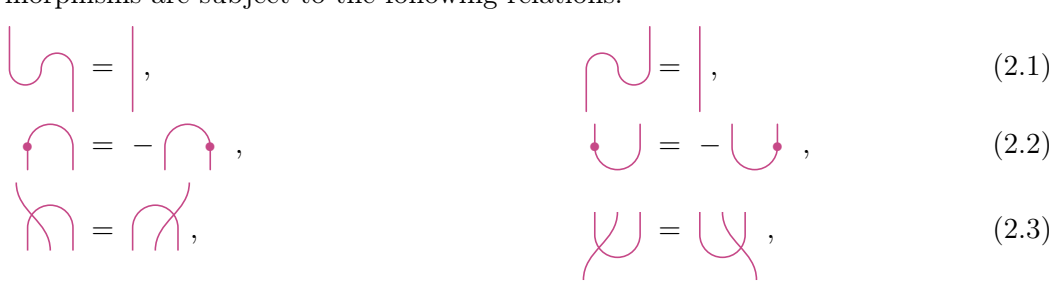
Proof. For existence, apply Lemma 1.1 to $f(u)$ and $g(u)$ to obtain $f_i(u)$ and $g_i(u)$, then set $R_i(u) := f_i(u)/g_i(u)$, which is in $u^{\mathrm{ord}_i(R)} + u^{\mathrm{ord}_i(R)-1}\mathfrak{m}[u^{-1}]$ as $g_i(u) = u^n(1-k(u))$ for $n := \deg g_i(u)$ and $k(u) \in u^{-1}\mathfrak{m}[u^{-1}]$, and $(1-k(u))^{-1} = 1 + k(u) + \cdots + k(u)^N$ for some N as $\mathfrak{m}$ is nilpotent. Uniqueness follows from the uniqueness assertion in Lemma 1.1 by an argument which involves cross multiplying to clear denominators. □

AI Statement. The foundational material about Brauer and iquantum categorifications in sections 4 and 5 was worked out using BI; the first author gave a talk about this at a conference in Uppsala in August 2026. The definition of the internal bubbles in section 7, the formulae in Theorem 8.2, and the choice of parameters for the 2-iquantum group in section 3 were also in place at that time, and we had almost finished checking the relations needed to prove Theorem 8.2. However, we were struggling to check the Reidemeister III relation in Lemma 8.13, especially when the 0 node was involved. We started using AI systematically in September, at which point ChatGPT suggested the amazingly short proof given below. We also used both ChatGPT and Anthropic's Claude extensively for the difficult convergence arguments in section 9, although their suggestions were rewritten and expanded by the authors. All of the other writing was done by the authors, but ChatGPT and Claude were used for proofreading purposes.

2. The affine Brauer category

We begin by recalling Rui and Song's definition of the affine Brauer category. We use string calculus with the standard conventions from the recent categorification literature.

Definition 2.1 ([RS19, Def. 1.2])**.** The *affine Brauer* category **AB** is the strict $\Bbbk$-linear monoidal category generated by an object B with identity endomorphism $|$ and the morphisms listed in Table 2. The generating morphisms are subject to the following relations:



(2.1)

(2.2)

(2.3)

Generator
$: B \otimes B \to \mathbb{1}$
$: \mathbb{1} \to B \otimes B$
$: B \otimes B \to B \otimes B$
$: B \to B$

TABLE 2. Generating morphisms for the affine Brauer category

(2.4)

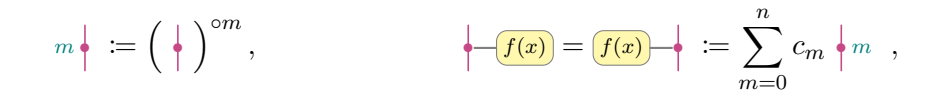
(2.5)

(2.6)

Remark 2.2. Our formulation of Definition 2.1 is convenient but not minimal: in (2.2) to (2.5), the second relation follows from the first one and the earlier relations. Sketch proof for (2.2), (2.3) and (2.5): attach a pair of cups at the bottom of the first relation, then simplify using the zig-zag relation; we refer to this sort of argument as rotation by 180°. Sketch proof for (2.4): attach a cup at the bottom and a cap at the top of one of the strings in the first relation, then use the zig-zag relation; this is rotation by 90°. See [RS19] for more details, though note that our definition differs from [RS19, Def. 1.2] by the sign of the dot.

We call the morphism a *dot*. It is clear from the defining relations that **AB** has symmetries $\mathtt{T}\colon \mathbf{AB}^{\mathrm{op}} \to \mathbf{AB}$ defined by reflecting string diagrams in a horizontal axis, and $\mathtt{R}\colon \mathbf{AB}^{\mathrm{rev}} \to \mathbf{AB}$ defined by reflecting string diagrams in a vertical axis then multiplying by $(-1)^d$ where d is the total number of dots in the diagram.

For a polynomial $f(x) = \sum_{m=0}^{n} c_m x^m \in \Bbbk[x]$, we use the notation



referring to the latter as *pinning* the polynomial $f(x)$ to the marked point on the string. More generally, one can consider pins involving polynomials $f(x) = \sum_{m=0}^{n} c_m x^m$ whose coefficients are elements of $\mathrm{End}_{\mathbf{AB}}(\mathbb{1})$. These endomorphisms become bubbles placed in the 2-cell in which the pinhead is placed. For example,

Now, the endomorphisms defined by the left pin and the right pin could be different!

Often, we will use power series in $\Bbbk[x]((u^{-1}))$ pinned to dots in string diagrams to denote generating functions of morphisms in **AB**. We introduce the shorthand notation

(2.7)

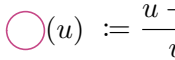
These are fixed by $\mathtt{T}$ and interchanged by $\mathtt{R}$. Also replacing u by $-u$ in one of them produces *minus* the other one. We have that

(2.8)

(2.9)

Applying $\mathtt{T}$ shows that the mirror images of these relations in a horizontal axis also hold.

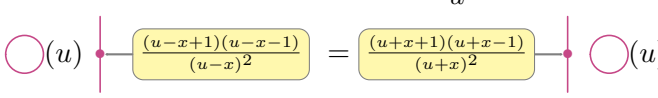
Recall that $h = \frac{1}{2} \in \Bbbk$. We will work with the following generating function for dotted bubbles:



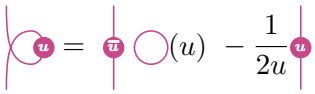
$$(u) \coloneqq \frac{u-h}{u}\,\mathrm{id}_{\mathbb{1}} + \quad \in \mathrm{id}_{\mathbb{1}} + u^{-1}\,\mathrm{End}_{\mathbf{AB}}(\mathbb{1})[\![u^{-1}]\!]\,. \tag{2.10}$$

It is the generating function used by Gao, Rui and Song in [GRS23a] multiplied by u^{-1}, and it is the generating function denoted in [SW26] multiplied by $\frac{u-h}{u}$. Since we have merely rescaled, the relations from [GRS23a, Lem. 4.1–4.2] and [SW26, Prop. 2.5] are easily translated to obtain the following relations:

$$(u)\,(-u) = \frac{(u+h)(u-h)}{u^2}\,\mathrm{id}_{\mathbb{1}}, \tag{2.11}$$



$$(u) \quad \frac{(u-x+1)(u-x-1)}{(u-x)^2} = \frac{(u+x+1)(u+x-1)}{(u+x)^2} \quad (u)\,, \tag{2.12}$$



$$= \quad (u) - \frac{1}{2u}\,. \tag{2.13}$$

Note also that the bubble generating function is invariant under both of the symmetries $\mathtt{T}$ and $\mathtt{R}$.

The fundamental theorem about $\mathbf{AB}$ proved by Rui and Song in [RS19] gives explicit bases for its morphism spaces. To state this briefly, let $Z \coloneqq \mathrm{End}_{\mathbf{AB}}(\mathbb{1})$. This is a commutative algebra generated by the dotted bubbles

$$z_r \coloneqq \quad r \tag{2.14}$$

for all $r \geq 0$. Now take $m, n \geq 0$. We make $\mathrm{Hom}_{\mathbf{AB}}(B^{\otimes n}, B^{\otimes m})$ into a right Z-module so that $z \in Z$ acts on a morphism θ by $\theta z \coloneqq \theta \otimes z$. An $m \times n$*-shape* means a (magenta) string diagram D for a morphism in $\mathrm{Hom}_{\mathbf{AB}}(B^{\otimes n}, B^{\otimes m})$ with no dots and no bubbles. Such shapes exist if and only if $m \equiv n \pmod 2$. We say that a shape is *reduced* if there are no zig-zags, bigons or loops, and two shapes are *equivalent* if they define the same combinatorial matching between their boundary points. Let $\mathrm{S}(m \times n)$ be a fixed choice of representatives for the equivalence classes of reduced $m \times n$ shapes. Also, for $D \in \mathrm{S}(m \times n)$ and $l \coloneqq \frac{m+n}{2}$, the number of strings in D, we fix a choice of normally ordered points $p_1, \ldots, p_l$ away from crossings and critical points, one on each string. For $f \in \Bbbk[x_1, \ldots, x_l]$, we let $D(f)\colon B^{\otimes n} \to B^{\otimes m}$ be the morphism obtained from D by pinning f to the points $p_1, \ldots, p_l$.

Theorem 2.3 ([RS19, Th. B])**.** *The algebra Z is freely generated by its even generators z_{2r} $(r \geq 0)$. Moreover, for $m, n \geq 0$, the morphism space $\mathrm{Hom}_{\mathbf{AB}}(B^{\otimes n}, B^{\otimes m})$ is 0 unless $m \equiv n \pmod 2$, in which case it is free as a right Z-module with basis*

$$\mathrm{B}(m \times n) \coloneqq \left\{ D(x_1^{e_1} \cdots x_l^{e_l}) \;\middle|\; D \in \mathrm{S}(m \times n), e_1, \ldots, e_l \geq 0 \right\}. \tag{2.15}$$

Remark 2.4. The first assertion of Theorem 2.3 implies that a full set of relations for the algebra Z may be obtained by equating coefficients in (2.11). The proof of the theorem has two parts. The easy part is a straightening algorithm showing that Z is spanned by monomials in z_{2r} $(r \geq 0)$ and that $\mathrm{Hom}_{\mathbf{AB}}(B^{\otimes n}, B^{\otimes m})$ is generated as a Z-module by $\mathrm{B}(m \times n)$. The hard part is to prove linear independence. Linear independence could also be reproved using the monoidal functor $\Psi\colon \mathbf{AB} \to \widehat{\mathbf{AB}}$ from Theorem 8.2, whose target is built from the completed 2-iquantum group $\widehat{\mathfrak{U}}^\imath$:

given the easy part, the arguments in the proofs of Lemmas 9.2 and 9.3 below show both that Z is freely generated by the z_{2r} and that Ψ maps any non-trivial Z-linear combination of the elements of $\mathrm{B}(m \times n)$ to a non-zero morphism. However, that is a much more complicated argument than Rui and Song's approach, which exploits some natural actions of $\mathbf{AB}$ on the categories $\mathcal{O}$ for classical Lie algebras.

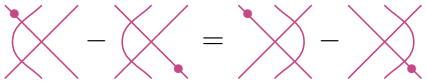
Remark 2.5. In [GRS26, Sec. 2], Gao, Rui and Song also consider some variants of $\mathbf{AB}$ in which some of the relations are dropped. Let $\mathbf{AB}_{\mathrm{I}}$ be the strict $\Bbbk$-linear monoidal category defined in the same way as $\mathbf{AB}$ above but omitting the Reidemeister I, II and III relations (we mean the relations in (2.5) and (2.6)). Similarly, let $\mathbf{AB}_{\mathrm{II}}$ be defined in the same way but omitting Reidemeister II and III, and let $\mathbf{AB}_{\mathrm{III}}$ be defined in the same way but only omitting Reidemeister III. The following relations hold in $\mathbf{AB}_{\mathrm{I}}$:

(2.16)

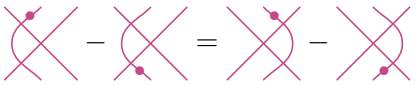
We leave the easy proof of this as an exercise for the reader. Less trivially, in [GRS26, Cor. 2.4], it is shown that the following relations hold in $\mathbf{AB}_{\mathrm{II}}$:

(2.17)

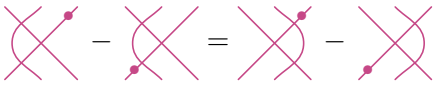
Also, in [GRS26, Cor. 2.7], it is shown that the following relations hold in $\mathbf{AB}_{\mathrm{III}}$:



(2.18)



(2.19)



(2.20)

The relations (2.17) to (2.20) were used to simplify the proofs of the Reidemeister II and III relations in [GRS26], and we will use them in exactly the same way in the proofs of Lemmas 8.11 and 8.13 below.

3. The 2-iquantum group

Next, we introduce the 2-iquantum group relevant for the constructions in this paper. It is a special case of the general definition in [BWW25]. Let $I := \Bbbk$ and $\tau\colon I \to I$ be the involution with $\tau i = -i$. We usually prefer to write $-i$ instead of τi but the latter is the notation used in the general setup of iquantum groups which we are following.

Let X be the free Abelian group $\bigoplus_{i \in I} \mathbb{Z}\varpi_i$ and $h_i\colon X \to \mathbb{Z}$ be the homomorphism of Abelian groups with $h_i(\varpi_j) := \delta_{i,j}$. Let $\alpha_i := 2\varpi_i - \varpi_{i-1} - \varpi_{i+1}$ so that

$$a_{i,j} := h_i(\alpha_j) = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } i = j \pm 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

is a symmetric Cartan matrix whose connected components are all of type A_∞ if $p = 0$ or $A^{(1)}_{p-1}$ if $p > 0$. For $i \in I$, let

$$\varsigma_i := \begin{cases} 1 & \text{if } i = h \\ -1 & \text{if } i = 0 \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

so that $\varsigma_i + \varsigma_{-i} = -a_{i,-i}$. Let $t\colon X \to \mathbb{Z}/2$ and $p\colon X \to \mathbb{Z}/2$ be the homomorphisms of Abelian groups mapping λ to

$$t(\lambda) := h_0(\lambda) \bmod 2, \qquad p(\lambda) := \sum_{i \in I} h_i(\lambda) \bmod 2. \tag{3.3}$$

We call $t(\lambda)$ the *type* and $p(\lambda)$ the *parity* of λ. Simple roots are even, so $p(\lambda - \alpha_i) = p(\lambda)$. However, $t(\lambda - \alpha_i)$ is different from $t(\lambda)$ when $i = \pm 1$.

Let $\tau\colon X \to X$ be the homomorphism with $\tau\varpi_i := \varpi_{-i}$. Let $X^\imath$ be the quotient of X by the subgroup spanned by $\varpi_i + \varpi_{-i}$ $(i \in I)$, denoting the quotient map by $\pi\colon X \twoheadrightarrow X^\imath$. Elements of $X^\imath$ are called *iweights*. For $\lambda \in X^\imath$ and $\mu \in X$, we allow ourselves to write expressions like $\lambda + \mu$ instead of $\lambda + \pi(\mu)$. The homomorphisms t and p factor through the quotient to induce homomorphisms we also denote by $t, p\colon X^\imath \to \mathbb{Z}/2$. The set $X^\imath$ decomposes as

$$X^\imath = X^\imath_0 \sqcup X^\imath_1, \tag{3.4}$$

where $X^\imath_0$ consists of the iweights with even parity, and $X^\imath_1$ consists of the odd ones. One more important notation is

$$\lambda_i := h_i\big((\mathrm{id} - \tau)(\hat{\lambda})\big) = h_i(\hat{\lambda}) - h_{-i}(\hat{\lambda}) \tag{3.5}$$

for $\lambda \in X^\imath$ with pre-image $\hat{\lambda} \in X$. This is well defined independent of the choice of $\hat{\lambda}$. Note also that $p(\lambda) \equiv t(\lambda) + \sum_{i \in I'} \lambda_i \pmod 2$ for any $I' \subset I$ chosen so that $I = I' \sqcup \{0\} \sqcup (-I')$.

Lemma 3.1. *For weights $\mu, \nu \in X$ of the same parity, we have that $(\mathrm{id} - \tau)(\mu) = (\mathrm{id} - \tau)(\nu)$ in X if and only if $\pi(\mu) = \pi(\nu)$ in $X^\imath$.*

Proof. An even weight $\lambda \in X$ is a τ-fixed point if and only if $\pi(\lambda) = 0$ in $X^\imath$. □

We now have the data needed to define a quasi-split iquantum group $\dot{\mathrm{U}}^\imath$ as in [BWW25, Sec. 3.1]. Since this will not play a major role, we will not repeat its generators-and-relations definition here, but note that it is a locally unital $\mathbb{Q}(q)$-algebra with distinguished idempotents $\{1_\lambda \mid \lambda \in X^\imath\}$ and generators $b_i 1_\lambda = 1_{\lambda - \alpha_i} b_i$ for $i \in I$ and $\lambda \in X^\imath$.

We proceed to the main goal of the section: the definition of a graded 2-category $\mathfrak{U}^\imath$ which is a categorification of an integral form of $\dot{\mathrm{U}}^\imath$. For this, we need some additional parameters, which involve some choices. Recall that $I = \Bbbk$. Let $\sim$ be the equivalence relation on I generated by the relations $i \sim -i$ and $i \sim i + 1$. We make a henceforth fixed choice K of $\sim$-equivalence class representatives. The equivalence classes $[k]$ for $k \in K$ are the vertex sets of the Satake diagrams in Table 1. If $p = 0$, there are three types: $[0]$, $[h]$, and all the others. If $p > 2$, there are only two types: $[0] = [h]$, and all others. We define two partitions

$$I = I_+ \sqcup \{0\} \sqcup I_- = I_{\mathrm{lo}} \sqcup \{0\} \sqcup I_{\mathrm{hi}} \tag{3.6}$$

of the set I by letting I_+ and I_- be the sets indicated by the $+$ and $-$ labels in the nodes in Table 1, and I_{lo} and I_{hi} be the sets on the bottom and top rows of these diagrams. Important properties of these partitions are that $I_- = -I_+, I_{\mathrm{hi}} = -I_{\mathrm{lo}}$ and $h \in I_+ \cap I_{\mathrm{lo}}$. Note also that $1 \in I_{\mathrm{hi}}$, but it could either be in I_+ or I_- depending on the characteristic p (it is in I_- if and only if $p \equiv 3 \pmod 4$).

For $i \in I$, we let $c_i \colon (X,+) \to \Bbbk^\times$ be the homomorphism of Abelian groups defined by

$$c_i(\lambda) := \begin{cases} 1 & \text{if } i \in I_+ \cup \{0\} \\ (-1)^{h_i(\lambda)} & \text{if } i \in I_-. \end{cases} \tag{3.7}$$

In [BWW25, Hyp. 3.3], these functions are called *normalization functions*, and they are required to satisfy the properties $c_i(\alpha_i) = 1$ for all $i \in I$, and

$$c_{-i}(\tau\lambda) = (-1)^{h_i(\lambda)} c_i(\lambda) \tag{3.8}$$

for $i \in I - \{0\}$ and $\lambda \in X$. Our choice for these functions clearly satisfies both of these properties. In particular, we have that

$$c_{i,j} := c_i(\alpha_j) = \begin{cases} 1 & \text{if } i \in I_+ \cup \{0\} \\ (-1)^{a_{i,j}} & \text{if } i \in I_- \end{cases} \tag{3.9}$$

for $i, j \in I$, and $c_{i,i} = 1, c_{i,-i} = (-1)^{\delta_{i,-h}}$. Let

$$t_{i,j} := \begin{cases} -1 & \text{if there is a single arrow } \overset{i}{\circ}\!\!\leftarrow\!\!\overset{j}{\circ} \text{ in Table 1} \\ 1 & \text{if there is a single arrow } \overset{i}{\circ}\!\!\rightarrow\!\!\overset{j}{\circ} \text{ in Table 1} \\ -\sqrt{-1} & \text{if there is a double arrow } \overset{i}{\circ}\!\!\Leftarrow\!\!\overset{j}{\circ} \text{ in Table 1} \\ \sqrt{-1} & \text{if there is a double arrow } \overset{i}{\circ}\!\!\Rightarrow\!\!\overset{j}{\circ} \text{ in Table 1} \\ 1 & \text{in all other cases,} \end{cases} \tag{3.10}$$

$$Q_{i,j}(x,y) := \begin{cases} t_{i,j}(x-y)^{-a_{i,j}} & \text{if } i \neq j \\ 0 & \text{if } i = j, \end{cases} \qquad Q^\imath_{i,j}(x,y) := (-1)^{\delta_{i,-j}} Q_{i,j}(x,y). \tag{3.11}$$

The last parameter choice to be made is

$$\zeta_i := \begin{cases} 1 & \text{if } i \in I_+ \\ -\frac{1}{2} & \text{if } i = 0 \\ -2 & \text{if } i = -h \\ -1 & \text{for all other } i \in I_-. \end{cases} \tag{3.12}$$

Because $\overset{-h}{\circ}\!\!\leftarrow\!\!\overset{h}{\circ}$, $\varsigma_h = 1$ and $\varsigma_{-h} = 0$, these satisfy

$$\zeta_0 = -\tfrac{1}{2}, \qquad Q_{-i,i}(x,-x) = (-1)^{\varsigma_i+1}\zeta_i\zeta_{-i}x^{-a_{i,-i}} \tag{3.13}$$

for all $i \in I - \{0\}$, which is another requirement from [BWW25, Hyp. 3.3].

There are several further pieces of notation dependent on the choices made. Let $\gamma_i \colon (X^\imath, +) \to \Bbbk^\times$ be the homomorphism of Abelian groups defined by $\gamma_i(\lambda) := (-1)^{h_{-i}(\hat\lambda)} c_i(\hat\lambda)/c_{-i}(\hat\lambda)$ for $\lambda \in X^\imath$ with pre-image $\hat\lambda \in X$. Explicitly, we have that

$$\gamma_i(\lambda) = \begin{cases} 1 & \text{if } i \in I_+ \\ (-1)^{t(\lambda)} & \text{if } i = 0 \\ (-1)^{\lambda_i} & \text{if } i \in I_-. \end{cases} \tag{3.14}$$

Also let

$$\gamma_{i,j} := \gamma_i(\pi(\alpha_j)) = (-1)^{a_{-i,j}} c_{i,j}/c_{-i,j} = \begin{cases} 1 & \text{if } i \in I_+ \\ (-1)^{a_{0,j}} & \text{if } i = 0 \\ (-1)^{a_{i,j}+a_{i,-j}} & \text{if } i \in I_-. \end{cases} \tag{3.15}$$

In particular, $\gamma_{i,i} = \gamma_{i,-i} = (-1)^{\delta_{i,-h}}$. We use the shorthand

$$\zeta_i(\lambda) \coloneqq \zeta_i\gamma_i(\lambda) = \begin{cases} 1 & \text{for } i \in I_+ \\ (-1)^{t(\lambda)+1}\frac{1}{2} & \text{if } i = 0 \\ (-1)^{\lambda_i+1}2 & \text{if } i = -h \\ (-1)^{\lambda_i+1} & \text{for all other } i \in I_- \end{cases} \tag{3.16}$$

for $\lambda \in X^\imath$. Let

$$r_{i,j} \coloneqq c_{i,j}/t_{i,j}. \tag{3.17}$$

We have that $r_{i,i} = r_{-i,i} = 1$ for all $i \in I$ as required by [BWW25, Hyp. 3.3]. To see this for $i \in \{\pm h\}$, note that $c_{h,-h} = t_{h,-h} = 1$ and $c_{-h,h} = t_{-h,h} = -1$. The following lemma checks the final constraint on parameters from [BWW25, Hyp. 3.3]:

Lemma 3.2. *For $i, j \in I$, we have that $Q_{-i,-j}(x, y) = r_{i,j}r_{j,i}r_{i,-j}r_{j,-i}Q_{i,j}(-x, -y)$.*

Proof. It holds if $i = j$ as both sides are zero, so assume that $i \neq j$. As $r_{i,j} = c_{i,j}/t_{i,j}$, $Q_{-i,-j}(x, y) = t_{-i,-j}(x - y)^{-a_{i,j}}$ and $Q_{i,j}(-x, -y) = t_{i,j}(-1)^{a_{i,j}}(x - y)^{-a_{i,j}}$, the identity is equivalent to

$$t_{-i,-j}t_{j,i}t_{i,-j}t_{j,-i} = (-1)^{a_{i,j}}c_{i,j}c_{j,i}c_{i,-j}c_{j,-i}.$$

Now one needs to consider the various cases for $\{\pm i, \pm j\}$, i.e., it is a rank-two calculation. Because $t_{i,j} = (-1)^{a_{i,j}}t_{j,i}$ and $(-1)^{a_{i,j}}t_{-j,-i} = t_{-i,-j}$, we have that $t_{-j,-i}t_{i,j} = t_{-i,-j}t_{j,i}$. From this, it follows that the identity holds for (i, j) if and only if it holds for (j, i), which cuts down the cases to be considered.

- If $i \notin \{-j, \pm j \pm 1\}$ the identity holds trivially.
- If $i = -j$ it holds trivially except if $i = \pm h$ and $j = \mp h$, and this case follows by checking it for $i = h, j = -h$ using $(-1)^{a_{h,-h}}c_{h,-h}c_{-h,h} = 1$.
- If $i = 0$ and $j = \pm 1$, or $j = 0$ and $i = \pm 1$, it suffices to check when $i = 0$ and $j = \pm 1$. The $j = -1$ case requires $t_{0,1}t_{-1,0}t_{0,1}t_{-1,0} = -c_{0,-1}c_{-1,0}c_{0,1}c_{-1,0}$ and the $j = 1$ case requires $t_{0,-1}t_{1,0}t_{0,-1}t_{1,0} = -c_{0,1}c_{1,0}c_{0,-1}c_{1,0}$. This is true—both sides of both equations equal -1.

We are now reduced to the situation that $i, j \neq 0$, $i \neq -j$ and $i = \pm j \pm 1$.

- If $i, j \neq 0$, $i \neq -j$ and $i \in \{-j \pm 1\}$ then $i \notin \{j \pm 1\}$, and the identity to be checked reduces to $t_{i,-j}t_{j,-i} = c_{i,-j}c_{j,-i}$. If $i \in I_+$ and $j \in I_-$ then the right-hand side is -1, as is the left-hand side because in this situation the direction of the arrow connecting i to $-j$ is the same as the direction of the arrow connecting $-i$ to j. If $i, j \in I_+$ or $i, j \in I_-$ then the right-hand side is 1, as is the left-hand side because in this situation the direction of the arrow connecting i to $-j$ is the opposite of the direction of the arrow connecting $-i$ to j.
- If $i, j \neq 0$, $i \neq -j$ and $i \notin \{-j \pm 1\}$ then $i \in \{j \pm 1\}$, and the identity to be checked reduces to $t_{-i,-j}t_{j,i} = -c_{i,j}c_{j,i}$. If $i \in I_+$ and $j \in I_-$ then the right-hand side is 1, as is the left-hand side because in this situation the direction of the arrow connecting i to j is opposite to the direction of the arrow connecting $-i$ to $-j$. If $i, j \in I_+$ or $i, j \in I_-$ then the right-hand side is -1, as is the left-hand side because in this situation the direction of the arrow connecting i to j is the same as the direction of the arrow connecting $-i$ to $-j$. □

Lemma 3.3. *For $i, j \in I$, we have that $r_{j,i}r_{-i,-j}r_{j,-i}r_{i,j} = \gamma_{i,j}\, t_{i,-j}/t_{i,j}$.*

Proof. Suppose first that $i = j$. Then the left-hand side is 1, as is the right-hand side because $\gamma_{i,i} = (-1)^{\delta_{i,-h}} = t_{i,-i}$ and $t_{i,i} = 1$.

Now suppose that $i \neq j$. As $Q_{-i,-j}(x, y) = t_{-i,-j}(x - y)^{-a_{i,j}}$ and $Q_{i,j}(-x, -y) = (-1)^{a_{i,j}}t_{i,j}(x - y)^{-a_{i,j}}$, Lemma 3.2 gives that $r_{i,j}r_{j,i}r_{i,-j}r_{j,-i} = (-1)^{a_{i,j}}t_{-i,-j}/t_{i,j}$. Multiplying by $r_{-i,-j}/r_{i,-j} =$

Generator	Degree
$: B_i\mathbb{1}_\lambda \Rightarrow B_i\mathbb{1}_\lambda$	2
$: B_{-i}B_i\mathbb{1}_\lambda \Rightarrow \mathbb{1}_\lambda$	$1-\lambda_i+\varsigma_i$
$: \mathbb{1}_\lambda \Rightarrow B_{-i}B_i\mathbb{1}_\lambda$	$1-\lambda_i+\varsigma_i$
$: \mathbb{1}_\lambda \Rightarrow \mathbb{1}_\lambda$ for $0 \le r \le \varsigma_i - \lambda_i$	$2r$
$: B_iB_j\mathbb{1}_\lambda \Rightarrow B_jB_i\mathbb{1}_\lambda$	$-a_{i,j}$

TABLE 3. Generating 2-morphisms for the 2-iquantum group

$c_{-i,-j}t_{i,-j}/c_{i,-j}t_{-i,-j}$, we deduce that

$$r_{j,i}r_{-i,-j}r_{j,-i}r_{i,j} = (-1)^{a_{i,j}}\frac{c_{-i,-j}}{c_{i,-j}} \cdot \frac{t_{i,-j}}{t_{i,j}}.$$

It remains to observe that $(-1)^{a_{i,j}}c_{-i,-j}/c_{i,-j} = \gamma_{i,j}$ by (3.8) and (3.15). □

Lemma 3.4. *For $i, j \in I$, we have that $r_{i,-j}r_{-i,j}r_{j,i}r_{-j,-i} = (-1)^{a_{i,j}}\gamma_{j,i}/\gamma_{i,j}$.*

Proof. Lemma 3.3 with i and j interchanged gives that $r_{i,j}r_{-j,-i}r_{i,-j}r_{j,i} = \gamma_{j,i}t_{j,-i}/t_{j,i}$. Hence,

$$r_{i,-j}r_{-i,j}r_{j,i}r_{-j,-i} = \gamma_{j,i}\frac{t_{j,-i}}{t_{j,i}} \cdot \frac{r_{-i,j}}{r_{i,j}} = \gamma_{j,i}\frac{t_{i,j}}{t_{j,i}} \cdot \frac{t_{j,-i}}{t_{-i,j}} \cdot \frac{c_{-i,j}}{c_{i,j}} = \gamma_{j,i}(-1)^{a_{i,j}}(-1)^{a_{j,-i}} \cdot (-1)^{a_{-i,j}}\gamma_{i,j}^{-1},$$

using $t_{i,j} = (-1)^{a_{i,j}}t_{j,i}$ (twice) and $\gamma_{i,j} = (-1)^{a_{-i,j}}c_{i,j}/c_{-i,j}$ from (3.15). As $a_{j,-i} = a_{-i,j}$, this is $(-1)^{a_{i,j}}\gamma_{j,i}/\gamma_{i,j}$. □

Now we have all of the required data in hand to be able to define the 2-iquantum group:

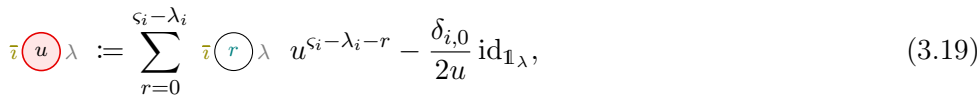
Definition 3.5 ([BWW25, Def. 3.5])**.** The *2-iquantum group* $\mathfrak{U}^\imath$ is the graded 2-category with object set $X^\imath$, generating 1-morphisms $B_i\mathbb{1}_\lambda = \mathbb{1}_{\lambda-\alpha_i}B_i \colon \lambda \to \lambda - \alpha_i$ for $\lambda \in X^\imath$ and $i \in I$, with identity 2-endomorphisms denoted by unoriented strings , and generating 2-morphisms listed in Table 3 for all $i, j \in I$ and $\lambda \in X^\imath$. The degrees of the generating 2-morphisms are also listed in this table. The generating 2-morphisms are subject to certain relations. Before writing these down, we explain some further conventions.

- We usually label strings just at one end, but note that the string label changes sign across critical points of cups and caps.
- We usually only label one of the 2-cells by the iweight λ, and if we omit all labels we are writing something that is true for all possible labels.
- We use pins defined in exactly the same way as in the previous section; the coefficients of a polynomial labelling a pinhead drawn in a 2-cell labelled λ should be elements of $\mathrm{End}_{\mathfrak{U}^\imath}(\mathbb{1}_\lambda)$. We adopt analogous shorthands to (2.7):

(3.18)

We extend the pin notation to pins with two or three dots, using additional variables y and z as needed. The alphabetical ordering of the variables x, y, z corresponds to the lexicographical ordering of the Cartesian coordinates of the dots.

- Let

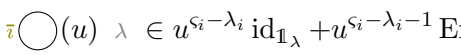


$$\overset{\imath}{\bar{\imath}}\,\overset{\lambda}{u}\,\lambda \;:=\; \sum_{r=0}^{\varsigma_i-\lambda_i} \overset{\imath}{\bar{\imath}}\,\overset{\lambda}{r}\,\lambda \;\; u^{\varsigma_i-\lambda_i-r} - \frac{\delta_{i,0}}{2u}\,\mathrm{id}_{\mathbb{1}_\lambda}, \tag{3.19}$$

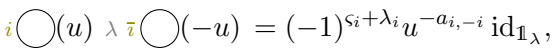
When $i=0$, the summation in (3.19) is empty (zero), so it is simply $-\frac{1}{2u}\,\mathrm{id}_{\mathbb{1}_\lambda} \in u^{-1}\,\mathrm{End}_{\mathfrak{U}^\imath}(\mathbb{1}_\lambda)$. When $i \neq 0$, $\overset{\imath}{\bar{\imath}}\,\overset{\lambda}{u}\,\lambda$ is a polynomial in $\mathrm{End}_{\mathfrak{U}^\imath}(\mathbb{1}_\lambda)[u]$ which we call the *fake bubble polynomial*; it is zero if $\lambda_i > \varsigma_i$. Then we define the *bubble generating function*:

$$\overset{\imath}{\bar{\imath}}\,(u)\,\lambda \;:=\; \zeta_i(\lambda)^{-1}\left(\overset{\imath}{\bar{\imath}}\,\overset{\lambda}{u}\,\lambda \;+\; \overset{\imath}{\bar{\imath}}\,\overset{\lambda}{u}\,\lambda\right) \in \mathrm{End}_{\mathfrak{U}^\imath}(\mathbb{1}_\lambda)((u^{-1})). \tag{3.20}$$

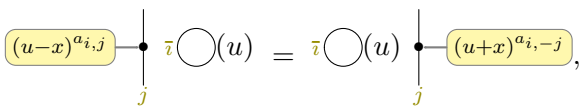
The defining relations are as follows:



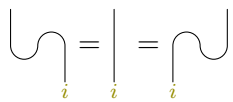
$$\overset{\imath}{\bar{\imath}}\,(u)\,\lambda \;\in\; u^{\varsigma_i-\lambda_i}\,\mathrm{id}_{\mathbb{1}_\lambda} + u^{\varsigma_i-\lambda_i-1}\,\mathrm{End}_{\mathfrak{U}^\imath}(\mathbb{1}_\lambda)[\![u^{-1}]\!], \tag{3.21}$$



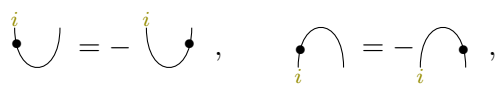
$$\overset{\imath}{i}\,(u)\,\lambda\,\overset{\imath}{\bar{\imath}}\,(-u) \;=\; (-1)^{\varsigma_i+\lambda_i}u^{-a_{i,-i}}\,\mathrm{id}_{\mathbb{1}_\lambda}, \tag{3.22}$$



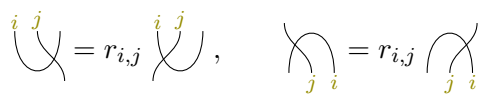
$$(u-x)^{a_{i,j}}\;\overset{j}{|}\;\overset{\imath}{\bar{\imath}}\,(u) \;=\; \overset{\imath}{\bar{\imath}}\,(u)\;\overset{j}{|}\;(u+x)^{a_{i,-j}}, \tag{3.23}$$



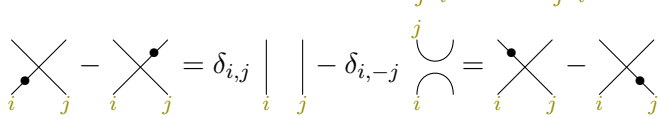
$$\overset{i}{\cup} = \overset{i}{|} = \overset{i}{\cap}, \tag{3.24}$$



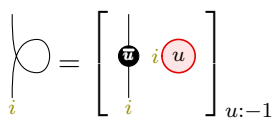
$$\overset{i}{\cap} = -\overset{i}{\cap}, \qquad \overset{i}{\cap} = -\overset{i}{\cap}, \tag{3.25}$$



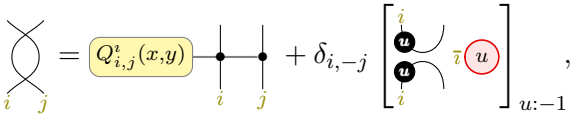
$$\overset{i\;j}{\cup} = r_{i,j}\,\overset{i\;j}{\times}, \qquad \overset{j\;i}{\cap} = r_{i,j}\,\overset{j\;i}{\cap}, \tag{3.26}$$



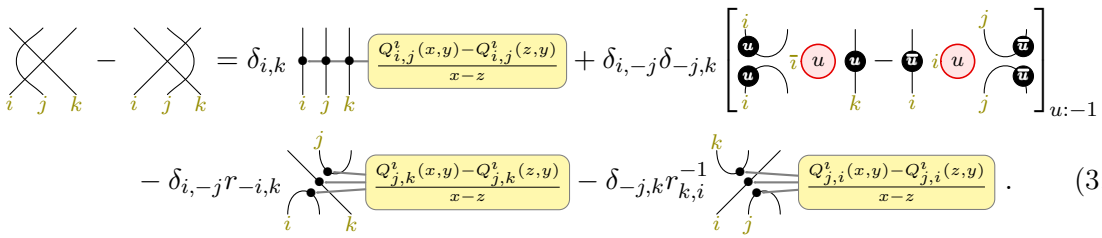
$$\overset{i\;j}{\times} - \overset{i\;j}{\times} = \delta_{i,j}\,\overset{i\;j}{|\;|} - \delta_{i,-j}\,\overset{j}{\cap}\overset{i}{|} = \overset{i\;j}{\times} - \overset{i\;j}{\times}, \tag{3.27}$$



$$\overset{i}{\cap} = \left[\overset{i}{|}\;\overset{\imath}{\bar{\imath}}\,\overset{\lambda}{u}\right]_{u:-1}, \tag{3.28}$$



$$\overset{i\;j}{\times} = Q^\imath_{i,j}(x,y)\;\overset{i\;j}{|\;|} + \delta_{i,-j}\left[\overset{i}{\cap}\;\overset{\imath}{\bar{\imath}}\,\overset{\lambda}{u}\right]_{u:-1}, \tag{3.29}$$

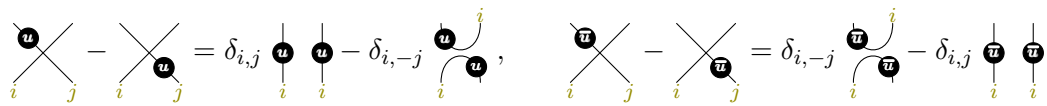


$$\overset{i\;j\;k}{\times} - \overset{i\;j\;k}{\times} = \delta_{i,k}\,\overset{i\;j\;k}{|\;|\;|}\;\frac{Q^\imath_{i,j}(x,y)-Q^\imath_{i,j}(z,y)}{x-z} + \delta_{i,-j}\delta_{-j,k}\left[\overset{i}{\cap}\;\overset{\imath}{\bar{\imath}}\,\overset{\lambda}{u}\;\overset{k}{\cap} - \overset{i}{\cap}\;\overset{\imath}{\bar{\imath}}\,\overset{\lambda}{u}\;\overset{j}{\cap}\right]_{u:-1}$$

$$- \delta_{i,-j}r_{-i,k}\,\overset{j}{\cap}\overset{i\;k}{\times}\;\frac{Q^\imath_{j,k}(x,y)-Q^\imath_{j,k}(z,y)}{x-z} - \delta_{-j,k}r^{-1}_{k,i}\,\overset{k}{\cap}\overset{i\;j}{\times}\;\frac{Q^\imath_{j,i}(x,y)-Q^\imath_{j,i}(z,y)}{x-z}. \tag{3.30}$$

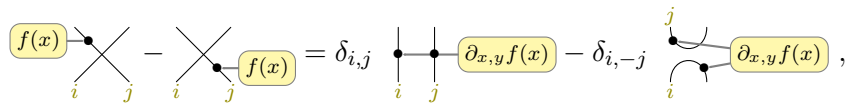
As usual, there are some further relations which are consequences of the defining relations. The following are analogous to (2.8) and (2.9):

$$\overset{i}{\cap} = \overset{i}{\cap}, \qquad \overset{i}{\cap} = \overset{i}{\cap}, \tag{3.31}$$

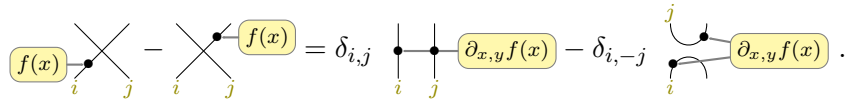


(3.32)

The mirror images of these relations in a horizontal axis also hold. Recall the Demazure operator $\partial_{x,y}$ from (1.2). We refer to (3.27) (and the analogous relation (2.4) in **AB**) as the *Demazure relation* because it implies

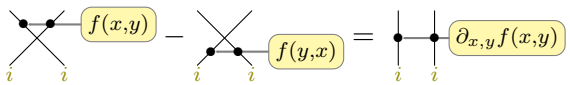


(3.33)



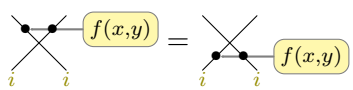
(3.34)

One way to deduce these identities is to use the first relation from (3.32) and its mirror image plus (1.1) and (1.4). Also, if $i \neq 0$ then



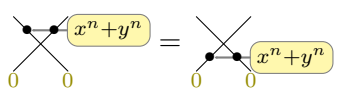
(3.35)

This implies that



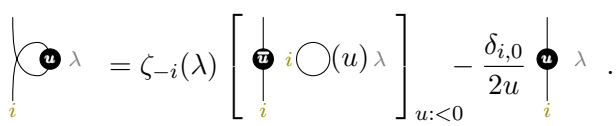
(3.36)

if $i \neq 0$ and $f(x,y) = f(y,x)$. Also a direct calculation shows that



(3.37)

for any odd $n \geq 1$; see [BWW26, Sec. 3.4] for a more general treatment of central elements in the $i = 0$ case. Note also the following relation, which is [BWW25, (3.43)]:



(3.38)

For any $\lambda \in X^\imath$, the bubble $0 \bigcirc \lambda$ is simply equal to 0 if $t(\lambda) = \bar{0}$ or $\mathrm{id}_{\mathbb{1}_\lambda}$ if $t(\lambda) = \bar{1}$. For all non-zero $i \in I$, the relation (3.21) implies that $\bar{\imath} \bigcirc 0 \lambda = \zeta_i(\lambda)\,\mathrm{id}_{\mathbb{1}_\lambda}$ if $\varsigma_i - \lambda_i \geq 0$, and

$$\bar{\imath} \bigcirc r \lambda = \begin{cases} \zeta_i(\lambda)\,\mathrm{id}_{\mathbb{1}_\lambda} & \text{if } r = \lambda_i - \varsigma_i - 1 \geq 0 \\ 0 & \text{if } 0 \leq r < \lambda_i - \varsigma_i - 1. \end{cases} \tag{3.39}$$

The generators $\bar{\imath} \bigcirc r \lambda$ for $0 < r \leq \varsigma_i - \lambda_i$ are called *fake bubbles*. When $\varsigma_i - \lambda_i \geq 1$, it is necessarily the case that $\varsigma_{-i} - \lambda_{-i} \leq 0$, hence, the relation (3.22) implies that any fake bubble $\bar{\imath} \bigcirc r \lambda$ can be expressed as a polynomial in the genuinely dotted bubbles $i \bigcirc r \lambda$. This is similar to the situation for 2-quantum groups. It seems worth pointing out that in [BWW25] more general 2-iquantum groups are defined in which it is possible that $\varsigma_i + \varsigma_{\tau i} \geq 2$, in which case it might not be possible to express all fake bubbles as polynomials in genuine dotted bubbles.

We use the notation

$$Z_\lambda \coloneqq \mathrm{End}_{\mathfrak{U}^\imath}(\mathbb{1}_\lambda). \tag{3.40}$$

It will sometimes be useful to have a more algebraic notation available to denote the renormalized dotted bubbles which generate Z_λ: for any $i \in I$, we define

$$O_{\lambda;i}(u) = \sum_{r\geq 0} z_{\lambda;i}^{(r)} u^{\varsigma_i - \lambda_i - r} := \ \bar{\imath}\bigcirc(u)\ \lambda\ . \tag{3.41}$$

More explicitly, $z_{\lambda;i}^{(0)} = \mathrm{id}_{\mathbb{1}_\lambda}$, and for $r > 0$ we have that

$$z_{\lambda;i}^{(r)} = \begin{cases} \zeta_i(\lambda)^{-1}\ \bar{\imath}\ (r)\ \lambda & \text{if } r \leq \varsigma_i - \lambda_i \\ \zeta_i(\lambda)^{-1}\ \bar{\imath}\ \bigcirc\!\bullet\ r+\lambda_i-\varsigma_i-1\ \ \lambda & \text{if } r > \varsigma_i - \lambda_i. \end{cases} \tag{3.42}$$

Note that $z_{\lambda;i}^{(r)}$ is of degree $2r$. Let

$$R := \Bbbk\big[z_i^{(r)} \,\big|\, i \in I,\ r \geq 1\big] \ / \ J, \tag{3.43}$$

where J is the ideal generated by the relations that are the coefficients of powers of u in the identities $z_{-i}(u)z_i(-u) = 1$ for all $i \in I$, where

$$z_i(u) := \sum_{r\geq 0} z_i^{(r)} u^{-r} \in R[\![u^{-1}]\!], \tag{3.44}$$

and $z_i^{(0)} := 1$ by convention. We view R as a graded algebra so that each $z_i^{(r)}$ is in degree $2r$. Comparing with (3.22), there is a graded algebra homomorphism

$$\varphi_\lambda \colon R \to Z_\lambda, \qquad z_i^{(r)} \mapsto z_{\lambda;i}^{(r)}, \qquad z_i(u) \mapsto u^{\lambda_i - \varsigma_i} O_{\lambda;i}(u). \tag{3.45}$$

We note also that the ring R is freely generated by the elements $z_0^{(2r-1)}$ and $z_i^{(r)}$ for all $r \geq 1$ and $i \in I' \subset I$ chosen so that $I = I' \sqcup \{0\} \sqcup (-I')$.

The fundamental theorem about $\mathfrak{U}^\imath$ proved in [BWW25] is its non-degeneracy. This gives an explicit diagrammatic basis for each of its 2-morphism spaces, as follows. Let $\langle I \rangle$ be the set of words $\boldsymbol{i} = i_1 \cdots i_s$ ($s \geq 0, i_1, \ldots, i_s \in I$) in the alphabet I. For such a word $\boldsymbol{i}$, we use the shorthand $B_{\boldsymbol{i}} := B_{i_1} \cdots B_{i_s}$. Thus, the 1-morphisms in $\mathrm{Hom}_{\mathfrak{U}^\imath}(\lambda, \mu)$ are the monomials $B_{\boldsymbol{i}}\mathbb{1}_\lambda$ for $\boldsymbol{i} \in \langle I \rangle$ such that $\lambda - \alpha_{\boldsymbol{i}} = \mu$, where $\alpha_{\boldsymbol{i}} := \alpha_{i_1} + \cdots + \alpha_{i_s} \in X$. For two words $\boldsymbol{i}, \boldsymbol{j} \in \langle I \rangle$, we denote their concatenation simply by $\boldsymbol{ij}$. We define an equivalence relation $\approx$ on $\langle I \rangle$ by declaring that $\boldsymbol{i} \approx \boldsymbol{j}$ if

- $\alpha_{\boldsymbol{i}}$ and $\alpha_{\boldsymbol{j}}$ have the same image in $X^\imath$;
- the lengths of the words $\boldsymbol{i}$ and $\boldsymbol{j}$ are of the same parity.

The first property implies for any $\lambda \in X^\imath$ that $B_{\boldsymbol{i}}\mathbb{1}_\lambda$ and $B_{\boldsymbol{j}}\mathbb{1}_\lambda$ are objects of the same morphism category of $\mathfrak{U}^\imath$. In that case, the 2-morphism space $\mathrm{Hom}_{\mathfrak{U}^\imath}(B_{\boldsymbol{j}}\mathbb{1}_\lambda, B_{\boldsymbol{i}}\mathbb{1}_\lambda)$ is 0 unless the second property holds. We use the homomorphism $\varphi_\lambda \colon R \to Z_\lambda$ to make the 2-morphism space $\mathrm{Hom}_{\mathfrak{U}^\imath}(B_{\boldsymbol{j}}\mathbb{1}_\lambda, B_{\boldsymbol{i}}\mathbb{1}_\lambda)$ into a graded right R-module so that $z \in R$ acts by horizontal composition on the right with $\varphi_\lambda(z)$.

Assuming that $\boldsymbol{i} \approx \boldsymbol{j}$, an $\boldsymbol{i} \times \boldsymbol{j}$-*shape* is a (black) string diagram D with no dots and no bubbles such that $D\ \lambda \colon B_{\boldsymbol{j}}\mathbb{1}_\lambda \Rightarrow B_{\boldsymbol{i}}\mathbb{1}_\lambda$ is a 2-morphism in $\mathfrak{U}^\imath$ for any $\lambda \in X^\imath$. We say that a shape is *reduced* if there are no zig-zags, bigons or loops, and two shapes are *equivalent* if they define the same combinatorial matching between their boundary points. Let $\mathrm{S}(\boldsymbol{i} \times \boldsymbol{j})$ be a fixed choice of representatives for the equivalence classes of reduced $\boldsymbol{i} \times \boldsymbol{j}$ shapes. Also, for $D \in \mathrm{S}(\boldsymbol{i} \times \boldsymbol{j})$ with l strings (l is the average of the lengths of the words $\boldsymbol{i}$ and $\boldsymbol{j}$), we fix a choice of normally ordered points $p_1, \ldots, p_l$ away from crossings and critical points, one on each string. For $f \in \Bbbk[x_1, \ldots, x_l]$, we let

$$D(f)\ \lambda \colon B_{\boldsymbol{j}}\mathbb{1}_\lambda \Rightarrow B_{\boldsymbol{i}}\mathbb{1}_\lambda \tag{3.46}$$

be the 2-morphism in $\mathfrak{U}^\imath$ obtained from D by pinning f to the points $p_1, \ldots, p_l$.

Theorem 3.6 (Non-degeneracy)**.** *For $\lambda \in X^\imath$ and $\boldsymbol{i}, \boldsymbol{j} \in \langle I \rangle$ such that $\boldsymbol{i} \approx \boldsymbol{j}$, let l be the average of the lengths of $\boldsymbol{i}$ and $\boldsymbol{j}$. The 2-morphism space $\mathrm{Hom}_{\mathfrak{U}^\imath}(B_{\boldsymbol{j}}\mathbb{1}_\lambda, B_{\boldsymbol{i}}\mathbb{1}_\lambda)$ is free as a right R-module with basis*

$$\mathrm{B}_\lambda(\boldsymbol{i} \times \boldsymbol{j}) := \left\{ D(x_1^{e_1} \cdots x_l^{e_l})\ {}_\lambda \,\middle|\, D \in \mathrm{S}(\boldsymbol{i} \times \boldsymbol{j}), e_1, \dots, e_l \geq 0 \right\}. \tag{3.47}$$

In particular, $\varphi_\lambda \colon R \to Z_\lambda$ is an isomorphism for every $\lambda \in X^\imath$.

Proof. This is [BWW25, Th. 5.10], which is applicable in the present situation because the Cartan matrix is symmetric and our parameter choices from (3.11) are of the required geometric type. □

4. Brauer categorifications

For a $\Bbbk$-linear category $\mathbf{R}$, we denote the strict $\Bbbk$-linear monoidal category of $\Bbbk$-linear endofunctors and natural transformations by $\mathbf{End}_\Bbbk(\mathbf{R})$. Then $\mathbf{R}$ is a *strict left* $\mathbf{AB}$*-module category* if and only if there is a strict $\Bbbk$-linear monoidal functor $\mathbf{AB} \to \mathbf{End}_\Bbbk(\mathbf{R})$. We use the special terminology *Abelian Brauer categorification* to mean a strict left $\mathbf{AB}$-module category which is also a locally finite Abelian category, that is, it is Abelian, all objects are of finite length, and all morphism spaces are finite-dimensional. These are the main objects of study in the remainder of the paper.

We say that a $\Bbbk$-linear category $\mathbf{R}$ is *locally finite Karoubian* if it is a Karoubian category, i.e., it is additive and idempotent complete, its object set is small, and its morphism spaces are finite-dimensional vector spaces. We denote the additive Karoubi envelope of a $\Bbbk$-linear category $\mathbf{R}$ by $\mathrm{Kar}(\mathbf{R})$; if $\mathbf{R}$ is small with finite-dimensional morphism spaces then $\mathrm{Kar}(\mathbf{R})$ is a locally finite Karoubian category. By a *Karoubian Brauer categorification*, we mean a strict left $\mathbf{AB}$-module category which is also a locally finite Karoubian category. Abelian Brauer categorifications are Karoubian, but not vice versa.

Of course, we prefer the Abelian case because we like to think in terms of irreducible objects, composition series, and so forth. This can be salvaged to some extent in the Karoubian case by passing from $\mathbf{R}$ to the Abelian category $\mathrm{Mod}_{\mathrm{lfd}}(\mathbf{R})$ of contravariant[1] $\Bbbk$-linear functors $V : \mathbf{R} \to \mathbf{Vec}_{\mathrm{fd}}$. We also refer to objects of $\mathrm{Mod}_{\mathrm{lfd}}(\mathbf{R})$ as *locally finite-dimensional* $\mathbf{R}$*-modules*. Since the morphism spaces of $\mathbf{R}$ are finite-dimensional, the category $\mathrm{Mod}_{\mathrm{lfd}}(\mathbf{R})$ is well behaved—it is what the authors call a *Schurian category*. Basic facts about such categories can be found in the expository article [BD17]. In particular, all finitely generated modules are locally finite-dimensional and have projective covers in $\mathrm{Mod}_{\mathrm{lfd}}(\mathbf{R})$. There also is a natural notion of blocks. By Yoneda, the category $\mathbf{R}$ embeds into $\mathrm{Mod}_{\mathrm{lfd}}(\mathbf{R})$, with objects of $\mathbf{R}$ being identified with the *distinguished* projective $\mathbf{R}$-modules. The indecomposable objects of $\mathbf{R}$ are indecomposable projective $\mathbf{R}$-modules. Some caution is needed, however: a locally finite-dimensional $\mathbf{R}$-module need not be finitely generated, and finitely generated $\mathbf{R}$-modules need not be of finite length.

We assume for the remainder of the section that

$$\boxed{\mathbf{R} \text{ is an Abelian or a Karoubian Brauer categorification.}}$$

The generating object B of $\mathbf{AB}$ acts as an endofunctor $B \colon \mathbf{R} \to \mathbf{R}$, and morphisms in $\mathbf{AB}$ act as natural transformations. Since B is a self-dual object of $\mathbf{AB}$, the endofunctor $B \colon \mathbf{R} \to \mathbf{R}$ is self-adjoint. Hence, in the Abelian case, B is exact.

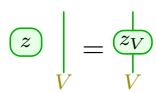
In the Karoubian case, the action of $\mathbf{AB}$ on $\mathbf{R}$ extends in a canonical way to make $\mathrm{Mod}_{\mathrm{lfd}}(\mathbf{R})$ into a strict left $\mathbf{AB}$-module category; e.g., this standard construction is explained in [BV22, Sec. 2.5]. The extended endofunctor B is again exact, and it takes finitely generated $\mathbf{R}$-modules to finitely generated $\mathbf{R}$-modules. To avoid too much duplication as we are discussing the Abelian and Karoubian cases simultaneously subsequently, we use the shorthand notation $\mathbf{R}^{\mathrm{fg}}$ to denote $\mathbf{R}$ in the Abelian case, or the full subcategory of $\mathrm{Mod}_{\mathrm{lfd}}(\mathbf{R})$ consisting of all finitely generated $\mathbf{R}$-modules in the Karoubian case. In the Karoubian case, morphism spaces in $\mathbf{R}^{\mathrm{fg}}$ are finite-dimensional, indeed, the space of $\mathbf{R}$-module homomorphisms from a finitely generated module to a locally finite-dimensional

[1] This word means we are working with *right* modules!

module is finite-dimensional. By an irreducible object $L \in \mathbf{R}^{\mathrm{fg}}$, we mean an object which is irreducible as an $\mathbf{R}$-module in the usual sense, i.e., an irreducible object of the Abelian category $\mathrm{Mod}_{\mathrm{lfd}}(\mathbf{R})$.

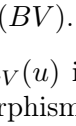
Remark 4.1. In the Karoubian case, we do not assume that the $\Bbbk$-linear category $\mathbf{R}$ is locally Noetherian. Consequently, submodules of finitely generated $\mathbf{R}$-modules need not be finitely generated. This means that the category $\mathbf{R}^{\mathrm{fg}}$ may not itself be Abelian, and we will never assume that this is the case below. It seems reasonable to us to conjecture that $\mathbf{R}$ is locally Noetherian in the important examples $\mathbf{cB}(A, f)$ constructed in Definition 10.1. One small piece of evidence for this: in [SS22, Th. 5.9], Sam and Snowden proved that the Brauer category $\mathbf{B}_\delta$ over the field $\mathbb{C}$ is locally Noetherian for any $\delta \in \mathbb{C}$. Since $\mathrm{Kar}(\mathbf{B}_\delta)$ is equivalent to the level one cyclotomic quotient $\mathbf{cB}\big(u - \frac{1-\delta}{2}\big)$, the latter is locally Noetherian too.

In string diagrams, we add a green string labelled by an object $V \in \mathbf{R}^{\mathrm{fg}}$ on the right-hand boundary of a string diagram of a morphism in $\mathbf{AB}$ to indicate the morphism in $\mathbf{R}^{\mathrm{fg}}$ obtained by evaluating the natural transformation on V. Morphisms in $\mathbf{R}^{\mathrm{fg}}$ can be indicated by coupons on the green string. An element z of the center $Z(\mathbf{R}^{\mathrm{fg}}) = \mathrm{End}(\mathrm{id}_{\mathbf{R}^{\mathrm{fg}}})$ can be represented by a bubble labelled z in the 2-cell immediately to the left of the green string, so that



where z_V is the evaluation of z on $V \in \mathbf{R}^{\mathrm{fg}}$. This is the usual extension of string calculus for (strict) monoidal categories to (strict) module categories.

Remark 4.2. In the previous paragraph, we restricted V to being an object of $\mathbf{R}^{\mathrm{fg}}$. However, in the Karoubian case, the same string diagrams also make sense with the green string labelled by any locally finite-dimensional $\mathbf{R}$-module. Since any $\mathbf{R}$-module is a union of finitely generated $\mathbf{R}$-modules, and the endofunctor B is exact and preserves finite generation, naturality implies that the resulting morphism is determined uniquely by its restriction to all finitely generated submodules.

We also use the notation Z_V to denote the center of the endomorphism algebra $\mathrm{End}_{\mathbf{R}^{\mathrm{fg}}}(V)$. This is a finite-dimensional commutative $\Bbbk$-algebra. Let $m_V(u) \in \Bbbk[u]$ be the (monic) minimal polynomial of  $\in \mathrm{End}_{\mathbf{R}^{\mathrm{fg}}}(BV)$. This definition makes sense because $\mathrm{End}_{\mathbf{R}^{\mathrm{fg}}}(BV)$ is finite-dimensional and $\Bbbk[x]$ is a PID; $m_V(u)$ is the unique monic polynomial which generates the non-zero kernel of the algebra homomorphism $\Bbbk[u] \to \mathrm{End}_{\mathbf{R}^{\mathrm{fg}}}(BV), u \mapsto$. Then we define the *β-degree* of V by

$$\beta(V) := \deg m_V(u). \tag{4.1}$$

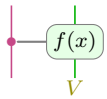
The image of the bubble generating function $(u) \in \mathrm{End}_{\mathbf{AB}}(\mathbb{1})[\![u^{-1}]\!]$ under the monoidal functor $\mathbf{AB} \to \mathbf{End}_{\Bbbk}(\mathbf{R}^{\mathrm{fg}})$ is an element of $Z(\mathbf{R}^{\mathrm{fg}})(\!(u^{-1})\!)$. On any $V \in \mathbf{R}^{\mathrm{fg}}$, this evaluates to a formal power series, which we denote by $O_V(u) \in Z_V[\![u^{-1}]\!]$. This formal power series encodes useful information about the central character of V. To explain what we mean, for a monic polynomial $f(u)$, we use the notation

$$\widetilde{f}(u) := (-1)^{\deg f(u)} f(-u). \tag{4.2}$$

Let

$$\chi_f(u) := \begin{cases} \dfrac{u-h}{u} \cdot \dfrac{\widetilde{f}(u)}{f(u)} & \text{if } \deg f(u) \text{ is even} \\[2ex] \dfrac{u+h}{u} \cdot \dfrac{\widetilde{f}(u)}{f(u)} & \text{if } \deg f(u) \text{ is odd.} \end{cases} \tag{4.3}$$

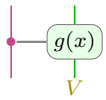
Theorem 4.3. *For any $V \in \mathbf{R}^{\mathrm{fg}}$, there exists a monic polynomial $f(u) \in Z_V[u]$ such that*



$$= 0, \qquad\qquad O_V(u) = \chi_f(u). \tag{4.4}$$

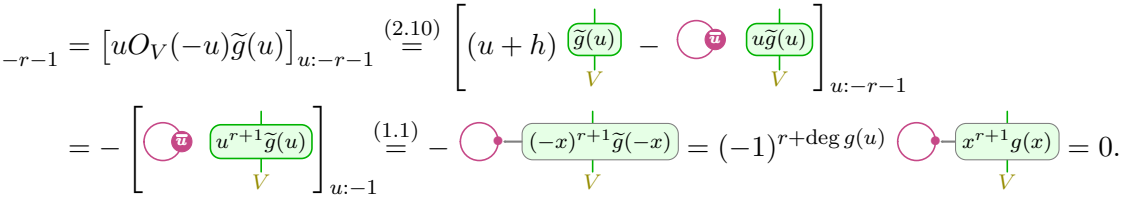
If V is a brick, that is, $\mathrm{End}_{\mathbf{R}^{\mathrm{fg}}}(V) = \Bbbk$, then $f(u) := m_V(u)$ has these properties.

Proof. Suppose that $g(u)$ is a monic polynomial in $Z_V[u]$ such that

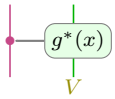


$$= 0.$$

For example, we could take $g(u) := m_V(u) \in \Bbbk[u] \subseteq Z_V[u]$. Let $g^*(u) := uO_V(-u)\widetilde{g}(u) \in Z_V(\!(u^{-1})\!)$. The following shows that $g^*(u)$ is a polynomial: for $r \geq 0$ we have that

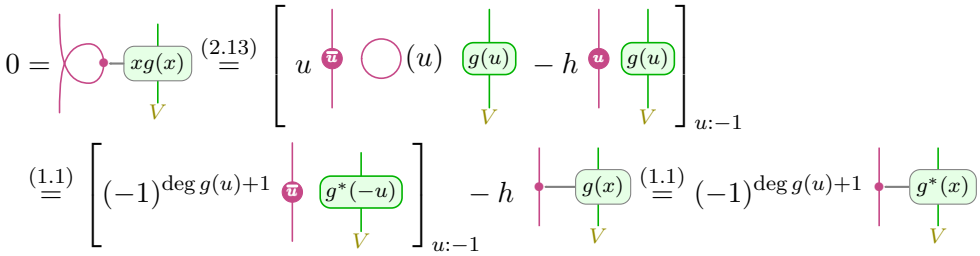


It is clear that $g^*(u) \in Z_V[u]$ is monic, of degree one more than $g(u)$. Also, we have that



$$= 0.$$

This follows from the following calculation:



Now we consider the linear combination

$$f(u) := \frac{g^*(u) + \big(u + (-1)^{\deg g(u)}h\big)g(u)}{2}.$$

This satisfies the first property from (4.4). To see that it satisfies the second property, note that

$$g^{**}(u) = u^2 O_V(u)O_V(-u)g(u) = (u-h)(u+h)g(u)$$

by (2.11). So

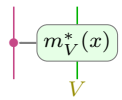
$$f^*(u) = \frac{g^{**}(u) + \big(u - (-1)^{\deg g(u)}h\big)g^*(u)}{2} = \frac{(u-h)(u+h)g(u) + \big(u - (-1)^{\deg g(u)}h\big)g^*(u)}{2}$$
$$= \big(u - (-1)^{\deg g(u)}h\big)\frac{\big(u + (-1)^{\deg g(u)}h\big)g(u) + g^*(u)}{2} = \big(u + (-1)^{\deg f(u)}h\big)f(u).$$

This shows that $uO_V(-u)\widetilde{f}(u) = \big(u + (-1)^{\deg f(u)}h\big)f(u)$. Multiplying both sides by $O_V(u)/f(u)$ and using (2.11), it follows that

$$\frac{(u-h)(u+h)}{u} \cdot \frac{\widetilde{f}(u)}{f(u)} = \big(u + (-1)^{\deg f(u)}h\big)O_V(u),$$

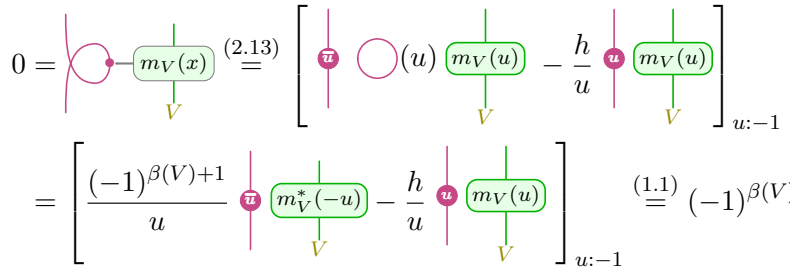
which implies the second property on cancelling a linear factor.

It remains to prove that $m_V(u)$ satisfies (4.4) in the case that V is a brick. Of course, it satisfies the first property by definition. For the second property, it is obvious if $BV = 0$ since then $m_V(u) = 1$ and $O_V(u) = \frac{u-h}{u}$ by (2.10). So we may assume that $m_V(u)$ is of positive degree. The construction in the first paragraph gives us a monic polynomial $m_V^*(u) \in \Bbbk[u]$ of degree one more than $m_V(u)$ such that



$= 0.$

It follows that $m_V(u) \mid m_V^*(u)$, so we have that $m_V^*(u) = m_V(u)(u-c)$ for some $c \in \Bbbk$. Also



where $q(u) \in \Bbbk[u]$ is the polynomial defined by

$$q(u) := \left[\frac{m_V^*(u) - (-1)^{\beta(V)} h m_V(u)}{u}\right]_{u:\geq 0} = m_V(u) - \left[\frac{\big(c + (-1)^{\beta(V)} h\big) m_V(u)}{u}\right]_{u:\geq 0}.$$

We have that $m_V(u) \mid q(u)$. Since $m_V(u)$ is of positive degree, this implies that $c + (-1)^{\beta(V)} h = 0$. Thus, we have shown that $m_V^*(u) = \big(u + (-1)^{\beta(V)} h\big) m_V(u)$. From this, the second property from (4.4) for $f(u) := m_V(u)$ follows in the same way as at the end of the second paragraph of the proof. □

Corollary 4.4. *For any $V \in \mathbf{R}^{\mathrm{fg}}$, the formal Laurent series $O_V(u) \in Z_V(\!(u^{-1})\!)$ is a rational function.*

Corollary 4.5 ([SW26, Th. 4.2])**.** *If $L \in \mathbf{R}^{\mathrm{fg}}$ is an irreducible object then $O_L(u) = \chi_{m_L}(u) \in \Bbbk(u)$, i.e.,*

$$O_L(u) = \begin{cases} \dfrac{u-h}{u} \cdot \dfrac{\widetilde{m}_L(u)}{m_L(u)} & \text{if } \beta(L) \text{ is even} \\[2ex] \dfrac{u+h}{u} \cdot \dfrac{\widetilde{m}_L(u)}{m_L(u)} & \text{if } \beta(L) \text{ is odd.} \end{cases} \tag{4.5}$$

Let Ξ be the set of all rational functions $\chi(u) \in \Bbbk(u)$ such that the expansion of $\chi(u)$ at $u = \infty$ is in $1 + u^{-1}\Bbbk[\![u^{-1}]\!]$ and $\chi(u)\chi(-u) = \frac{(u-h)(u+h)}{u^2}$. In view of (2.11) and Corollary 4.5, the rational functions $O_L(u)$ for irreducible $L \in \mathbf{R}^{\mathrm{fg}}$ are of this form. Recall that $I = \Bbbk$. Let $X = \bigoplus_{i \in I} \mathbb{Z}\varpi_i$ be the weight lattice and $X^\imath = X/\operatorname{im}(\mathrm{id} + \tau)$ be the Abelian group of iweights introduced in the previous section. For a monic polynomial $f(u) = \prod_{i \in I} (u-i)^{\beta_i} \in \Bbbk[u]$, let

$$\mathrm{wt}^\imath(f) := \pi(\mathrm{wt}(f)) \qquad \text{where} \qquad \mathrm{wt}(f) := \begin{cases} \displaystyle\sum_{i \in I} \beta_i \varpi_i & \text{if } \deg f(u) \text{ is even} \\[2ex] \varpi_h + \displaystyle\sum_{i \in I} \beta_i \varpi_i & \text{if } \deg f(u) \text{ is odd.} \end{cases} \tag{4.6}$$

Note that $\mathrm{wt}(f) \in X$ is even, hence, $\mathrm{wt}^\imath(f) \in X^\imath$ is even. For any $\lambda \in X^\imath$, let

$$\chi_\lambda(u) := \prod_{i \in I} (u-i)^{\varsigma_i - \lambda_i} \in \Xi. \tag{4.7}$$

The following lemma also uses the notation $\chi_f(u)$ from (4.3):

Lemma 4.6. *The function $X_{\bar{0}}^{\imath} \to \Xi$, $\lambda \mapsto \chi_\lambda(u)$ is a bijection. Moreover:*
(i) *Every iweight $\lambda \in X_{\bar{0}}^{\imath}$ is of the form $\mathrm{wt}^{\imath}(f)$ for some monic $f(u) \in \Bbbk[u]$.*
(ii) *Every rational function $\chi(u) \in \Xi$ is of the form $\chi_f(u)$ for some monic $f(u) \in \Bbbk[u]$.*
(iii) *For monic $f(u) \in \Bbbk[u]$ and $\lambda \in X_{\bar{0}}^{\imath}$, $\mathrm{wt}^{\imath}(f) = \lambda$ if and only if $\chi_f(u) = \chi_\lambda(u)$.*

Proof. The opening statement follows easily from the other three parts, which are proved as follows:

(i) Let $\lambda \in X^{\imath}$ be an even iweight. It is the image of $\sum_{i \in I,\ \lambda_i > 0} \lambda_i \varpi_i$ if $t(\lambda) = \bar{0}$ or $\varpi_0 + \sum_{i \in I,\ \lambda_i > 0} \lambda_i \varpi_i$ if $t(\lambda) = \bar{1}$. This is $\mathrm{wt}(f)$ for the monic polynomial of even degree

$$f(u) := \begin{cases} \displaystyle\prod_{i \in I,\ \lambda_i > 0} (u-i)^{\lambda_i} & \text{if } t(\lambda) = \bar{0} \\ \displaystyle u \prod_{i \in I,\ \lambda_i > 0} (u-i)^{\lambda_i} & \text{if } t(\lambda) = \bar{1}. \end{cases}$$

(ii) Take $\chi(u) \in \Xi$ and let $\mathrm{ord}(\chi) := \sum_{i \in I} m_i \varpi_i$, where m_i is the order of i as a zero or a pole of $\chi(u)$. The assumption that $\chi(u)\chi(-u) = \frac{(u-h)(u+h)}{u^2}$ implies that $m_i + m_{-i} = 0$ for $i \in I - \{0, \pm h\}$, $m_0 = -1$, and $m_h + m_{-h} = 1$. It follows that

$$\chi(u) = \begin{cases} \displaystyle\frac{u+h}{u} \cdot \prod_{i \in I - \{0\},\ m_i \leq 0} \left(\frac{u+i}{u-i}\right)^{-m_i} & \text{if } m_h \leq 0 \\ \displaystyle\frac{u-h}{u} \cdot \prod_{i \in I - \{0\},\ m_i \leq 0} \left(\frac{u+i}{u-i}\right)^{-m_i} & \text{if } m_{-h} \leq 0. \end{cases}$$

This shows that there is a monic $f(u) \in \Bbbk[u]$ such that $\chi(u) = \frac{u \pm h}{u} \cdot \frac{\widetilde{f}(u)}{f(u)}$. Then we either have that $\chi(u) = \chi_f(u)$ or $\chi(u) = \chi_g(u)$ where $g(u) := uf(u)$.

(iii) Take a monic polynomial $f(u) \in \Bbbk[u]$ and $\lambda \in X_{\bar{0}}^{\imath}$ with pre-image $\hat{\lambda} \in X$. One checks from (3.2) and (3.5) and the definitions (4.3) and (4.7) that

$$\mathrm{ord}(\chi_f) = \varpi_h - \varpi_0 - (\mathrm{id} - \tau)(\mathrm{wt}(f)), \qquad \mathrm{ord}(\chi_\lambda) = \varpi_h - \varpi_0 - (\mathrm{id} - \tau)(\hat{\lambda}). \tag{4.8}$$

Hence, $\chi_f(u) = \chi_\lambda(u) \Leftrightarrow \mathrm{ord}(\chi_f) = \mathrm{ord}(\chi_\lambda) \Leftrightarrow (\mathrm{id} - \tau)(\mathrm{wt}(f)) = (\mathrm{id} - \tau)(\hat{\lambda})$. Since $\mathrm{wt}(f)$ and $\hat{\lambda}$ are both even, Lemma 3.1 implies that this holds if and only if $\mathrm{wt}^{\imath}(f) = \lambda$. □

For any object $V \in \mathbf{R}^{\mathrm{fg}}$ and $i \in I$, let $\beta_i(V)$ be the multiplicity of i as a root of $m_V(u)$, so that

$$m_V(u) = \prod_{i \in I} (u-i)^{\beta_i(V)}. \tag{4.9}$$

We call the numbers $(\beta_i(V))_{i \in I}$ the *β-numbers* of V; their sum is the β-degree defined earlier. We define the *iweight* $\mathrm{wt}^{\imath}(L)$ of an irreducible object $L \in \mathbf{R}^{\mathrm{fg}}$ to be $\mathrm{wt}^{\imath}(m_L)$ as defined above, that is, the image in $X_{\bar{0}}^{\imath}$ of the even weight

$$\mathrm{wt}(L) := \begin{cases} \displaystyle\sum_{i \in I} \beta_i(L) \varpi_i & \text{if } \beta(L) \text{ is even} \\ \displaystyle\varpi_h + \sum_{i \in I} \beta_i(L) \varpi_i & \text{if } \beta(L) \text{ is odd.} \end{cases} \tag{4.10}$$

Corollary 4.5 and Lemma 4.6 imply:

Corollary 4.7. *For irreducibles $K, L \in \mathbf{R}^{\mathrm{fg}}$, $O_K(u) = O_L(u)$ if and only if $\mathrm{wt}^{\imath}(K) = \mathrm{wt}^{\imath}(L)$.*

From Corollary 4.7, it follows that irreducible objects of $\mathbf{R}^{\mathrm{fg}}$ of different iweight have different central character. In the Abelian case, letting $\mathbf{R}_\lambda$ be the Serre subcategory of $\mathbf{R}$ generated by all irreducible objects $L \in \mathbf{R}$ with $\mathrm{wt}^\imath(L) = \lambda$, we obtain the decomposition

$$\mathbf{R} = \bigoplus_{\lambda \in X_{\bar{0}}^\imath} \mathbf{R}_\lambda \tag{4.11}$$

of $\mathbf{R}$ into "blocks". We mean that there are no non-zero homomorphisms between different "blocks". We are calling $\mathbf{R}_\lambda$ a "block" rather than a block because it may decompose further, and it could also be zero.

There is also a "block" decomposition in the Karoubian case, which warrants a bit more discussion. Given $\lambda \in X_{\bar{0}}^\imath$, it still makes sense to consider the Serre subcategory $\mathrm{Mod}_{\mathrm{lfd}}(\mathbf{R})_\lambda$ of $\mathrm{Mod}_{\mathrm{lfd}}(\mathbf{R})$ consisting of the locally finite-dimensional $\mathbf{R}$-modules whose irreducible subquotients L satisfy $\mathrm{wt}^\imath(L) = \lambda$. Then $\mathbf{R}_\lambda$ and $\mathbf{R}_\lambda^{\mathrm{fg}}$ may be defined to be the full subcategories with objects that are all of the objects of $\mathbf{R}$ or $\mathbf{R}^{\mathrm{fg}}$, respectively, that belong to $\mathrm{Mod}_{\mathrm{lfd}}(\mathbf{R})_\lambda$. Equivalently, $\mathbf{R}_\lambda$ is the additive subcategory of $\mathbf{R}$ generated by the indecomposable objects $P \in \mathbf{R}$ which, when viewed as indecomposable projective modules, have unique irreducible quotients L satisfying $\mathrm{wt}^\imath(L) = \lambda$. The Krull-Schmidt theorem implies that

$$\mathbf{R} = \bigoplus_{\lambda \in X_{\bar{0}}^\imath} \mathbf{R}_\lambda. \tag{4.12}$$

Since finitely generated modules have projective covers, there is a similar decomposition of $\mathbf{R}^{\mathrm{fg}}$.

Remark 4.8. There is another more direct way to determine whether an indecomposable object $V \in \mathbf{R}^{\mathrm{fg}}$ belongs to the "block" $\mathbf{R}_\lambda^{\mathrm{fg}}$. Indecomposability implies that Z_V is a finite-dimensional commutative local $\Bbbk$-algebra with unique maximal ideal $\mathfrak{m}$ such that $Z_V/\mathfrak{m} = \Bbbk$. For $f(u) \in Z_V[u]$, we write $\bar{f}(u)$ for the image of $f(u)$ in $\Bbbk[u]$ under the canonical quotient map. Similarly, for a rational function $R(u) \in Z_V(u)$, we write $\bar{R}(u)$ for its image in $\Bbbk(u)$. By Theorem 4.3, $O_V(u) = \chi_f(u)$ for some monic $f(u) \in Z_V[u]$. It follows that $\bar{O}_V(u) = \chi_{\bar{f}}(u)$. Then we have that $V \in \mathbf{R}_\lambda^{\mathrm{fg}}$ where $\lambda \in X_{\bar{0}}^\imath$ is $\mathrm{wt}^\imath(\bar{f})$; equivalently, by Lemma 4.6(iii), $\bar{O}_V(u) = \chi_\lambda(u)$. This agrees with the description in the previous paragraph because $\bar{O}_V(u) = O_L(u)$ for any irreducible quotient L of V. We leave the proof of this statement to the reader.

Now we are going to play the same game as in [BSW20, Sec. 4] (see also [BSW26, Sec. 4]). String diagrams in the remainder of the section will be string diagrams for natural transformations between endofunctors of $\mathbf{R}^{\mathrm{fg}}$. The endofunctor B decomposes as

$$B = \bigoplus_{i \in I} B_i, \tag{4.13}$$

where B_i is the subfunctor of B such that $B_i V$ is the generalized i-eigenspace of the endomorphism $: BV \to BV$ for each $V \in \mathbf{R}^{\mathrm{fg}}$. The identity endomorphism of B_i will be represented using the labelled string . We use the string diagrams

$: B_i \Rightarrow B,$ $: B \Rightarrow B_i$ (4.14)

to denote the inclusions and projections. We then have that

$= \delta_{i,j}$, $= \sum_{i \in I}$. (4.15)

Note that the infinite sum on the right-hand side of (4.15) makes sense as a natural transformation because all but finitely many terms are zero after evaluation on any object $V \in \mathbf{R}^{\mathrm{fg}}$ (in the Karoubian case, this uses that such modules V are finitely generated).

The restriction of the dot gives an endomorphism of the functor B_i such that

(4.16)

We refer to these natural transformations as *projected dots*. In section 2, we introduced a diagrammatic shorthand "pinning" a polynomial $f(x) \in \Bbbk[x]$ to a dot in a Brauer string diagram. This makes sense for projected dots as well. In section 3, we went a bit further, extending the convention to allow polynomials in two variables x, y to be pinned to two dots—double pins—and polynomials in x, y, z to be pinned to three dots—triple pins. This was in the iquantum setting, but double and triple pins make sense in the Brauer setting too.

Composing the cup and cap with projectors gives natural transformations

$: \mathrm{id}_{\mathbf{R}^{\mathrm{fg}}} \Rightarrow B_i \circ B_j,$ $: B_i \circ B_j \Rightarrow \mathrm{id}_{\mathbf{R}^{\mathrm{fg}}}.$ (4.17)

These are zero if $i \neq -j$, as may be proved by a similar argument to the proof of [BSW20, Lem. 4.1] using the dot slide relations (2.2). It follows that

(4.18)

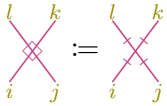
This shows that the *projected cups and caps*

(4.19)

are the unit and counit of an adjunction making (B_{-i}, B_i) into an adjoint pair for each $i \in I$.

When pinned to a string of color i, $x - i$ is *locally nilpotent*, that is, it is nilpotent on evaluation at any object of $\mathbf{R}^{\mathrm{fg}}$ (although the nilpotency degree may not be bounded as the object varies). This means that the pin notation makes sense not just for polynomials $f(x) \in \Bbbk[x]$ pinned to a string of color i but also for any power series $f(x) \in \Bbbk[\![x - i]\!]$. This observation extends in the obvious way to double and triple pins too.

We have already introduced projected dots, caps and cups. There are also *projected crossings*

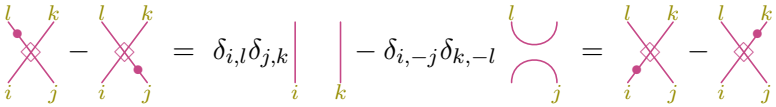


(4.20)

The same argument as used to prove [BSW20, Lem. 4.1] shows that the projected crossing is zero unless at least one of the following holds:

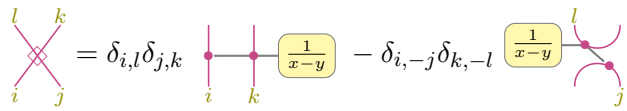
- $i = k$ and $j = l$;
- $i = l$ and $j = k$;
- $i = -j$ and $k = -l$.

The dot slide relation (2.4) implies that



(4.21)

The following is the Brauer analog of [BSW20, Lem. 4.2] and is proved in the same way: if $i \neq k$ and $j \neq l$ (required for the pins in the equation below to make sense) then



(4.22)

We will not use this subsequently, but it explains the first two terms on the right-hand side of the key definition (8.6) appearing later.

Theorem 4.9. *If K is an irreducible subquotient of B_iL for some irreducible $L \in \mathbf{R}^{\mathrm{fg}}$ and $i \in I$ then* $\mathrm{wt}^\imath(K) = \mathrm{wt}^\imath(L) - \alpha_i$.

Proof. Recall the definition of $\mathrm{ord}(\chi)$ from the proof of Lemma 4.6(ii). The bubble slide relation (2.12) implies that

$$O_K(u) = \frac{(u-i)^2}{(u-i+1)(u-i-1)} \cdot \frac{(u+i+1)(u+i-1)}{(u+i)^2} \cdot O_L(u). \tag{4.23}$$

The order of the rational function appearing in front of $O_L(u)$ in this equation is $(\mathrm{id} - \tau)(\alpha_i)$, so $\mathrm{ord}(O_K) = \mathrm{ord}(O_L) + (\mathrm{id} - \tau)(\alpha_i)$. Since $\mathrm{ord}(O_L) = \varpi_h - \varpi_0 - (\mathrm{id} - \tau)(\mathrm{wt}(L))$ by Corollary 4.5 and (4.8), it follows that

$$(\mathrm{id} - \tau)(\mathrm{wt}(K)) = (\mathrm{id} - \tau)(\mathrm{wt}(L)) - (\mathrm{id} - \tau)(\alpha_i) = (\mathrm{id} - \tau)(\mathrm{wt}(L) - \alpha_i).$$

Since $\mathrm{wt}(K)$ and $\mathrm{wt}(L) - \alpha_i$ are even, Lemma 3.1 implies finally that $\mathrm{wt}^\imath(K) = \mathrm{wt}^\imath(L) - \alpha_i$. □

Corollary 4.10. *The eigenfunctor $B_i \colon \mathbf{R}^{\mathrm{fg}} \to \mathbf{R}^{\mathrm{fg}}$ maps objects in $\mathbf{R}^{\mathrm{fg}}_\lambda$ to objects in $\mathbf{R}^{\mathrm{fg}}_{\lambda - \alpha_i}$.*

Proof. The Abelian case follows from exactness and Theorem 4.9. In the Karoubian case, take $V \in \mathbf{R}^{\mathrm{fg}}_\lambda$. Since V is finitely generated, so is B_iV. If B_iV has a non-zero component in $\mathbf{R}^{\mathrm{fg}}_\mu$ for some $\mu \in X^\imath_{\bar{0}}$, that component is finitely generated, hence it has an irreducible quotient K. By adjunction $0 \neq \mathrm{Hom}_{\mathbf{R}^{\mathrm{fg}}}(B_iV, K) \cong \mathrm{Hom}_{\mathbf{R}^{\mathrm{fg}}}(V, B_{-i}K)$. The image of a non-zero morphism $f : V \to B_{-i}K$ has an irreducible quotient L. Since L is a quotient of V, we have $\mathrm{wt}^\imath(L) = \lambda$. Since L is a subquotient of $B_{-i}K$, Theorem 4.9 implies that $\lambda = \mu - \alpha_{-i} = \mu + \alpha_i$. Thus $\mu = \lambda - \alpha_i$, as required. □

Remark 4.11. Corollaries 4.7 and 4.10 are analogs of the results [BSW20] about Heisenberg categorifications, and the proofs are only slightly more difficult. Both results are also proved in [GRS26]. We proved them independently before [GRS26] became available.

By the *spectrum* of the categorification $\mathbf{R}$, we mean the set $I_{\mathbf{R}}$ of roots of the minimal polynomials $m_L(u)$ for all irreducible objects $L \in \mathbf{R}^{\mathrm{fg}}$. In other words,

$$I_{\mathbf{R}} = \{i \in I \mid \beta_i(L) > 0 \text{ for some irreducible } L \in \mathbf{R}^{\mathrm{fg}}\}.$$

This is an invariant of $\mathbf{R}$ which is interesting to determine in examples. Recall from the previous section that we denote the $\sim$-equivalence class of $k \in I$ by $[k]$; it is the vertex set of one of the Satake diagrams in Table 1.

Theorem 4.12. *The spectrum $I_{\mathbf{R}}$ is the union of sets chosen from the $\sim$-equivalence classes $[k]$ for $k \in I$, the exceptional set $\{0\}$, and the exceptional set $[h] - \{\pm h\}$.*

Proof. The theorem follows from the following claims.

<u>Claim 1</u>: $i \in I_{\mathbf{R}} \Rightarrow -i \in I_{\mathbf{R}}$.

Proof: There exists an irreducible $L \in \mathbf{R}^{\mathrm{fg}}$ such that $\beta_i(L) > 0$, hence, $B_iL \neq 0$. Since B_iL is finitely generated, it has an irreducible quotient K. Then

$$0 \neq \mathrm{Hom}_{\mathbf{R}^{\mathrm{fg}}}(B_iL, K) \cong \mathrm{Hom}_{\mathbf{R}^{\mathrm{fg}}}(L, B_{-i}K),$$

proving that $-i \in I_{\mathbf{R}}$.

Claim 2: $i \in I_{\mathbf{R}}, i \notin \{0, -1, -h, -h-1\} \Rightarrow i+1 \in I_{\mathbf{R}}$.

Proof: Take an irreducible $L \in \mathbf{R}^{\mathrm{fg}}$ with $\beta_i(L) > 0$. This implies that $B_i L$ is non-zero. Let K be an irreducible quotient of $B_i L$. The equality (4.23) plus the explicit formula from Corollary 4.5 implies that

$$\big(u - (-1)^{\beta(K)}h\big)(u-i+1)(u+i)^2(u-i-1)\widetilde{m}_K(u)m_L(u) = \\ \big(u - (-1)^{\beta(L)}h\big)(u+i+1)(u-i)^2(u+i-1)\widetilde{m}_L(u)m_K(u).$$

Since $u-i-1$ divides the left-hand side, and it does not equal $u \pm h$, $u+1+i$, $u-i$ or $u-1+i$ by the assumption on i, it must divide $\widetilde{m}_L(u)m_K(u)$. If $u-i-1$ divides $\widetilde{m}_L(u)$ then $u+i+1$ divides $m_L(u)$, i.e., $\beta_{-i-1}(L) > 0$. Hence, $-i-1 \in I_{\mathbf{R}}$ and $i+1 \in I_{\mathbf{R}}$ by Claim 1. If $u-i-1$ divides $m_K(u)$ then $\beta_{i+1}(K) > 0$, so again $i+1 \in I_{\mathbf{R}}$.

Claim 3: $1 \in I_{\mathbf{R}} \Rightarrow 0 \in I_{\mathbf{R}}$.

Proof: Let L be an irreducible in $\mathbf{R}^{\mathrm{fg}}$ with $B_1 L \neq 0$ and K be an irreducible quotient of $B_1 L$. Theorem 4.9 implies that $\mathrm{wt}^\imath(K) = \mathrm{wt}^\imath(L) - \alpha_1$. Recalling (3.3), it follows that one of the weights $\mathrm{wt}^\imath(K)$ or $\mathrm{wt}^\imath(L)$ is of odd type, so either $\beta_0(K)$ or $\beta_0(L)$ is odd. This shows that $0 \in I_{\mathbf{R}}$. □

Example 4.13. As we mentioned already in the introduction, the main source of interesting examples of Abelian Brauer categorifications comes from representations of the algebraic groups $\mathrm{Sp}_{2n}(\Bbbk)$ $(n \geq 1), \mathrm{SO}_n(\Bbbk)$ $(n \geq 2)$ or $\mathrm{Spin}_n(\Bbbk)$ $(n \geq 2)$, or their Lie algebras. In the Lie algebra case over the field $\mathbb{C}$, this is developed in detail in [RS19]: the functor B is defined by tensoring with the natural representation V, and the dot acts as the natural transformation $V \otimes - \Rightarrow V \otimes -$ defined by $\varepsilon\left(\Omega + \frac{\dim V - \varepsilon}{2}\right)$ where Ω is the Casimir tensor and $\varepsilon = -1$ in the symplectic case or 1 in the orthogonal or spin cases. The spectra for the categories of rational representations of these groups over $\mathbb{C}$ are as follows (by "spin representation" we mean a rational representation of the spin group on which the non-trivial element of the kernel of the homomorphism to the special orthogonal group acts as -1):

Category	Spectrum
Rational representations of $\mathrm{Sp}_{2n}(\mathbb{C})$	$[h] - \{\pm h\}$
Rational representations of $\mathrm{SO}_{2n}(\mathbb{C})$	$[h]$
Rational representations of $\mathrm{SO}_{2n+1}(\mathbb{C})$	$[0]$
Spin representations of $\mathrm{Spin}_{2n}(\mathbb{C})$	$[0]$
Spin representations of $\mathrm{Spin}_{2n+1}(\mathbb{C})$	$\{0\} \cup [h] - \{\pm h\}$

This follows from the results of [RS19] using the classical lemma recalled in [RS19, Lem. 4.8].

Example 4.14. A trivial example of a Brauer categorification: the category $\mathbf{SVec}_{\mathrm{fd}}$ of finite-dimensional vector superspaces with morphisms that are even linear maps. The endofunctor B is the parity-switching functor, so $B^2 = \mathrm{id}_{\mathbf{SVec}_{\mathrm{fd}}}$. The cup, the cap and the crossing act as identity natural transformations, and $\dot{|} : B \Rightarrow B$ is zero. Up to isomorphism, there are only two irreducible objects $L_0 := \Bbbk^{1|0}$ and $L_1 := \Bbbk^{0|1}$, with minimal polynomials $m_{L_0}(u) = m_{L_1}(u) = u$. It follows that $\mathrm{wt}^\imath(L_0) = \mathrm{wt}^\imath(L_1) = \varpi_0 + \varpi_h$, and its spectrum is $\{0\}$.

Example 4.15. Another example in positive characteristic comes from the category $\mathrm{Ver}_p(\mathrm{SL}_2)$, the semisimplification of the category of tilting modules for $\mathrm{Sp}_2(\Bbbk) = \mathrm{SL}_2(\Bbbk)$ in characteristic $p \geq 5$. The endofunctor B is induced by $V \otimes -$ for the natural module V, and the dot acts as in Example 4.13. The irreducible objects are $L(0), \dots, L(p-2)$, and $V \otimes L(k) \cong L(k-1) \oplus L(k+1)$ with $L(-1) := 0$ and $L(p-1)$ being negligible, hence 0. The spectrum is $[h] - \{\pm h\}$.

5. Iquantum categorifications

Now we are going to look in the other direction for a while. Let $\mathfrak{U}^\imath$ be the 2-iquantum group defined in section 3. We will usually ignore the grading, viewing it simply as a $\Bbbk$-linear 2-category.

A *2-representation* of $\mathfrak{U}^\imath$ is a strict $\Bbbk$-linear 2-functor from $\mathfrak{U}^\imath$ to the 2-category of $\Bbbk$-linear categories. Thus:

- For each $\lambda \in X^\imath$, there is given a $\Bbbk$-linear category $\mathbf{R}_\lambda$.
- For each $i \in I$ and $\lambda \in X^\imath$, there is given a $\Bbbk$-linear functor $B_i \mathbb{1}_\lambda = \mathbb{1}_{\lambda-\alpha_i} B_i \colon \mathbf{R}_\lambda \to \mathbf{R}_{\lambda-\alpha_i}$.
- For each of the generating 2-morphisms of $\mathfrak{U}^\imath$ listed in Table 3 there is given a natural transformation between these functors satisfying the defining relations from Definition 3.5.

We simply say that $(\mathbf{R}_\lambda)_{\lambda\in X^\imath}$ is a 2-representation of $\mathfrak{U}^\imath$ when all of this data is in our hands.

We say that $(\mathbf{R}_\lambda)_{\lambda\in X^\imath}$ is a *locally finite Abelian 2-representation* or a *locally finite Karoubian 2-representation* of $\mathfrak{U}^\imath$ if in addition each of the categories $\mathbf{R}_\lambda$ is a locally finite Abelian or Karoubian category, respectively. Then it is convenient to set

$$\mathbf{R} \coloneqq \bigoplus_{\lambda\in X^\imath} \mathbf{R}_\lambda \tag{5.1}$$

and extend the functors arising from the action of $\mathfrak{U}^\imath$ so that the 1-morphism $\mathbb{1}_\lambda$ acts as the identity on $\mathbf{R}_\lambda$ and as the zero functor on other $\mathbf{R}_\mu$'s, and $B_i \coloneqq \bigoplus_{\lambda\in X^\imath} B_i \mathbb{1}_\lambda \colon \mathbf{R} \to \mathbf{R}$.

Since $p(\lambda - \alpha_i) = p(\lambda)$, the subcategories

$$\mathbf{R}_{\bar{0}} \coloneqq \bigoplus_{\substack{\lambda\in X^\imath \\ p(\lambda)=\bar{0}}} \mathbf{R}_\lambda, \qquad \mathbf{R}_{\bar{1}} \coloneqq \bigoplus_{\substack{\lambda\in X^\imath \\ p(\lambda)=\bar{1}}} \mathbf{R}_\lambda \tag{5.2}$$

are stable under the action of the endofunctor B_i. We say that $\mathbf{R}$ is *even* if $\mathbf{R}_{\bar{1}}$ is trivial and *odd* if $\mathbf{R}_{\bar{0}}$ is trivial.

Remark 5.1. There is a strict $\Bbbk$-linear 2-isomorphism $S^{-1} \circ \Omega^\imath \colon \mathfrak{U}^\imath \xrightarrow{\sim} \mathfrak{U}^\imath$ defined by composing the inverse of the symmetry S from [BWW25, Lem. 3.7] with the symmetry $\Omega^\imath$ from [BWW25, Sec. 3.3]. This composition is a strict $\Bbbk$-linear 2-functor taking the object $\lambda \in X^\imath$ to $\varpi_h - \lambda$ and the generating 1-morphism $B_i \mathbb{1}_\lambda$ to $B_{-i} \mathbb{1}_{\varpi_h-\lambda}$. We omit the precise description of $S^{-1} \circ \Omega^\imath$ on 2-morphisms since it is not needed here. The pull-back of an even (resp., odd) iquantum categorification along this isomorphism is odd (resp., even). Thus, nothing is lost by restricting attention just to the study of even iquantum categorifications. We will do this later on, but there is no need to do it yet.

Definition 5.2. We say that $\mathbf{R}$ is an *Abelian iquantum categorification* or a *Karoubian iquantum categorification* if $\mathbf{R}$ is as in (5.1) for a locally finite Abelian or locally finite Karoubian 2-representation $(\mathbf{R}_\lambda)_{\lambda\in X^\imath}$, respectively, such that in addition[2] $\big|\{i \in I \mid B_i V \neq 0\}\big| < \infty$ for all $\lambda \in X^\imath$ and $V \in \mathbf{R}_\lambda$.

Assume for the remainder of the section that

$$\boxed{\mathbf{R} \text{ is an Abelian or Karoubian iquantum categorification.}}$$

As the 1-morphism $B_i \mathbb{1}_\lambda$ is left and right dual to $\mathbb{1}_\lambda B_{-i}$, the endofunctor B_i is left and right adjoint to B_{-i}. In particular, in the Abelian case, all of the endofunctors B_i are exact. In the Karoubian case, we use the Yoneda embedding to identify each $\mathbf{R}_\lambda$ with a full subcategory of the Schurian category $\mathrm{Mod}_{\mathrm{lfd}}(\mathbf{R}_\lambda)$, hence, $\mathbf{R}$ is identified with a full subcategory of the Schurian category $\mathrm{Mod}_{\mathrm{lfd}}(\mathbf{R})$. The categorical action of $\mathfrak{U}^\imath$ extends in a canonical way to make $(\mathrm{Mod}_{\mathrm{lfd}}(\mathbf{R}_\lambda))_{\lambda\in X^\imath}$ into a 2-representation in its own right. Each $B_i \colon \mathrm{Mod}_{\mathrm{lfd}}(\mathbf{R}_\lambda) \to \mathrm{Mod}_{\mathrm{lfd}}(\mathbf{R}_{\lambda-\alpha_i})$ is exact, and it takes finitely generated modules to finitely generated.

As in the previous section, to unify the exposition of the Abelian and Karoubian cases as much as possible from now on, we let $\mathbf{R}^{\mathrm{fg}} = \bigoplus_{\lambda\in X^\imath} \mathbf{R}^{\mathrm{fg}}_\lambda$ be defined so that $\mathbf{R}^{\mathrm{fg}} = \mathbf{R}$ and $\mathbf{R}^{\mathrm{fg}}_\lambda = \mathbf{R}_\lambda$ in the Abelian case, while in the Karoubian case we let $\mathbf{R}^{\mathrm{fg}}$ and $\mathbf{R}^{\mathrm{fg}}_\lambda$ be the full subcategories of $\mathrm{Mod}_{\mathrm{lfd}}(\mathbf{R})$ and $\mathrm{Mod}_{\mathrm{lfd}}(\mathbf{R}_\lambda)$ consisting of all finitely generated modules. The functor B_i maps objects of $\mathbf{R}^{\mathrm{fg}}_\lambda$ to objects of $\mathbf{R}^{\mathrm{fg}}_{\lambda-\alpha_i}$. Of course, we will also use string calculus to denote morphisms in $\mathbf{R}^{\mathrm{fg}}$, with

[2]This additional hypothesis was not present in [BSW20]. However, the correct formulation of [BSW20, Th. 5.22] requires it. We are grateful to Gao, Rui and Song whose paper [GRS26] drew our attention to this oversight.

green strings labelled by objects of $\mathbf{R}^{\mathrm{fg}}$ on the right-hand boundary as in section 4, and a label λ in a 2-cell indicating the subcategory $\mathbf{R}^{\mathrm{fg}}_\lambda$.

The next fundamental theorem is the iquantum counterpart of Corollary 4.5. We prove it just for the case in hand, but there is a generalization to all 2-iquantum groups in the setup of [BWW25]. In diagonal type, it recovers the analogous result for 2-quantum groups, which was established already in [BSW20, Sec. 5.1]. The bubble generating function $\bar{\imath}\bigcirc(u)\lambda \in \mathrm{End}_{\mathfrak{U}^\imath}(\mathbb{1}_\lambda)(\!(u^{-1})\!)$ acts as a formal Laurent series in $Z(\mathbf{R}^{\mathrm{fg}}_\lambda)(\!(u^{-1})\!)$. In turn, the evaluation of this on any object $V \in \mathbf{R}^{\mathrm{fg}}_\lambda$ is a formal Laurent series denoted

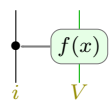
$$O_{V;i}(u) \in u^{\varsigma_i-\lambda_i}\,\mathrm{id}_V + u^{\varsigma_i-\lambda_i-1} Z_V[\![u^{-1}]\!], \tag{5.3}$$

where Z_V denotes the center of $\mathrm{End}_{\mathbf{R}^{\mathrm{fg}}_\lambda}(V)$ as before. Also let $m_{V;i}(u) \in \Bbbk[u]$ be the (monic) minimal polynomial of $\underset{i}{\bullet}\!\!\underset{V}{|}$. We define the *$\beta$-numbers* $(\beta_i(V))_{i\in I}$ by $\beta_i(V) := \deg m_{V;i}(u)$. The additional hypothesis in Definition 5.2 ensures that $\beta_i(V) = 0$ for all but finitely many $i \in I$, hence, it also makes sense to define

$$\beta(V) := \sum_{i\in I} \beta_i(V). \tag{5.4}$$

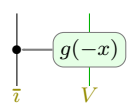
We call this the *β-degree* of V. In the following lemma, we use an obvious variation of the pin notation in which the pin label is a polynomial in $Z_V[x]$, not merely $\Bbbk[x]$. We indicate this by drawing the pinhead as a coupon on the green string.

Lemma 5.3. *Suppose that $V \in \mathbf{R}^{\mathrm{fg}}_\lambda$ for some $\lambda \in X^\imath$, and $0 \neq i \in I$. Suppose that $f(u) \in Z_V[u]$ is a monic polynomial such that*



$= 0.$

Then $\deg f(u) \geq \lambda_i - \varsigma_i$, *and* $g(u) := f(u)O_{V;i}(u)$ *is a monic polynomial in* $Z_V[u]$ *of degree* $\deg f(u) + \varsigma_i - \lambda_i$ *such that*

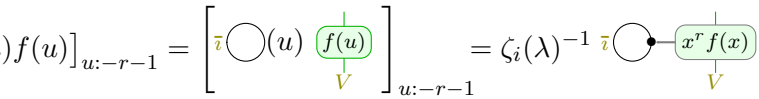
$$O_{V;i}(u) = \frac{g(u)}{f(u)} \qquad \textit{and} \qquad$$


$= 0.$

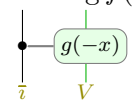
Proof. Let $g(u) := f(u)O_{V;i}(u)$, so that $O_{V;i}(u) = \frac{g(u)}{f(u)}$. By (3.21), we have that

$$g(u) \in u^{\deg f(u)+\varsigma_i-\lambda_i}\,\mathrm{id}_V + u^{\deg f(u)+\varsigma_i-\lambda_i-1} Z_V[\![u^{-1}]\!].$$

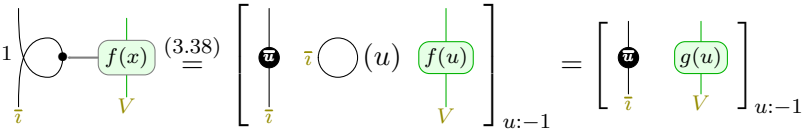
Using a version of (1.1), we have for $r \geq 0$ that

$$\big[g(u)\big]_{u:-r-1} = \big[O_{V;i}(u)f(u)\big]_{u:-r-1} =$$


$= 0.$

Since the leading term of $g(u)$ is $u^{\deg f(u)+\varsigma_i-\lambda_i}\,\mathrm{id}_V$, it follows that $\deg f(u) \geq \lambda_i - \varsigma_i$, and $g(u)$ is a monic polynomial in u of this degree. It remains to show that


$= 0$. This follows because

$$0 = \zeta_i(\lambda)^{-1}$$


.

□

Recall that we write $\widetilde{f}(u)$ for $(-1)^{\deg f(u)} f(-u)$.

Theorem 5.4. *For $\lambda \in X^\imath$, and an irreducible $L \in \mathbf{R}^{\mathrm{fg}}_\lambda$, the following hold:*

(i) $O_{L;0}(u) = u^{-1}\dfrac{\widetilde{m}_{L;0}(u)}{m_{L;0}(u)}$ *and* $t(\lambda) \equiv \beta_0(L) \pmod 2$.

(ii) *If* $i \in I - \{0, \pm h\}$ *then* $O_{L;i}(u) = \dfrac{\widetilde{m}_{L;-i}(u)}{m_{L;i}(u)}$ *and* $\lambda_i = \beta_i(L) - \beta_{-i}(L)$.

(iii) Either

(a) $O_{L;h}(u) = \dfrac{\widetilde{m}_{L;-h}(u)}{m_{L;h}(u)}$, $O_{L;-h}(u) = u\dfrac{\widetilde{m}_{L;h}(u)}{m_{L;-h}(u)}$, *and* $\lambda_h = \beta_h(L) - \beta_{-h}(L) + 1$*; or*

(b) $O_{L;h}(u) = u\dfrac{\widetilde{m}_{L;-h}(u)}{m_{L;h}(u)}$, $O_{L;-h}(u) = \dfrac{\widetilde{m}_{L;h}(u)}{m_{L;-h}(u)}$ *and* $\lambda_h = \beta_h(L) - \beta_{-h}(L)$.

Proof. Part (i) is a restatement of [BWW26, Cor. 3.11]. To prove (ii) and (iii), take $i \in I - \{0\}$. Applying Lemma 5.3 once with $f(u) = m_{L;i}(u)$ and again with $f(u) = m_{L;-i}(u)$, we get monic polynomials $g(u)$ and $q(u)$ such that

- $\deg g(u) = \beta_i(L) + \varsigma_i - \lambda_i$, $O_{L;i}(u) = \frac{g(u)}{m_{L;i}(u)}$ and $m_{L;-i}(-u) \mid g(u)$;
- $\deg q(u) = \beta_{-i}(L) + \varsigma_{-i} + \lambda_i$, $O_{L;-i}(u) = \frac{q(u)}{m_{L;-i}(u)}$ and $m_{L;i}(-u) \mid q(u)$.

If $i \neq \pm h$ then $\varsigma_i = \varsigma_{-i} = 0$, and $O_{L;i}(-u)O_{L;-i}(u) = (-1)^{\lambda_i}$ by (3.22). It follows that $\frac{g(-u)q(u)}{m_{L;i}(-u)m_{L;-i}(u)} = (-1)^{\lambda_i}$, so $\deg g(u) + \deg q(u) = \deg m_{L;-i}(u) + \deg m_{L;i}(u)$. Since $\widetilde{m}_{L;-i}(u) \mid g(u)$ and $\widetilde{m}_{L;i}(u) \mid q(u)$ and all are monic, it must be that $g(u) = \widetilde{m}_{L;-i}(u)$ and $q(u) = \widetilde{m}_{L;i}(u)$. So we have that $O_{L;i}(u) = \frac{\widetilde{m}_{L;-i}(u)}{m_{L;i}(u)}$, and comparing degrees gives that $\beta_i(L) - \beta_{-i}(L) = \lambda_i$.

Instead, if $i = h$ then $\varsigma_h = 1, \varsigma_{-h} = 0$, and (3.22) implies that $O_{L;h}(-u)O_{L;-h}(u) = (-1)^{\lambda_h+1}u$. So $(-1)^{\lambda_h+1}u = \frac{g(-u)q(u)}{m_{L;h}(-u)m_{L;-h}(u)}$. Now $\deg g(u) + \deg q(u) = 1 + \deg m_{L;-h}(u) + \deg m_{L;h}(u)$. So *either* (a) $g(u) = \widetilde{m}_{L;-h}(u)$, *or* (b) $q(u) = \widetilde{m}_{L;h}(u)$. In the former case, $q(u) = u\widetilde{m}_{L;h}(u)$, $O_{L;h}(u) = \frac{\widetilde{m}_{L;-h}(u)}{m_{L;h}(u)}$, $O_{L;-h}(u) = \frac{q(u)}{m_{L;-h}(u)} = u\frac{\widetilde{m}_{L;h}(u)}{m_{L;-h}(u)}$, and comparing degrees gives that $\beta_h(L) - \beta_{-h}(L) + 1 = \lambda_h$. In the latter case, $g(u) = u\widetilde{m}_{L;-h}(u)$, $O_{L;h}(u) = u\frac{\widetilde{m}_{L;-h}(u)}{m_{L;h}(u)}$, $O_{L;-h}(u) = \frac{\widetilde{m}_{L;h}(u)}{m_{L;-h}(u)}$, and comparing degrees gives that $\beta_h(L) - \beta_{-h}(L) = \lambda_h$. Thus, (iii) is proved. □

Corollary 5.5. *For an irreducible* $L \in \mathbf{R}_\lambda^{\mathrm{fg}}$*, the iweight* λ *is equal to the canonical image in* $X^\imath$ *of*

$$\hat{\lambda} := \begin{cases} \sum_{i \in I} \beta_i(L)\varpi_i & \text{if } \lambda_h \equiv \beta_h(L) - \beta_{-h}(L) \pmod 2 \\ \varpi_h + \sum_{i \in I} \beta_i(L)\varpi_i & \text{if } \lambda_h \not\equiv \beta_h(L) - \beta_{-h}(L) \pmod 2. \end{cases}$$

Proof. This follows from the properties relating λ to the β-numbers established in Theorem 5.4. □

Definition 5.6. We say that the Abelian or Karoubian iquantum categorification $\mathbf{R}$ is *nilpotent* if the endomorphism $: B_iV \to B_iV$ is nilpotent for all $i \in I$, $\lambda \in X^\imath$ and all $V \in \mathbf{R}_\lambda$. In the Karoubian case, it is then automatic that is nilpotent also for $V \in \mathbf{R}_\lambda^{\mathrm{fg}}$.

Nilpotency implies the following for $i \in I$, $\lambda \in X^\imath$ and an irreducible object $L \in \mathbf{R}_\lambda^{\mathrm{fg}}$:

$$m_{L;i}(u) = u^{\beta_i(L)}, \qquad O_{L;i}(u) = u^{\varsigma_i - \lambda_i}\,\mathrm{id}_L\,. \tag{5.5}$$

Lemma 5.7. *If* $\mathbf{R}$ *is nilpotent then, for every* $\lambda \in X^\imath$*, the evaluation of any dotted bubble of positive degree on any object* $V \in \mathbf{R}_\lambda^{\mathrm{fg}}$ *is a nilpotent element of* Z_V*. In the Abelian case, its nilpotency degree is bounded above by the composition length of* V*.*

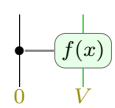
Proof. Let z be a dotted bubble of positive degree. Since $O_{L;i}(u) = u^{\varsigma_i - \lambda_i}\,\mathrm{id}_L$, we have that $z_L = 0$ for every irreducible $L \in \mathbf{R}^{\mathrm{fg}}_\lambda$. In the Abelian case, naturality now implies that z_V is nilpotent with nilpotency degree bounded by the length of V. In the Karoubian case, we may assume that V is indecomposable. Then $A := \mathrm{End}_{\mathbf{R}^{\mathrm{fg}}_\lambda}(V)$ is a finite-dimensional local $\Bbbk$-algebra. Choose an irreducible quotient $\pi : V \twoheadrightarrow L$. Naturality gives $\pi \circ z_V = z_L \circ \pi = 0$, so z_V is not invertible. Hence, z_V is in the Jacobson radical of A, which is nilpotent. □

Nilpotency of dots implies that it makes sense to pin a power series $f(x) \in \Bbbk[\![x]\!]$, not merely a polynomial, to a dot on a string. More generally, one can pin any $f(x) \in Z(\mathbf{R}^{\mathrm{fg}}_\lambda)[\![x]\!]$ to any dot provided that the pinhead is placed in a 2-cell labelled by λ. In the Karoubian case, all such diagrams make sense on evaluation on any finitely generated $\mathbf{R}$-module. The diagram could also be interpreted as a natural transformation when evaluated on an arbitrary locally finite-dimensional $\mathbf{R}$-module since any such module is a direct limit of finitely generated objects. However, we will never need to use this observation.

We end the section with a lemma which gives a replacement for Lemma 5.3 in the case $i = 0$.

Lemma 5.8. *For $\lambda \in X^\imath$, and $V \in \mathbf{R}^{\mathrm{fg}}_\lambda$, there exists a monic polynomial $f(u) \in Z_V[u]$ with* $\deg f(u) \equiv t(\lambda) \pmod 2$ *such that*

$$O_{V;0}(u) = u^{-1}\frac{\widetilde{f}(u)}{f(u)} \qquad \text{and} \qquad$$



$$= 0.$$

If $\mathbf{R}$ is nilpotent then $f(u)$ can be chosen so that all of its coefficients apart from the coefficient of the top degree term are nilpotent elements of Z_V.

Proof. We apply [BWW26, Th. 3.10] taking the polynomial $f(u)$ there to be the minimal polynomial $m_{V;0}(u) \in \Bbbk[u]$. The theorem produces another polynomial $g(u)$ which satisfies exactly the requirements for $f(u)$ in the present lemma. For the final assertion, in the nilpotent case, the minimal polynomial $m_{V;0}(u)$ is a monomial, and the $g(u)$ constructed from $f(u) := m_{V;0}(u)$ in [BWW26, Th. 3.10] is monic with lower degree coefficients that are linear combinations of dotted bubbles of positive degree. So, by Lemma 5.7, all of the lower degree coefficients of $g(u)$, which is the $f(u)$ in the present lemma, are nilpotent. □

6. Completions

Next, we introduce a completion $\widehat{\mathfrak{U}}^\imath$ of $\mathfrak{U}^\imath$ whose topological 2-representations capture the finiteness properties of nilpotent iquantum categorifications. Recall the graded ring $R = R_0 \oplus R_1 \oplus \cdots$ from (3.43). Let $R_{\geq d} := \bigoplus_{d' \geq d} R_{d'}$. Also for $J \subseteq I$, let U_J be the ideal of R generated by $\{z_j^{(n)} \mid j \in J, n \geq 1\}$. We put a linear topology on R, which we call the *GC topology* ("graded and cofinite"), by declaring that the ideals

$$U_{d,J} := R_{\geq d} + U_J \tag{6.1}$$

for all $d \in \mathbb{Z}$ and all cofinite subsets $J \subseteq I$ are a system of neighborhoods of 0. Let $\widehat{R} = \varprojlim_{d,J} R/U_{d,J}$ be the completion of R with respect to this topology. The GC topology on R is obviously separated, so the canonical algebra homomorphism $R \to \widehat{R}$ is injective. Also

$$\widehat{R}/\overline{U}_{d,J} \cong R/U_{d,J}, \tag{6.2}$$

where $\overline{U}_{d,J}$ is the closure of $U_{d,J}$ in $\widehat{R}$. Similarly, $\overline{U}_J$ denotes the closure of U_J.

Lemma 6.1. *The neighborhoods $U_{d,J}$ for $d \geq 0$ and cofinite $J \subseteq I$ with $J = -J$ already form a system of neighborhoods of 0 in R.*

Proof. Since $z_{-j}(u) = z_j(-u)^{-1}$, the ideal $U_J \trianglelefteq R$ contains $z_{-j}^{(n)}$ for all $j \in J$ and $n \geq 1$. This shows that $U_J = U_{J\cup(-J)}$. The lemma follows using also that R is positively graded. □

Let $M \subset R$ be the set of monomials in the generators $\{z_0^{(2n-1)}, z_i^{(n)} \mid i \in I_{\mathrm{hi}}, n \geq 1\}$ of R. As noted earlier, this set is a linear basis for R. It is also a topological linear basis for $\widehat{R}$ in the strongest possible sense: any element $z \in \widehat{R}$ can be written as $z = \sum_{m\in M} c_m m$ for unique $c_m \in \Bbbk$, and any such infinite linear combination converges. This follows because all but finitely many monomials in M are in $U_{d,J}$ for any given $d \in \mathbb{Z}$ and cofinite $J \subseteq I$, and for every finite family of monomials there exists $d \in \mathbb{Z}$ and cofinite $J \subseteq I$ such that the images of the monomials in the family are linearly independent in $R/U_{d,J}$. Thus, the function

$$\prod_{m\in M} \Bbbk \to \widehat{R}, \qquad (c_m)_{m\in M} \mapsto \sum_{m\in M} c_m m \tag{6.3}$$

is a homeomorphism if we give $\Bbbk$ the discrete topology and $\prod_{m\in M} \Bbbk$ the product topology.

Now take $\lambda \in X^\imath$ and $\boldsymbol{i}, \boldsymbol{j} \in \langle I \rangle$ such that $\alpha_{\boldsymbol{i}}$ and $\alpha_{\boldsymbol{j}}$ have the same images in $X^\imath$. We will use the shorthand

$$H_\lambda(\boldsymbol{j}, \boldsymbol{i}) \coloneqq \mathrm{Hom}_{\mathfrak{U}^\imath}(B_{\boldsymbol{j}}\mathbb{1}_\lambda, B_{\boldsymbol{i}}\mathbb{1}_\lambda). \tag{6.4}$$

This is a free graded right R-module with degree d component $H_\lambda(\boldsymbol{j}, \boldsymbol{i})_d$. It is 0 unless $\boldsymbol{i} \approx \boldsymbol{j}$, i.e., the lengths of the words $\boldsymbol{i}$ and $\boldsymbol{j}$ have the same parity. In the homogeneous basis $\mathrm{B}_\lambda(\boldsymbol{i} \times \boldsymbol{j})$ for $H_\lambda(\boldsymbol{j}, \boldsymbol{i})$ as a free right R-module from (3.47), the degrees of the basis vectors are bounded below, and there are only finitely many basis vectors in each degree. Let $H_\lambda(\boldsymbol{j}, \boldsymbol{i})_{\geq d} \coloneqq \sum_{d'\geq d} H_\lambda(\boldsymbol{j}, \boldsymbol{i})_{d'}$.

The *GC topology* on $H_\lambda(\boldsymbol{j}, \boldsymbol{i})$ is the linear topology with a system of neighborhoods of 0 given by the subspaces

$$H_\lambda(\boldsymbol{j}, \boldsymbol{i})_{d,J} \coloneqq H_\lambda(\boldsymbol{j}, \boldsymbol{i})_{\geq d} + H_\lambda(\boldsymbol{j}, \boldsymbol{i})U_J \tag{6.5}$$

for all $d \in \mathbb{Z}$ and all cofinite subsets $J \subseteq I$. We denote the completion $\varprojlim_{d,J} H_\lambda(\boldsymbol{j}, \boldsymbol{i})/H_\lambda(\boldsymbol{j}, \boldsymbol{i})_{d,J}$ with respect to this topology by $\widehat{H}_\lambda(\boldsymbol{j}, \boldsymbol{i})$. The action of R on $H_\lambda(\boldsymbol{j}, \boldsymbol{i})$ extends by continuity to make $\widehat{H}_\lambda(\boldsymbol{j}, \boldsymbol{i})$ into a topological right $\widehat{R}$-module. The following lemma shows that $\mathrm{B}_\lambda(\boldsymbol{i} \times \boldsymbol{j})$, the linear basis for $H_\lambda(\boldsymbol{j}, \boldsymbol{i})$ as a right R-module, is also a topological basis for $\widehat{H}_\lambda(\boldsymbol{j}, \boldsymbol{i})$ as a topological right $\widehat{R}$-module.

Lemma 6.2. *In the above notation, the linear map*

$$\prod_{\theta\in \mathrm{B}_\lambda(\boldsymbol{i}\times\boldsymbol{j})} \widehat{R} \to \widehat{H}_\lambda(\boldsymbol{j}, \boldsymbol{i}), \qquad (z_\theta)_{\theta\in \mathrm{B}_\lambda(\boldsymbol{i}\times\boldsymbol{j})} \mapsto \sum_{\theta\in \mathrm{B}_\lambda(\boldsymbol{i}\times\boldsymbol{j})} \theta z_\theta \tag{6.6}$$

is a homeomorphism for the linear topology on the domain that is the product of the GC topologies, and the GC topology on the codomain.

Proof. This follows from the following equivalent description of the GC topology on $H_\lambda(\boldsymbol{j}, \boldsymbol{i})$: it is the linear topology with a system of neighborhoods of 0 given by the subspaces $\bigoplus_{\theta\in \mathrm{B}_\lambda(\boldsymbol{i}\times\boldsymbol{j})} \theta U_{d(\theta),J(\theta)}$ for $d(\theta) \in \mathbb{Z}$ and cofinite subsets $J(\theta) \subseteq I$ such that $d(\theta) \leq 0$ (hence, $U_{d(\theta),J(\theta)} = R$) for all but finitely many θ. For this new basis-dependent way of defining the GC topology on $H_\lambda(\boldsymbol{j}, \boldsymbol{i})$, it is clear that (6.6) is a homeomorphism.

To prove that the new description of the GC topology is equivalent to the old one, we observe that $H_\lambda(\boldsymbol{j}, \boldsymbol{i})_{d,J} = \bigoplus_{\theta\in \mathrm{B}_\lambda(\boldsymbol{i}\times\boldsymbol{j})} \theta U_{d-\deg(\theta),J}$. Only finitely many θ have $\deg(\theta) < d$, so this is one of the new neighborhoods. Conversely, given one of the new neighborhoods $N \coloneqq \bigoplus_{\theta\in \mathrm{B}_\lambda(\boldsymbol{i}\times\boldsymbol{j})} \theta U_{d(\theta),J(\theta)}$, let $F \coloneqq \{\theta \in \mathrm{B}_\lambda(\boldsymbol{i} \times \boldsymbol{j}) \mid d(\theta) > 0\}$, $d \coloneqq \max(0, \deg(\theta) + d(\theta) \mid \theta \in F)$ and $J \coloneqq I \cap \bigcap_{\theta\in F} J(\theta)$. Then $H_\lambda(\boldsymbol{j}, \boldsymbol{i})_{d,J} \subseteq N$. □

The GC topology on $H_\lambda(\boldsymbol{j}, \boldsymbol{i})$ is separated, so the canonical map $H_\lambda(\boldsymbol{j}, \boldsymbol{i}) \to \widehat{H}_\lambda(\boldsymbol{j}, \boldsymbol{i})$ is injective. This is also clear from Lemma 6.2 and Theorem 3.6.

Definition 6.3. The *completed 2-iquantum group* $\widehat{\mathfrak{U}}^\imath$ is the $\Bbbk$-linear 2-category with the following objects, 1-morphisms and 2-morphisms:

- Its object set is $X^\imath$, the same as $\mathfrak{U}^\imath$.
- For $\lambda, \mu \in X^\imath$, the objects of the morphism category $\mathbf{Hom}_{\widehat{\mathfrak{U}}^\imath}(\lambda, \mu)$ are the formal direct sums
$$B_{\boldsymbol{I}}\mathbb{1}_\lambda = \mathbb{1}_\mu B_{\boldsymbol{I}} := \bigoplus_{\boldsymbol{i}\in\boldsymbol{I}} B_{\boldsymbol{i}}\mathbb{1}_\lambda \tag{6.7}$$
for (possibly infinite) sets $\boldsymbol{I}$ such that $\boldsymbol{I} \subseteq I^n$ for some $n \in \mathbb{N}$ and $\lambda - \alpha_{\boldsymbol{i}} = \mu$ for all $\boldsymbol{i} \in \boldsymbol{I}$. If $\boldsymbol{I} = \{\varnothing\}$ (hence, $\lambda = \mu$) then we denote $B_{\boldsymbol{I}}\mathbb{1}_\lambda$ simply by $\mathbb{1}_\lambda$, and if $\boldsymbol{I} = \{i\}$ (hence, $\lambda - \alpha_i = \mu$) then we denote $B_{\boldsymbol{I}}\mathbb{1}_\lambda$ simply by $B_i\mathbb{1}_\lambda$.
- For $\boldsymbol{I}, \boldsymbol{J}$ of this form, the vector space $\widehat{H}_\lambda(\boldsymbol{J}, \boldsymbol{I}) := \mathrm{Hom}_{\widehat{\mathfrak{U}}^\imath}(B_{\boldsymbol{J}}\mathbb{1}_\lambda, B_{\boldsymbol{I}}\mathbb{1}_\lambda)$ consists of matrices $\theta = (\theta_{\boldsymbol{i},\boldsymbol{j}})_{\boldsymbol{i}\in\boldsymbol{I},\boldsymbol{j}\in\boldsymbol{J}}$ such that $\theta_{\boldsymbol{i},\boldsymbol{j}} \in \widehat{H}_\lambda(\boldsymbol{j}, \boldsymbol{i})$ for each $\boldsymbol{i} \in \boldsymbol{I}$, $\boldsymbol{j} \in \boldsymbol{J}$. This space is 0 unless the lengths of the words in $\boldsymbol{I}$ and the words in $\boldsymbol{J}$ are of the same parity.

Vertical composition is by matrix multiplication:
$$\big(\theta_{\boldsymbol{i},\boldsymbol{j}}\big)_{\boldsymbol{i}\in\boldsymbol{I},\boldsymbol{j}\in\boldsymbol{J}} \circ \big(\theta_{\boldsymbol{j},\boldsymbol{k}}\big)_{\boldsymbol{j}\in\boldsymbol{J},\boldsymbol{k}\in\boldsymbol{K}} = \Big(\sum_{\boldsymbol{j}\in\boldsymbol{J}} \theta_{\boldsymbol{i},\boldsymbol{j}} \circ \theta_{\boldsymbol{j},\boldsymbol{k}}\Big)_{\boldsymbol{i}\in\boldsymbol{I},\boldsymbol{k}\in\boldsymbol{K}}.$$
The infinite sums that are the matrix entries here make sense thanks to Lemma 6.4 below. The horizontal composition of 1-morphisms $B_{\boldsymbol{I}'}\mathbb{1}_\mu \colon \mu \to \nu$ and $B_{\boldsymbol{I}''}\mathbb{1}_\lambda \colon \lambda \to \mu$ is $B_{\boldsymbol{I}}\mathbb{1}_\lambda \colon \lambda \to \nu$ where $\boldsymbol{I} := \{\boldsymbol{i}'\boldsymbol{i}'' \,|\, \boldsymbol{i}' \in \boldsymbol{I}', \boldsymbol{i}'' \in \boldsymbol{I}''\}$. The horizontal composition of 2-morphisms represented by the matrices $(\theta_{\boldsymbol{i}',\boldsymbol{j}'})_{\boldsymbol{i}'\in\boldsymbol{I}',\boldsymbol{j}'\in\boldsymbol{J}'}$ and $(\theta_{\boldsymbol{i}'',\boldsymbol{j}''})_{\boldsymbol{i}''\in\boldsymbol{I}'',\boldsymbol{j}''\in\boldsymbol{J}''}$ is the matrix with $(\boldsymbol{i}'\boldsymbol{i}'', \boldsymbol{j}'\boldsymbol{j}'')$-entry $\theta_{\boldsymbol{i}',\boldsymbol{j}'}\theta_{\boldsymbol{i}'',\boldsymbol{j}''}$ as defined in Lemma 6.5 below. Finally, we equip each 2-morphism space
$$\widehat{H}_\lambda(\boldsymbol{J}, \boldsymbol{I}) = \prod_{\boldsymbol{i}\in\boldsymbol{I},\boldsymbol{j}\in\boldsymbol{J}} \widehat{H}_\lambda(\boldsymbol{j}, \boldsymbol{i}) \tag{6.8}$$
with the topology that is the product of the GC topologies on each hom space, thereby making $\widehat{\mathfrak{U}}^\imath$ into a 2-category enriched in topological vector spaces.

We will not go through the checks of the 2-category axioms in detail. The only one that is slightly delicate is the associativity of vertical composition. This follows because the infinite sums defining vertical composition reduce to finite sums on projection to the quotient modulo a GC neighborhood of 0 like in the proof of Lemma 6.4(ii), and for those associativity is clear.

Lemma 6.4. *Suppose given $\lambda \in X^\imath$, $\boldsymbol{i}, \boldsymbol{j} \in \langle I \rangle$ and $\boldsymbol{K} \subseteq I^n$ such that $\mu := \lambda - \alpha_{\boldsymbol{i}} = \lambda - \alpha_{\boldsymbol{j}} = \lambda - \alpha_{\boldsymbol{k}}$ for all $\boldsymbol{k} \in \boldsymbol{K}$.*

(i) *For any $\boldsymbol{k} \in \boldsymbol{K}$, the vertical composition in $\mathfrak{U}^\imath$ extends by continuity to define a bilinear map*
$$\widehat{H}_\lambda(\boldsymbol{k}, \boldsymbol{i}) \times \widehat{H}_\lambda(\boldsymbol{j}, \boldsymbol{k}) \to \widehat{H}_\lambda(\boldsymbol{j}, \boldsymbol{i}), \qquad (f, g) \mapsto f \circ g.$$
(ii) *Given 2-morphisms $\theta_{\boldsymbol{i},\boldsymbol{k}} \in \widehat{H}_\lambda(\boldsymbol{k}, \boldsymbol{i})$ and $\theta_{\boldsymbol{k},\boldsymbol{j}} \in \widehat{H}_\lambda(\boldsymbol{j}, \boldsymbol{k})$ for $\boldsymbol{k} \in \boldsymbol{K}$, the infinite sum $\sum_{\boldsymbol{k}\in\boldsymbol{K}} \theta_{\boldsymbol{i},\boldsymbol{k}} \circ \theta_{\boldsymbol{k},\boldsymbol{j}}$ converges in $\widehat{H}_\lambda(\boldsymbol{j}, \boldsymbol{i})$.*

Proof. (i) The result is trivial if $H_\lambda(\boldsymbol{k}, \boldsymbol{i}) = 0$ or $H_\lambda(\boldsymbol{j}, \boldsymbol{k}) = 0$. Assuming both are non-zero, let a be minimal such that $H_\lambda(\boldsymbol{j}, \boldsymbol{k})_a \neq 0$ and b be minimal such that $H_\lambda(\boldsymbol{k}, \boldsymbol{i})_b \neq 0$. For any $d \in \mathbb{Z}$ and cofinite $J \subseteq I$, we have that
$$H_\lambda(\boldsymbol{k}, \boldsymbol{i})_{d-a,J} \circ H_\lambda(\boldsymbol{j}, \boldsymbol{k}) \subseteq H_\lambda(\boldsymbol{j}, \boldsymbol{i})_{d,J}, \qquad H_\lambda(\boldsymbol{k}, \boldsymbol{i}) \circ H_\lambda(\boldsymbol{j}, \boldsymbol{k})_{d-b,J} \subseteq H_\lambda(\boldsymbol{j}, \boldsymbol{i})_{d,J}.$$
So the linear map $H_\lambda(\boldsymbol{k}, \boldsymbol{i}) \otimes H_\lambda(\boldsymbol{j}, \boldsymbol{k}) \to H_\lambda(\boldsymbol{j}, \boldsymbol{i})/H_\lambda(\boldsymbol{j}, \boldsymbol{i})_{d,J}$ induced by composition factors through the quotient $(H_\lambda(\boldsymbol{k}, \boldsymbol{i})/H_\lambda(\boldsymbol{k}, \boldsymbol{i})_{d-a,J}) \otimes (H_\lambda(\boldsymbol{j}, \boldsymbol{k})/H_\lambda(\boldsymbol{j}, \boldsymbol{k})_{d-b,J})$. Taking the inverse limit over target neighborhoods then gives the required continuous extension.

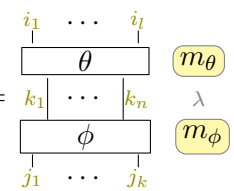
(ii) By (i), the composition $\theta_{\boldsymbol{i},\boldsymbol{k}} \circ \theta_{\boldsymbol{k},\boldsymbol{j}}$ is a well-defined element of $\widehat{H}_\lambda(\boldsymbol{j}, \boldsymbol{i})$ for each $\boldsymbol{k} \in \boldsymbol{K}$. To prove that the sum $\sum_{\boldsymbol{k}\in\boldsymbol{K}} \theta_{\boldsymbol{i},\boldsymbol{k}} \circ \theta_{\boldsymbol{k},\boldsymbol{j}}$ converges, it suffices to show for each GC neighborhood $H_\lambda(\boldsymbol{j}, \boldsymbol{i})_{d,J}$

that the canonical projection of $\theta_{\boldsymbol{i},\boldsymbol{k}} \circ \theta_{\boldsymbol{k},\boldsymbol{j}}$ to $H_\lambda(\boldsymbol{j},\boldsymbol{i})/H_\lambda(\boldsymbol{j},\boldsymbol{i})_{d,J}$ is zero for all but finitely many $\boldsymbol{k} \in \boldsymbol{K}$. Since $\boldsymbol{K} \subseteq I^n$, this is a consequence of the following:

<u>Claim</u>: *For every cofinite $J \subseteq I$, there is a finite subset $\overline{J} \subset I$ such that $H_\lambda(\boldsymbol{k},\boldsymbol{i}) \circ H_\lambda(\boldsymbol{j},\boldsymbol{k}) \subseteq H_\lambda(\boldsymbol{j},\boldsymbol{i})U_J$ for every $\boldsymbol{k} \in I^n - \overline{J}^n$ with $\lambda - \alpha_{\boldsymbol{k}} = \mu$.*

Proof: Consider the Cartan graph G with vertex set I and edges i — $i{+}1$ for each $i \in I$. Let S be a finite subset of I closed under $\pm$ which contains $I - J$, all letters of the words $\boldsymbol{i}$ and $\boldsymbol{j}$, and all $i \in I$ such that $\varsigma_i - \lambda_i \neq 0$. Then define $\overline{J} := \{i \in I \mid d_G(i,S) \leq n\}$, where $d_G(i,S)$ is the distance in the graph G. Since S is closed under $\pm$, $d_G(i,S) = d_G(-i,S)$, so $\overline{J}$ is also closed under $\pm$.

Now we take $\boldsymbol{k} \in I^n - \overline{J}^n$ with $\lambda - \alpha_{\boldsymbol{k}} = \mu$, $\theta \in H_\lambda(\boldsymbol{k},\boldsymbol{i})$ and $\phi \in H_\lambda(\boldsymbol{j},\boldsymbol{k})$ and show that $\theta \circ \phi \in H_\lambda(\boldsymbol{j},\boldsymbol{i})U_J$. We may assume that θ and ϕ are string diagrams of reduced shapes multiplied on the right by some monomials m_θ and m_ϕ in dotted bubbles, so

$\theta \circ \phi =$ 

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At least one of the n letters of the word $\boldsymbol{k}$ is in $I - \overline{J}$. So at most $n-1$ of the letters of $\boldsymbol{k}$ are of distance $\leq n$ from S. Consequently, there exists $1 \leq t \leq n$ such that no letter of $\boldsymbol{k}$ is of distance exactly t from S. Then we let A (resp., B) be the set of $k \in I$ such that $\pm k$ is a letter of $\boldsymbol{k}$ and $d_G(k,S) > t$ (resp., $d_G(k,S) < t$). The union D of connected components of $\theta \circ \phi$ labelled by elements of A is non-empty because some letter of $\boldsymbol{k}$ is in A, and closed because the letters of $\boldsymbol{i}$ and $\boldsymbol{j}$ are not in A. All elements of A are separated from $S \cup B$ in the sense that they are not connected by an edge in the graph G. Hence, by the defining relations of $\mathfrak{U}^\imath$, D can be commuted past all other strings to move it to the right-hand boundary.

It remains to observe that any element of $H_\lambda(\boldsymbol{j},\boldsymbol{i})$ multiplied on the right by a non-empty closed string diagram D all of whose strings are labelled by elements of the set A lies in $H_\lambda(\boldsymbol{j},\boldsymbol{i})U_J$. This follows because D, which is located in a 2-cell of iweight λ, can be straightened using the relations to obtain a linear combination of non-trivial monomials in non-zero dotted bubbles labelled by elements of A. Since $\varsigma_k - \lambda_k = 0$ for $k \in A$, any dotted bubble of this form is of strictly positive degree, so it is a non-zero multiple of $z^{(r)}_{\lambda;k}$ for some $k \in A$ and $r \geq 1$. Since $A \subseteq J$, any linear combination of non-trivial monomials in such elements lies in $\varphi_\lambda(U_J)$ where φ_λ is the isomorphism from (3.45). □

Lemma 6.5. *Suppose given $\boldsymbol{i} = \boldsymbol{i}'\boldsymbol{i}''$ and $\boldsymbol{j} = \boldsymbol{j}'\boldsymbol{j}''$ for $\boldsymbol{i}' \in I^{m'}, \boldsymbol{i}'' \in I^{m''}$ and $\boldsymbol{j}' \in I^{n'}, \boldsymbol{j}'' \in I^{n''}$ such that $\mu = \lambda - \alpha_{\boldsymbol{i}''} = \lambda - \alpha_{\boldsymbol{j}''}$ and $\nu = \mu - \alpha_{\boldsymbol{i}'} = \mu - \alpha_{\boldsymbol{j}'}$ for some $\lambda, \mu, \nu \in X^\imath$. The horizontal composition in $\mathfrak{U}^\imath$ extends by continuity to define a bilinear map*

$$\widehat{H}_\mu(\boldsymbol{j}',\boldsymbol{i}') \times \widehat{H}_\lambda(\boldsymbol{j}'',\boldsymbol{i}'') \to \widehat{H}_\lambda(\boldsymbol{j},\boldsymbol{i}), \qquad (f',f'') \mapsto f'f''.$$

Proof. If $H_\mu(\boldsymbol{j}',\boldsymbol{i}')$ or $H_\lambda(\boldsymbol{j}'',\boldsymbol{i}'')$ is 0 then the result is trivial, so assume that they are both non-zero. Let a be minimal such that $H_\lambda(\boldsymbol{j}'',\boldsymbol{i}'')_a \neq 0$ and b be minimal such that $H_\mu(\boldsymbol{j}',\boldsymbol{i}')_b \neq 0$. Also let $K \subseteq I$ be the set of all $i \in I$ such that $a_{i,j} = a_{i,-j} = 0$ for all letters j appearing in $\boldsymbol{i}''$ or $\boldsymbol{j}''$. The bubble slide relation (3.23) implies that any dotted bubble labelled by $i \in K$ commutes freely past any string colored by one of the letters appearing in the words $\boldsymbol{i}''$ and $\boldsymbol{j}''$.

Now suppose that $J \subseteq I$ is a cofinite subset. Let $J' := J \cap K$. It is a cofinite subset of I contained in J. Then we have that $\varphi_\mu(U_{J'})H_\lambda(\boldsymbol{j}'',\boldsymbol{i}'') \subseteq H_\lambda(\boldsymbol{j}'',\boldsymbol{i}'')U_{J'} \subseteq H_\lambda(\boldsymbol{j}'',\boldsymbol{i}'')U_J$. From this, it follows that

$$H_\mu(\boldsymbol{j}',\boldsymbol{i}')_{d-a,J'}\ H_\lambda(\boldsymbol{j}'',\boldsymbol{i}'') \subseteq H_\lambda(\boldsymbol{j},\boldsymbol{i})_{d,J}, \qquad H_\mu(\boldsymbol{j}',\boldsymbol{i}')\ H_\lambda(\boldsymbol{j}'',\boldsymbol{i}'')_{d-b,J} \subseteq H_\lambda(\boldsymbol{j},\boldsymbol{i})_{d,J}.$$

The lemma follows from this like at the end of the proof of Lemma 6.4(i). □

The following 1-morphisms in $\widehat{\mathfrak{U}}^\imath$ will be particularly important: for $\lambda, \mu \in X^\imath$ and $n \geq 0$, let

$$\mathbb{1}_\mu B^n \mathbb{1}_\lambda := \bigoplus_{\substack{\boldsymbol{i} \in I^n \\ \lambda - \alpha_{\boldsymbol{i}} = \mu}} B_{\boldsymbol{i}} \mathbb{1}_\lambda. \tag{6.9}$$

This formal direct sum is the object $B_{\boldsymbol{I}} \mathbb{1}_\lambda = \mathbb{1}_\mu B_{\boldsymbol{I}}$ of the morphism category $\mathbf{Hom}_{\widehat{\mathfrak{U}}^\imath}(\lambda, \mu)$ from (6.7) with $\boldsymbol{I} := \{\boldsymbol{i} \in I^n \mid \lambda - \alpha_{\boldsymbol{i}} = \mu\}$.

Definition 6.6. Let $\widehat{\mathbf{AB}}$ be the strict $\Bbbk$-linear monoidal category with objects $B^{\otimes n}$ ($n \in \mathbb{N}$) and morphisms

$$\mathrm{Hom}_{\widehat{\mathbf{AB}}}(B^{\otimes n}, B^{\otimes m}) := \prod_{\lambda, \mu \in X_{\bar{0}}^\imath} \mathrm{Hom}_{\widehat{\mathfrak{U}}^\imath}(\mathbb{1}_\mu B^n \mathbb{1}_\lambda, \mathbb{1}_\mu B^m \mathbb{1}_\lambda). \tag{6.10}$$

We denote a morphism in this space by $\theta = (\theta^{\mu,\lambda})_{\lambda,\mu \in X_{\bar{0}}^\imath}$ for $\theta^{\mu,\lambda} \in \mathrm{Hom}_{\widehat{\mathfrak{U}}^\imath}(\mathbb{1}_\mu B^n \mathbb{1}_\lambda, \mathbb{1}_\mu B^m \mathbb{1}_\lambda)$. The vertical composition of $\theta \colon B^{\otimes m} \to B^{\otimes l}$ and $\phi \colon B^{\otimes n} \to B^{\otimes m}$ is defined in terms of vertical composition in $\widehat{\mathfrak{U}}^\imath$ by $(\theta \circ \phi)^{\mu,\lambda} := \theta^{\mu,\lambda} \circ \phi^{\mu,\lambda}$. Tensor product of objects is defined by $B^{\otimes n'} \otimes B^{\otimes n''} := B^{\otimes (n'+n'')}$. The tensor product $\theta \otimes \phi \colon B^{\otimes (n'+n'')} \to B^{\otimes (m'+m'')}$ of morphisms $\theta \colon B^{\otimes n'} \to B^{\otimes m'}$ and $\phi \colon B^{\otimes n''} \to B^{\otimes m''}$ is defined in terms of horizontal composition of 2-morphisms in $\widehat{\mathfrak{U}}^\imath$ by

$$(\theta \otimes \phi)^{\mu,\lambda}_{\boldsymbol{i},\boldsymbol{j}} := \begin{cases} \theta^{\mu,\nu}_{\boldsymbol{i}',\boldsymbol{j}'} \phi^{\nu,\lambda}_{\boldsymbol{i}'',\boldsymbol{j}''} & \text{if } \nu := \lambda - \alpha_{\boldsymbol{i}''} = \lambda - \alpha_{\boldsymbol{j}''} \\ 0 & \text{otherwise,} \end{cases} \tag{6.11}$$

for $\boldsymbol{i}, \boldsymbol{j}$ with $\lambda - \alpha_{\boldsymbol{i}} = \lambda - \alpha_{\boldsymbol{j}} = \mu$ such that $\boldsymbol{i} = \boldsymbol{i}'\boldsymbol{i}''$ for $\boldsymbol{i}' \in I^{m'}$, $\boldsymbol{i}'' \in I^{m''}$ and $\boldsymbol{j} = \boldsymbol{j}'\boldsymbol{j}''$ for $\boldsymbol{j}' \in I^{n'}$, $\boldsymbol{j}'' \in I^{n''}$. We equip each morphism space in $\widehat{\mathbf{AB}}$ with the topology that is the product of the GC topologies on each hom space. Then $\widehat{\mathbf{AB}}$ is a strict monoidal category enriched in topological vector spaces. Its identity object $\mathbb{1}$ is $B^{\otimes 0}$.

Remark 6.7. Unravelling definitions, $\mathrm{Hom}_{\widehat{\mathbf{AB}}}(B^{\otimes n}, B^{\otimes m})$ is identified with the completion of

$$\bigoplus_{\substack{\lambda \in X_{\bar{0}}^\imath,\ \boldsymbol{i} \in I^m, \boldsymbol{j} \in I^n \\ \lambda - \alpha_{\boldsymbol{i}} = \lambda - \alpha_{\boldsymbol{j}}}} H_\lambda(\boldsymbol{j}, \boldsymbol{i})$$

with respect to the linear topology which has a system of neighborhoods of 0 given by the subspaces

$$\bigoplus_{\substack{\lambda \in X_{\bar{0}}^\imath,\ \boldsymbol{i} \in I^m, \boldsymbol{j} \in I^n \\ \lambda - \alpha_{\boldsymbol{i}} = \lambda - \alpha_{\boldsymbol{j}}}} H_\lambda(\boldsymbol{j}, \boldsymbol{i})_{d(\lambda;\boldsymbol{i},\boldsymbol{j}), J(\lambda;\boldsymbol{i},\boldsymbol{j})}$$

for $d(\lambda; \boldsymbol{i}, \boldsymbol{j}) \in \mathbb{Z}$ and cofinite subsets $J(\lambda; \boldsymbol{i}, \boldsymbol{j}) \subseteq I$ such that $H_\lambda(\boldsymbol{j}, \boldsymbol{i})_{d(\lambda;\boldsymbol{i},\boldsymbol{j}), J(\lambda;\boldsymbol{i},\boldsymbol{j})} = H_\lambda(\boldsymbol{j}, \boldsymbol{i})$ for all but finitely many of the triples $(\lambda; \boldsymbol{i}, \boldsymbol{j})$.

Recall that $Z_\lambda = \mathrm{End}_{\mathfrak{U}^\imath}(\mathbb{1}_\lambda)$. We denote the algebra $\mathrm{End}_{\widehat{\mathfrak{U}}^\imath}(\mathbb{1}_\lambda) = \widehat{H}_\lambda(\varnothing, \varnothing)$, which is the completion of Z_λ, by $\widehat{Z}_\lambda$. For each λ, the isomorphism φ_λ from (3.45) induces an isomorphism

$$\hat{\varphi}_\lambda \colon \widehat{R} \xrightarrow{\sim} \widehat{Z}_\lambda. \tag{6.12}$$

We described the GC topology on $\widehat{R}$ rather explicitly at the beginning of the section. Using the isomorphisms φ_λ and $\hat{\varphi}_\lambda$, this description can be transferred to Z_λ and $\widehat{Z}_\lambda$. In particular, we will denote $\varphi_\lambda(U_J)$, $\varphi_\lambda(U_{d,J})$, $\hat{\varphi}_\lambda(\overline{U}_J)$ and $\hat{\varphi}_\lambda(\overline{U}_{d,J})$ by $U_{\lambda;J}$, $U_{\lambda;d,J}$, $\overline{U}_{\lambda;J}$ and $\overline{U}_{\lambda;d,J}$, respectively.

We denote $\mathrm{End}_{\widehat{\mathbf{AB}}}(\mathbb{1})$ by $\widehat{Z}$. By a special case of Remark 6.7, the complete topological algebra $\widehat{Z}$ is identified with the completion of $\bigoplus_{\lambda \in X_{\bar{0}}^\imath} Z_\lambda$ with respect to the linear topology defined by

neighborhoods of 0 given by $\bigoplus_{\lambda\in X_0^\imath} U_{\lambda;d(\lambda),J(\lambda)}$ for $d(\lambda)\in\mathbb{Z}$ and cofinite $J(\lambda)\subseteq I$ such that $U_{\lambda;d(\lambda),J(\lambda)}=\mathrm{End}_{\mathfrak{U}^\imath}(\mathbb{1}_\lambda)$ for all but finitely many λ. So we have that

$$\widehat{Z}=\prod_{\lambda\in X_0^\imath}\widehat{Z}_\lambda. \tag{6.13}$$

The infinite product $\prod_{i\in I}O_{\lambda;i}(u-i)$ makes sense in $\widehat{Z}_\lambda(\!(u^{-1})\!)$. This expression will play an important role in the next section.

Also of particular interest are the algebras $\mathrm{End}_{\widehat{\mathfrak{U}}^\imath}(B_i\mathbb{1}_\lambda)$ and $\mathrm{End}_{\widehat{\mathbf{AB}}}(B)$. By a special case of Remark 6.7, $\mathrm{End}_{\widehat{\mathbf{AB}}}(B)$ is the completion of $\bigoplus_{i\in I,\lambda\in X_0^\imath}\mathrm{End}_{\mathfrak{U}^\imath}(B_i\mathbb{1}_\lambda)$ with respect to the topology defined by neighborhoods of 0 given by $\bigoplus_{i\in I,\lambda\in X_0^\imath}H_\lambda(i,i)_{d(\lambda;i),J(\lambda;i)}$ for $d(\lambda;i)\in\mathbb{Z}$ and cofinite $J(\lambda;i)\subseteq I$ such that $H_\lambda(i,i)_{d(\lambda;i),J(\lambda;i)}=\mathrm{End}_{\mathfrak{U}^\imath}(B_i\mathbb{1}_\lambda)$ for all but finitely many $(\lambda;i)$. So we have that

$$\mathrm{End}_{\widehat{\mathbf{AB}}}(B)=\prod_{\substack{i\in I\\ \lambda\in X_0^\imath}}\mathrm{End}_{\widehat{\mathfrak{U}}^\imath}(B_i\mathbb{1}_\lambda). \tag{6.14}$$

Theorem 6.8. *The action of $\mathfrak{U}^\imath$ on any nilpotent Abelian or Karoubian iquantum categorification $\mathbf{R}$ extends by continuity to make $\mathbf{R}$ into a 2-representation of $\widehat{\mathfrak{U}}^\imath$.*

Proof. We first prove two claims:

Claim 1: *For each object $V\in\mathbf{R}_\lambda$, the kernel of the homomorphism $Z_\lambda\to\mathrm{End}_{\mathbf{R}}(V), z\mapsto z_V$ contains a GC neighborhood of 0 in Z_λ.*

Proof: Let K be a finite subset of I closed under $\pm$ which contains all $i\in I$ with $B_iV\neq 0$ or $\varsigma_i-\lambda_i\neq 0$. Then $\left(z^{(n)}_{\lambda;j}\right)_V=0$ for all $j\in J:=I-K$ and $n\geq 1$.

For each i and $n\geq 1$, Lemma 5.7 implies that $\left(z^{(n)}_{\lambda;i}\right)_V$ is in the Jacobson radical $J(Z_V)$ of the finite-dimensional commutative algebra Z_V. Choose $N\geq 1$ such that $J(Z_V)^N=0$. Also choose $M\geq 1$ so that $\left(z^{(n)}_{\lambda;k}\right)_V=0$ whenever $k\in K$ and $n>M$. Any monomial that can act nontrivially therefore has fewer than N factors, each of degree at most $2M$. Consequently every monomial of degree at least $d=2M(N-1)+1$ acts by zero, and $U_{\lambda;d,J}$ lies in the kernel of the homomorphism $Z_\lambda\to\mathrm{End}_{\mathbf{R}}(V)$, as required.

Claim 2: *For each object $V\in\mathbf{R}_\lambda$ and $\boldsymbol{i},\boldsymbol{j}\in\langle I\rangle$ with $\lambda-\alpha_{\boldsymbol{i}}=\lambda-\alpha_{\boldsymbol{j}}$, the evaluation map $H_\lambda(\boldsymbol{j},\boldsymbol{i})\to\mathrm{Hom}_{\mathbf{R}}(B_{\boldsymbol{j}}V,B_{\boldsymbol{i}}V)$ is continuous for the GC topology on the first hom space and the discrete topology on the second one.*

Proof: Let d and J be as in Claim 1. Consider the basis $\mathrm{B}_\lambda(\boldsymbol{i}\times\boldsymbol{j})$ for $H_\lambda(\boldsymbol{j},\boldsymbol{i})$. At each of the chosen dot positions on the finitely many shapes in $\mathrm{S}(\boldsymbol{i}\times\boldsymbol{j})$, the dot acts nilpotently on the relevant object obtained from V by applying the functors B_i for the strings to the right of that position. Hence, a basis diagram acts by zero on V providing it is of sufficiently high degree, say $\geq d'$. Then any element of $H_\lambda(\boldsymbol{j},\boldsymbol{i})_{d+d',J}$ acts as zero on V.

Claim 2 implies that the evaluation map $H_\lambda(\boldsymbol{j},\boldsymbol{i})\to\mathrm{Hom}_{\mathbf{R}}(B_{\boldsymbol{j}}V,B_{\boldsymbol{i}}V)$ extends to the completion $\widehat{H}_\lambda(\boldsymbol{j},\boldsymbol{i})$. To complete the proof of the theorem, it remains to observe that for a fixed $V\in\mathbf{R}$ and a fixed length $n\geq 0$, there are only finitely many words $\boldsymbol{i}\in I^n$ such that $B_{\boldsymbol{i}}V\neq 0$. This follows by induction on n because there are only finitely many $i\in I$ such that $B_iV\neq 0$. Consequently, the formal sum defining the 1-morphism $B_{\boldsymbol{I}}$ of $\widehat{\mathfrak{U}}^\imath$ acts on V as a finite direct sum, and the $\boldsymbol{I}\times\boldsymbol{J}$ matrices that are 2-morphisms in $\widehat{\mathfrak{U}}^\imath$ evaluate to finitely supported matrices. □

Corollary 6.9. *Let $\mathbf{R}$ be an even nilpotent Abelian or Karoubian iquantum categorification viewed as a 2-representation of $\widehat{\mathfrak{U}}^\imath$ as in Theorem 6.8. Then $\mathbf{R}$ can be made into a strict left $\widehat{\mathbf{AB}}$-module category as follows:*

(i) *The object $B^{\otimes n}$ of $\widehat{\mathbf{AB}}$, i.e., the formal direct sum (6.9) of 1-morphisms of $\widehat{\mathfrak{U}}^\imath$, acts so that the summand $\mathbb{1}_\mu B^n \mathbb{1}_\lambda$ acts as the functor $\mathbf{R}_\lambda \to \mathbf{R}_\mu$ defining the action of this 1-morphism of $\widehat{\mathfrak{U}}^\imath$, and as 0 on all other iweight subcategories.*

(ii) *A morphism $\theta = (\theta^{\mu,\lambda})_{\mu,\lambda \in X_0^\imath}$ in $\widehat{\mathbf{AB}}$, which by definition is a matrix of 2-morphisms in $\widehat{\mathfrak{U}}^\imath$, acts as the obvious matrix of natural transformations.*

Proof. This follows from the definitions. The only point to note is that only finitely many terms of (6.9) act in a non-zero way on a given object $V \in \mathbf{R}$, so the evaluation of the matrix $\theta = (\theta^{\mu,\lambda})_{\mu,\lambda \in X_0^\imath}$ on V has finite support. □

7. Internal bubbles

In this section, rather than working directly with a nilpotent iquantum categorification, we will work in the completed 2-iquantum group $\widehat{\mathfrak{U}}^\imath$ which was just defined. All convergence of formal Laurent series in u^{-1} with coefficients that are 2-morphisms in $\widehat{\mathfrak{U}}^\imath$ is pointwise, that is, the coefficients of each power of u converge in the GC topology.

For $\lambda \in X^\imath$, recall that $\widehat{Z}_\lambda := \mathrm{End}_{\widehat{\mathfrak{U}}^\imath}(\mathbb{1}_\lambda)$ and $O_{\lambda;i}(u) = \ \bar{\imath}\bigcirc(u)\ \lambda\ \in \widehat{Z}_\lambda(\!(u^{-1})\!)$. In the formal Laurent series $O_{\lambda;i}(u)$, we can replace u by $u-i$ to obtain $O_{\lambda;i}(u-i) \in \widehat{Z}_\lambda(\!(u^{-1})\!)$. For fixed λ, we have that $O_{\lambda;i}\big(u-i\big) \in \mathrm{id}_{\mathbb{1}_\lambda} + u^{-1}\widehat{Z}_\lambda[\![u^{-1}]\!]$ for all but finitely many $i \in I$.

Lemma 7.1. *For any $\lambda \in X^\imath$ and $i \in I$, we have that*

$$\prod_{j \in I - \{i\}} O_{\lambda;j}(u-j) O_{\lambda;-j}(j-u) = \begin{cases} \dfrac{1-4u^2}{4}\,\mathrm{id}_{\mathbb{1}_\lambda} & \text{if } i = 0 \\ (-1)^{\lambda_h}\dfrac{1+2u}{2u^2}\,\mathrm{id}_{\mathbb{1}_\lambda} & \text{if } i = h \\ (-1)^{\lambda_h}\dfrac{1-2u}{2u^2}\,\mathrm{id}_{\mathbb{1}_\lambda} & \text{if } i = -h \\ (-1)^{\lambda_i}\dfrac{4u^2-1}{4u^2}\,\mathrm{id}_{\mathbb{1}_\lambda} & \text{otherwise.} \end{cases} \tag{7.1}$$

Also, recalling the partition $I = I_{\mathrm{lo}} \cup \{0\} \cup I_{\mathrm{hi}}$ from (3.6), we have that

$$\prod_{j \in I_{\mathrm{lo}}} O_{\lambda;j}(u-j) O_{\lambda;-j}(j-u) = (-1)^{p(\lambda)} \zeta_0(\lambda)(1-2u)\,\mathrm{id}_{\mathbb{1}_\lambda}\,. \tag{7.2}$$

Proof. For each $j \in I$, we have by (3.22) that $O_{\lambda;-j}(u)O_{\lambda;j}(-u) = (-1)^{\varsigma_j + \lambda_j} u^{-a_{j,-j}}\,\mathrm{id}_{\mathbb{1}_\lambda}$. Equivalently, replacing u by $j-u$, we have that $O_{\lambda;j}(u-j)O_{\lambda;-j}(j-u) = (-1)^{\varsigma_j+\lambda_j}(j-u)^{-a_{j,-j}}\,\mathrm{id}_{\mathbb{1}_\lambda}$. Also, the only j such that either $a_{j,-j}$ or ς_j are non-zero are $j = 0, \pm h$. The identity (7.1) follows easily from these facts. For (7.2), the product in the statement of the lemma simplifies to $(-1)^{1+\sum_{j \in I_{\mathrm{lo}}} \lambda_j}(h-u)\,\mathrm{id}_{\mathbb{1}_\lambda} = (-1)^{p(\lambda)+t(\lambda)+1}(h-u)\,\mathrm{id}_{\mathbb{1}_\lambda}$, then we use that $\zeta_0(\lambda) = \frac{(-1)^{t(\lambda)+1}}{2}$. □

In order to state and prove Theorem 7.2, we also need something we call *fancy notation*, which we used originally in the proof of [BSW26, Th. 7.1]. First, we introduce the following algebraic notation encoding the fake bubbles from (3.19): let

$$P_{\lambda;i}(u) := \begin{cases} (-1)^{t(\lambda)} u^{-1}\,\mathrm{id}_{\mathbb{1}_\lambda} & \text{if } i = 0 \\ \zeta_i(\lambda)^{-1} \displaystyle\sum_{n=0}^{\varsigma_i - \lambda_i} \ \bar{\imath}\,\textcircled{n}\,\lambda\ \ (u-i)^{\varsigma_i - \lambda_i - n} & \text{otherwise.} \end{cases} \tag{7.3}$$

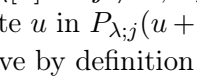
This is an element of $u^{-1}\widehat{Z}_\lambda[u]$; actually, it is a polynomial in $\widehat{Z}_\lambda[u]$ unless $i=0$. More generally, for any (possibly infinite) subset $J \subseteq I$, let

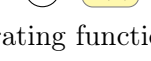
$$P_{\lambda;J}(u) := \prod_{j\in J} P_{\lambda;j}(u) \in u^{-1}\widehat{Z}_\lambda[u]. \tag{7.4}$$

The infinite product here makes sense because $P_{\lambda;j}(u) = \mathrm{id}_{\mathbb{1}_\lambda}$ for all but finitely many $j \in I$. This would not be true if we had not included the normalization factor $\zeta_i(\lambda)^{-1}$ (usually just a sign) in the definition (7.3)!

Next, let K be a finite subset of I, and consider the algebra $\widehat{Z}_\lambda[\![x_k \mid k \in K]\!]$. The fancy notation is the function

$$Q_{\lambda;K} \colon \widehat{Z}_\lambda[\![x_k \mid k \in K]\!] \to \widehat{Z}_\lambda \tag{7.5}$$

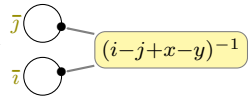
mapping $f(x_k \mid k \in K)$ to the element of $\widehat{Z}_\lambda$ obtained by pinning $f(x_k \mid k \in K)$ to the string diagram $\prod_{k\in K}\zeta_k(\lambda)^{-1}$  with the variable x_k corresponding to the dot on the kth bubble in the product. This map is a $\widehat{Z}_\lambda$-module homomorphism but it is *not* an algebra homomorphism. For example, in the most important special case that $K = \{i\}$, when we denote $Q_{\lambda;K}$ simply by $Q_{\lambda;i}$, we have that

$$Q_{\lambda;i}(f(x_i)) = \zeta_i(\lambda)^{-1}$$  $$\tag{7.6}$$

for $f(x_i) \in \widehat{Z}_\lambda[\![x_i]\!]$. Extending $P_{\lambda;i}$ and $Q_{\lambda;i}$ to generating functions in the formal variable u in the usual way, we have by (3.20) that

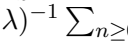
$$O_{\lambda;i}(u) = P_{\lambda;i}(u+i) + Q_{\lambda;i}\big(\tfrac{1}{u-x_i}\big) \in \widehat{Z}_\lambda(\!(u^{-1})\!). \tag{7.7}$$

Another example of fancy notation:

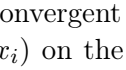
$$Q_{\lambda;\{i,j\}}\big(\tfrac{1}{i-j+x_i-x_j}\big) = \zeta_i(\lambda)^{-1}\zeta_j(\lambda)^{-1}$$  $$\text{for } i \neq j \text{ in } I. \tag{7.8}$$

Now we come to the most important observation of all: for any $c \in \Bbbk^\times$ and an indeterminate x, it makes sense to evaluate $O_{\lambda;j}(u)$ at $u = c+x$ to obtain a formal power series $O_{\lambda;j}(c+x) \in \widehat{Z}_\lambda[\![x]\!]$. To justify this, by (7.7), we have that

$$O_{\lambda;j}(c+x) = P_{\lambda;j}(c+x+j) + Q_{\lambda;j}\big(\tfrac{1}{c+x-x_j}\big) \in \widehat{Z}_\lambda[\![x]\!], \tag{7.9}$$

but we still need to explain why the two terms on the right-hand side here make sense. Since $P_{\lambda;j}(u+j)$ is a polynomial in $\widehat{Z}_\lambda[u]$ for $j \neq 0$, $P_{\lambda;0}(u) \in u^{-1}\widehat{Z}_\lambda[u]$, and $c+x$ is invertible in $\widehat{Z}_\lambda[\![x]\!]$, it makes sense for any j to evaluate u in $P_{\lambda;j}(u+j)$ at $c+x$ to obtain $P_{\lambda;j}(c+x+j)$. To understand the second term in (7.9), we have by definition that $Q_{\lambda;j}\big(\frac{1}{u-x_j}\big) = \zeta_j(\lambda)^{-1}\sum_{n\geq 0} u^{-n-1}$  , so

$$Q_{\lambda;j}\big(\tfrac{1}{c+x-x_j}\big) = \zeta_j(\lambda)^{-1}\sum_{n\geq 0}(c+x)^{-n-1}$$.

Its x^m-coefficient is $\zeta_j(\lambda)^{-1}\sum_{n\geq 0} c^{-m-n-1}\binom{-n-1}{m}$  , which is a convergent series in $\widehat{Z}_\lambda$ as the degree of the bubble increases with n. The expressions $O_{\lambda;j}(i-j+x_i)$ on the right-hand side of the identity in the following theorem are of this form; they are elements of the ring $\widehat{Z}_\lambda[\![x_i]\!]$:

Theorem 7.2. *For $\lambda \in X^\imath$, we have that*

$$\prod_{i\in I} O_{\lambda;i}(u-i) = \Big(1 - \tfrac{(-1)^{p(\lambda)}}{2u}\Big)\,\mathrm{id}_{\mathbb{1}_\lambda} + \sum_{i\in I} Q_{\lambda;i}\Big(\tfrac{\prod_{j\in I-\{i\}} O_{\lambda;j}(i-j+x_i)}{u-i-x_i}\Big).$$

Proof. We first observe that $\prod_{i\in I} O_{\lambda;i}(u-i)$ is a power series in $\mathrm{id}_{\mathbb{1}_\lambda} + u^{-1}\widehat{Z}_\lambda[\![u^{-1}]\!]$, so the constant terms agree on both sides of the identity. This follows because

$$O_{\lambda;i}(u-i) \in u^{\varsigma_i-\lambda_i}\,\mathrm{id}_{\mathbb{1}_\lambda} + u^{\varsigma_i-\lambda_i-1}\widehat{Z}_\lambda[\![u^{-1}]\!]$$

for each $i \in I$ by (3.21), and $\sum_{i\in I}(\varsigma_i - \lambda_i) = 0$ by (3.2) and (3.5). The proof is now reduced to showing that

$$\left[\prod_{i\in I} O_{\lambda;i}(u-i)\right]_{u:<0} = \frac{(-1)^{p(\lambda)+1}}{2u}\,\mathrm{id}_{\mathbb{1}_\lambda} + \sum_{i\in I} Q_{\lambda;i}\Big(\tfrac{\prod_{j\in I-\{i\}} O_{\lambda;j}(i-j+x_i)}{u-i-x_i}\Big). \tag{7.10}$$

The left-hand side of (7.10) expands to

$$\left[\sum_{\substack{J,K \text{ with } |K|<\infty\\ I=J\sqcup K}} \Big(\prod_{j\in J} P_{\lambda;j}(u)\Big)\Big(\prod_{k\in K} Q_{\lambda;k}\big(\tfrac{1}{u-k-x_k}\big)\Big)\right]_{u:<0}.$$

Using fancy notation, this can be rewritten more simply as

$$\sum_{\substack{J,K \text{ with } |K|<\infty\\ I=J\sqcup K}} Q_{\lambda;K}\left(\left[\frac{P_{\lambda;J}(u)}{\prod_{k\in K}(u-k-x_k)}\right]_{u:<0}\right). \tag{7.11}$$

Let $y_i := i + x_i$. By partial fractions, for any polynomial $f(u)$ in u and finite $K \subseteq I$, we have that

$$\left[\frac{f(u)}{\prod_{k\in K}(u-y_k)}\right]_{u:<0} = \left[\sum_{i\in K}\frac{f(u)}{(u-y_i)\prod_{k\in K-\{i\}}(y_i-y_k)}\right]_{u:<0} \overset{(1.1)}{=} \sum_{i\in K}\frac{f(y_i)}{(u-y_i)\prod_{k\in K-\{i\}}(y_i-y_k)},$$

$$\left[\frac{f(u)}{u\prod_{k\in K}(u-y_k)}\right]_{u:<0} = \cdots = \sum_{i\in K}\frac{f(y_i)}{y_i(u-y_i)\prod_{k\in K-\{i\}}(y_i-y_k)} + \frac{f(0)}{u\prod_{k\in K}(-y_k)}.$$

In (7.11), $P_{\lambda;J}(u)$ is a polynomial (so the first partial fractions identity can be applied taking $f(u) := P_{\lambda;J}(u)$) unless $0 \in J$, in which case $P_{\lambda;J}(u) = \frac{(-1)^{t(\lambda)}P_{\lambda;J-\{0\}}(u)}{u}$ with $P_{\lambda;J-\{0\}}(u)$ being a polynomial (so the second partial fractions identity makes sense taking $f(u) := (-1)^{t(\lambda)}P_{\lambda;J-\{0\}}(u)$). Using both partial fractions identities, we expand (7.11), simplifying slightly to obtain

$$\sum_{\substack{J,K \text{ with } |K|<\infty\\ I=J\sqcup K}} Q_{\lambda;K}\left(\sum_{i\in K}\frac{P_{\lambda;J}(i+x_i)}{(u-i-x_i)\prod_{k\in K-\{i\}}(i-k+x_i-x_k)}\right) + \sum_{\substack{J,K \text{ with } |K|<\infty\\ I-\{0\}=J\sqcup K}} Q_{\lambda;K}\left(\frac{(-1)^{t(\lambda)}P_{\lambda;J}(0)}{u\prod_{k\in K}(-k-x_k)}\right). \tag{7.12}$$

Both terms in this expression make sense as formal power series in u^{-1}. Rewriting the first term by moving the summation over $i \in K$ to the front produces

$$\begin{aligned}
&\sum_{i\in I}\sum_{\substack{J,K \text{ with } |K|<\infty\\ I-\{i\}=J\sqcup K}} Q_{\lambda;\{i\}\cup K}\left(\frac{P_{\lambda;J}(i+x_i)}{(u-i-x_i)\prod_{k\in K}(i-k+x_i-x_k)}\right)\\
&\qquad = \sum_{i\in I} Q_{\lambda;i}\left(\frac{1}{u-i-x_i}\sum_{\substack{J,K \text{ with } |K|<\infty\\ I-\{i\}=J\sqcup K}} Q_{\lambda;K}\left(\frac{P_{\lambda;J}(i+x_i)}{\prod_{k\in K}(i-k+x_i-x_k)}\right)\right)\\
&\qquad = \sum_{i\in I} Q_{\lambda;i}\left(\frac{1}{u-i-x_i}\prod_{j\in I-\{i\}}\left(P_{\lambda;j}(i+x_i) + Q_{\lambda;j}\big(\tfrac{1}{i-j+x_i-x_j}\big)\right)\right)\\
&\qquad = \sum_{i\in I} Q_{\lambda;i}\Big(\tfrac{\prod_{j\in I-\{i\}} O_{\lambda;j}(i-j+x_i)}{u-i-x_i}\Big),
\end{aligned}$$

which is the main term on the right-hand side of (7.10).

It just remains to show that the second term in (7.12) is equal to $-\frac{(-1)^{p(\lambda)}}{2u}\,\mathrm{id}_{\mathbb{1}_\lambda}$. It is

$$\frac{(-1)^{t(\lambda)}}{u}\sum_{\substack{J,K \text{ with } |K|<\infty\\ I-\{0\}=J\sqcup K}}\Big(\textstyle\prod_{j\in J}P_{\lambda;j}(0)\Big)\Big(\textstyle\prod_{k\in K}Q_{\lambda;k}\big(\frac{1}{-k-x_k}\big)\Big)$$
$$=\frac{(-1)^{t(\lambda)}}{u}\prod_{i\in I-\{0\}}\Big(P_{\lambda;i}(0)+Q_{\lambda;i}\big(\tfrac{1}{-i-x_i}\big)\Big)=\frac{(-1)^{t(\lambda)}}{u}\prod_{i\in I-\{0\}}O_{\lambda;i}(-i). \quad (7.13)$$

The identity (7.2) implies that

$$\prod_{i\in I_{\mathrm{lo}}}O_{\lambda;i}(x-i)O_{\lambda;-i}(i-x)=(-1)^{p(\lambda)}\zeta_0(\lambda)(1-2x)\,\mathrm{id}_{\mathbb{1}_\lambda}$$

in $\widehat{Z}_\lambda[\![x]\!]$. Evaluating at $x=0$ gives that $\prod_{i\in I-\{0\}}O_{\lambda;i}(-i)=(-1)^{p(\lambda)}\zeta_0(\lambda)\,\mathrm{id}_{\mathbb{1}_\lambda}$, which is exactly what is needed to deduce that (7.13) equals $-\frac{(-1)^{p(\lambda)}}{2u}\,\mathrm{id}_{\mathbb{1}_\lambda}$. □

Corollary 7.3. *For $\lambda\in X^\imath$ and $i\in I$, we have that*

$$P_{\lambda;i}(u)\prod_{j\in I-\{i\}}O_{\lambda;j}(u-j)+Q_{\lambda;i}\bigg(\frac{\prod_{j\in I-\{i\}}O_{\lambda;j}(u-j)-\prod_{j\in I-\{i\}}O_{\lambda;j}(i-j+x_i)}{u-i-x_i}\bigg)$$
$$=\Big(1-\frac{(-1)^{p(\lambda)}}{2u}\Big)\,\mathrm{id}_{\mathbb{1}_\lambda}+\sum_{j\in I-\{i\}}Q_{\lambda;j}\bigg(\frac{\prod_{k\in I-\{j\}}O_{\lambda;k}(j-k+x_j)}{u-j-x_j}\bigg).$$

Proof. We have that $P_{\lambda;i}(u)=O_{\lambda;i}(u-i)-Q_{\lambda;i}\big(\frac{1}{u-i-x_i}\big)$. Hence,

$$P_{\lambda;i}(u)\prod_{j\in I-\{i\}}O_{\lambda;j}(u-j)=\prod_{j\in I}O_{\lambda;j}(u-j)-Q_{\lambda;i}\bigg(\frac{\prod_{j\in I-\{i\}}O_{\lambda;j}(u-j)}{u-i-x_i}\bigg).$$

Using this, the left-hand side of the identity we are trying to prove becomes

$$\prod_{j\in I}O_{\lambda;j}(u-j)-Q_{\lambda;i}\bigg(\frac{\prod_{j\in I-\{i\}}O_{\lambda;j}(i-j+x_i)}{u-i-x_i}\bigg).$$

Now we use Theorem 7.2 to see that this equals

$$\Big(1-\frac{(-1)^{p(\lambda)}}{2u}\Big)\,\mathrm{id}_{\mathbb{1}_\lambda}+\sum_{j\in I}Q_{\lambda;j}\Big(\tfrac{\prod_{k\in I-\{j\}}O_{\lambda;k}(j-k+x_j)}{u-j-x_j}\Big)-Q_{\lambda;i}\bigg(\frac{\prod_{j\in I-\{i\}}O_{\lambda;j}(i-j+x_i)}{u-i-x_i}\bigg).$$

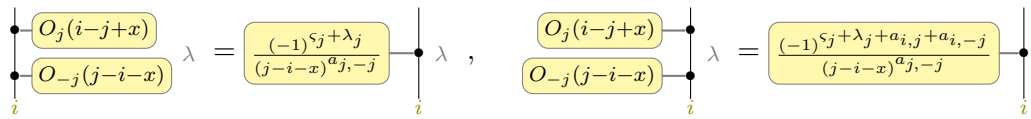
The last term cancels with the $j=i$ term in the summation, and the corollary follows. □

To further simplify notation, we adopt the convention of omitting the subscript λ when any of $O_{\lambda;i}, P_{\lambda;i}$ or $Q_{\lambda;i}$ are drawn in a pinhead in a string diagram, since the value of λ can be inferred from the label of the ambient 2-cell. Recall also that we omit 2-cell labels altogether when we are writing something that is true for all possible iweights. Theorem 7.2 suggests considering the following for $i\neq j$:

(7.14)

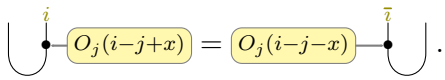
When the right-hand 2-cell is labelled λ, the first of these is the series $O_{\lambda;j}(i-j+x)\in\widehat{Z}_\lambda[\![x]\!]$ from (7.9) pinned to the i-string, and the others have an analogous meaning. We refer to these elements of $\mathrm{End}_{\widehat{\mathfrak{U}}^\imath}(B_i\mathbb{1}_\lambda)$ collectively as *internal bubbles*.

When pinned to the same string, it is obvious that all internal bubbles commute with each other and with dots. From (3.22), we get that



(7.15)

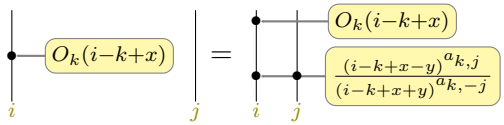
This is a place in which we cannot omit the 2-cell label λ since the coefficients on the right-hand side depend on it! Internal bubbles can be moved across cups and caps, simply negating the sign of the variable x. For example:



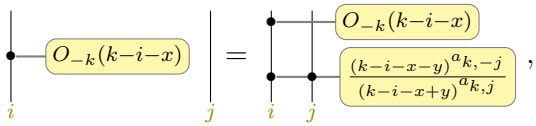
(7.16)

The next two lemmas give *internal bubble slides*.

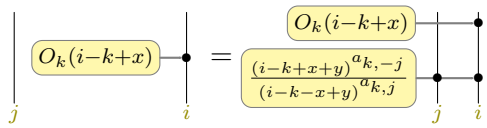
Lemma 7.4. *For $i, j, k \in I$ with $i \neq k$, we have that*



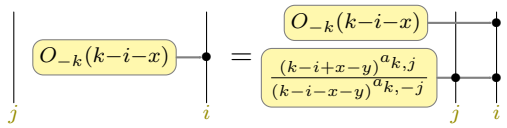
(7.17)



(7.18)

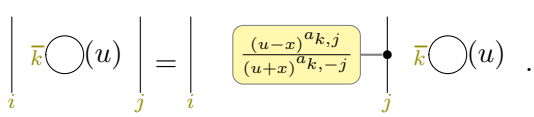


(7.19)



(7.20)

Proof. These are all proved in a similar way, so we just explain the first identity. By (3.23), we have that

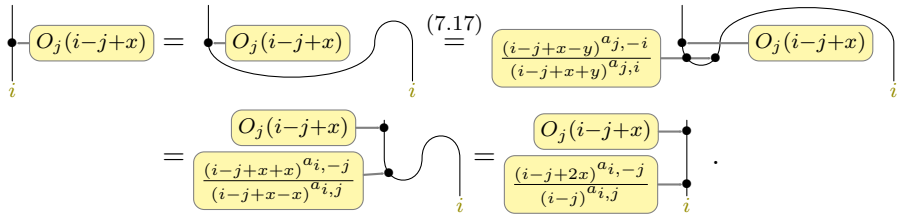


Then we replace u by $i - k + x$ pinned to the string of color i, also replacing the x pinned to the string of color j by y to take account of the pin changing to a double pin. □

Lemma 7.5. *For any $i \neq j$, we have that*

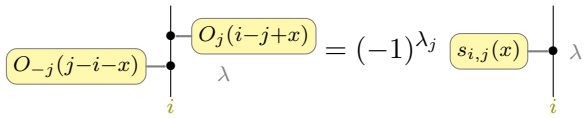
(7.21)

Proof. It suffices to prove the first equality. Then the second can be deduced using (3.15) and (7.15) (or using (3.24) and (7.16)). Here is the proof of the first equality:



□

From (7.15) and Lemma 7.5, it follows that



(7.22)

for any $i \neq j$ in I, where

$$s_{i,j}(x) := (-1)^{\varsigma_j} \frac{(j-i-2x)^{a_{i,-j}}}{(j-i)^{a_{i,j}}(j-i-x)^{a_{j,-j}}} \in \Bbbk[\![x]\!]. \tag{7.23}$$

As $\varsigma_j + \varsigma_{-j} = -a_{j,-j}$, we have that

$$s_{i,j}(x) = (-1)^{a_{i,j}+a_{i,-j}} s_{-i,-j}(-x). \tag{7.24}$$

Lemma 7.6. *For any $i \in I$, we have that* $\prod_{j \in I-\{i\}} s_{i,j}(x) = \zeta_i \zeta_{-i} \gamma_{-i,i}$.

Proof. This is a direct calculation. The product $\prod_{j \in I-\{i\}} s_{i,j}(x)$ is the product of the following four terms:

$$A := \prod_{j \in I-\{i\}} (-1)^{\varsigma_j}, \quad B := \prod_{j \in I-\{i\}} (j-i)^{-a_{i,j}}, \quad C := \prod_{j \in I-\{i\}} (j-i-2x)^{a_{i,-j}}, \quad D := \prod_{j \in I-\{i\}} (j-i-x)^{-a_{j,-j}}.$$

Only $j = i \pm 1$ contribute to B, so $B = -1$ always. For the other three products, only $j = 0, h$ contribute to A, only $j = -i, -i \pm 1$ contribute to C, and only $j = 0, \pm h$ contribute to D. When $i \notin \{0, \pm h\}$, we have that $A = 1, B = -1, C = \frac{4(i+x)^2}{4(i+x)^2-1}$ and $D = \frac{(i+x)^2-h^2}{(i+x)^2}$ so $ABCD = -1$, which equals $\zeta_i \zeta_{-i} \gamma_{-i,i} = -1$ in this case. We leave the check of the exceptional cases $i = 0$ and $i = \pm h$ to the reader. □

In the case $i = 0$, we need the following more refined statement:

Lemma 7.7. $\prod_{j \in I_{\mathrm{hi}}} s_{0,j}(x) = \zeta_0(h+x)/(h-x)$ *and* $\prod_{j \in I_{\mathrm{lo}}} s_{0,j}(x) = \zeta_0(h-x)/(h+x)$.

Proof. This is a similar calculation to the one in Lemma 7.6. In the notation from that proof, when $i = 0$, the only $j \neq 0$ that contribute are $j = h \in I_{\mathrm{lo}}$ (to A and D), $j = -h \in I_{\mathrm{hi}}$ (to D), and $j = \pm 1$ (to B and C), remembering that $1 \in I_{\mathrm{hi}}$ and $-1 \in I_{\mathrm{lo}}$. □

Now we are going to consider the following infinite products for $i \in I$ and $\lambda \in X^\imath$:

$$\Delta_{\lambda;i}(x) := \begin{cases} (-1)^{p(\lambda)} \prod_{j \in I-\{i\}} O_{\lambda;j}(i-j+x) & \text{if } i \in I_+ \\ \dfrac{1}{h+x} \prod_{j \in I_{\mathrm{hi}}} O_{\lambda;j}(-j+x) & \text{if } i = 0 \\ \mathrm{id}_{\mathbb{1}_\lambda} & \text{if } i \in I_-, \end{cases} \tag{7.25}$$

$$\nabla_{\lambda;i}(x) := \begin{cases} \mathrm{id}_{\mathbb{1}_\lambda} & \text{if } i \in I_+ \\ (-1)^{p(\lambda)} \zeta_0(\lambda)^{-1}(h+x) \prod_{j \in I_{\mathrm{lo}}} O_{\lambda;j}(-j+x) & \text{if } i = 0 \\ (-1)^{p(\lambda)} \zeta_i(\lambda)^{-1} \prod_{j \in I-\{i\}} O_{\lambda;j}(i-j+x) & \text{if } i \in I_-. \end{cases} \tag{7.26}$$

These are all elements of $\widehat{Z}_\lambda[\![x]\!]$. Also let

$$\diamondsuit_{\lambda;i}(x) := \Delta_{\lambda;i}(x) \nabla_{\lambda;i}(x) = (-1)^{p(\lambda)} \zeta_i(\lambda)^{-1} \prod_{j \in I-\{i\}} O_{\lambda;j}(i-j+x). \tag{7.27}$$

From the definitions, (3.16), (7.1) and (7.2), we have that

$$\Delta_{\lambda;-i}(-x)\nabla_{\lambda;i}(x) = \begin{cases} 1 & \text{if } i \in I_+ \\ 2x+1 & \text{if } i = 0 \\ \dfrac{x-1}{2(x-h)^2} & \text{if } i = -h \\ \dfrac{1}{4(i+x)^2} - 1 & \text{if } i \in I_- - \{-h\}. \end{cases} \tag{7.28}$$

Hence, $\Delta_{\lambda;i}(x)$, $\nabla_{\lambda;i}(x)$ and $\Diamond_{\lambda;i}(x)$ are all invertible in $\widehat{Z}_\lambda[\![x]\!]$. Moreover, setting $x = 0$ in the $i = 0$ case of (7.28), we have that

$$\Diamond_{\lambda;0}(0) = 1. \tag{7.29}$$

Again we use the convention of omitting the subscript λ when $\Delta_{\lambda;i}(x)$, $\nabla_{\lambda;i}(x)$ or $\Diamond_{\lambda;i}(x)$ is pinned to a dot in a string diagram, since the intended value of λ can be inferred from the label of the 2-cell containing the pinhead. We use the following additional shorthands, which we refer to as *glyphs*:

(7.30)

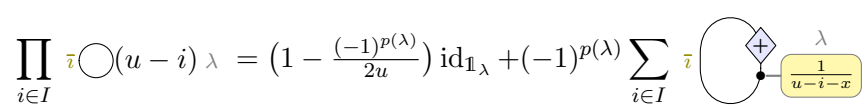
(7.31)

The convention here is that the glyphs labelled + are shorthand for a pin in the 2-cell immediately to the right of the glyph, while the glyphs labelled − are pins to the left. There are also the *double glyphs*:

(7.32)

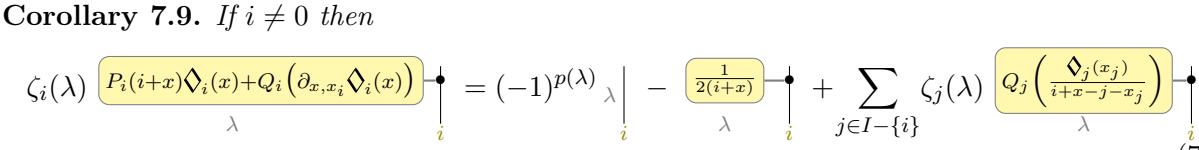
The following is a restatement of Theorem 7.2 using the new glyph notation:

Theorem 7.8. *For any $\lambda \in X^\imath$, we have that*



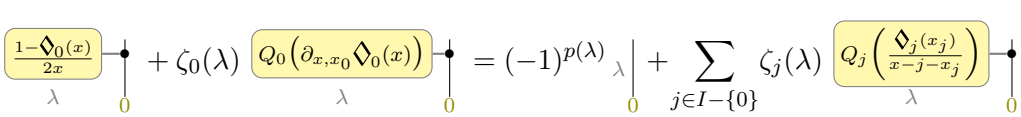
$$= \left(1 - \frac{(-1)^{p(\lambda)}}{2u}\right)\mathrm{id}_{\mathbb{1}_\lambda} + (-1)^{p(\lambda)} \sum_{i\in I} \tag{7.33}$$

Corollary 7.9. *If $i \neq 0$ then*



$$\zeta_i(\lambda) \quad = (-1)^{p(\lambda)} \quad - \quad + \sum_{j\in I-\{i\}} \zeta_j(\lambda) \quad . \tag{7.34}$$

Also



$$+ \zeta_0(\lambda) \quad = (-1)^{p(\lambda)} \quad + \sum_{j\in I-\{0\}} \zeta_j(\lambda) \quad . \tag{7.35}$$

Proof. The formula (7.34) when $i \neq 0$ follows by taking the identity from Corollary 7.3, multiplying by $(-1)^{p(\lambda)}$, then evaluating u at λ $i+x$ i. If $i = 0$, it does not make sense to evaluate u at x directly, so first we rearrange the identity from Corollary 7.3 using that $P_{\lambda;0}(u) = \frac{(-1)^{t(\lambda)}}{u} = -\frac{\zeta_0(\lambda)^{-1}}{2u}$ to obtain

$$(-1)^{p(\lambda)}\frac{\mathrm{id}_{\mathbb{1}_\lambda}-(-1)^{p(\lambda)}\zeta_0(\lambda)^{-1}\prod_{j\in I-\{0\}}O_{\lambda;j}(u-j)}{2u}$$
$$+Q_{\lambda;0}\left(\frac{\prod_{j\in I-\{0\}}O_{\lambda;j}(u-j)-\prod_{j\in I-\{0\}}O_{\lambda;j}(-j+x_0)}{u-x_0}\right)$$
$$=\mathrm{id}_{\mathbb{1}_\lambda}+\sum_{j\in I-\{0\}}Q_{\lambda;j}\left(\frac{\prod_{k\in I-\{j\}}O_{\lambda;k}(j-k+x_j)}{u-j-x_j}\right).$$

Then we multiply by $(-1)^{p(\lambda)}$ and evaluate u at as before. For the first term, this evaluation makes sense because (7.29) implies that the numerator is divisible by the denominator (after evaluation). This proves (7.35). □

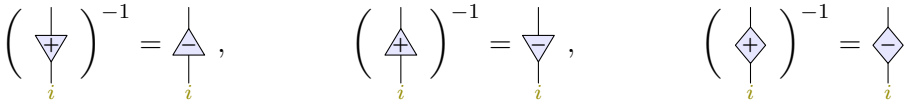
In the rest of the section, we derive some further properties of the glyphs. Using (7.16) and its variations, it follows that

(7.36)

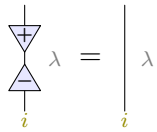
(7.37)

(7.38)

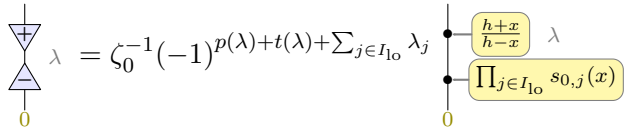
Lemma 7.10. *We have that*

 (7.39)

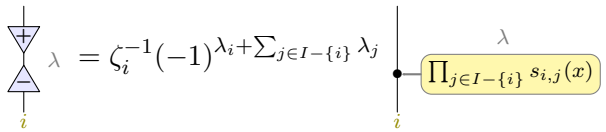
Proof. In view of (7.36), it suffices to prove the first equality, so we must show that



for every $\lambda\in X^\imath$. This is immediate from the definitions if $i\in I_+$. If $i=0$, the definitions and (7.22) imply that


$$=\zeta_0^{-1}(-1)^{p(\lambda)+t(\lambda)+\sum_{j\in I_{\mathrm{lo}}}\lambda_j}$$

which equals $\mathrm{id}_{B_0\mathbb{1}_\lambda}$ thanks to Lemma 7.7. If $i\in I_-$, we instead get that


$$=\zeta_i^{-1}(-1)^{\lambda_i+\sum_{j\in I-\{i\}}\lambda_j}$$

Since $\gamma_{-i,i}=\zeta_{-i}=1$, $\lambda_0=0$ and $\lambda_j+\lambda_{-j}=0$ for all $j\in I$, Lemma 7.6 implies that this equals $\mathrm{id}_{B_i\mathbb{1}_\lambda}$. □

Lemma 7.11. *For $i,j\in I$, we have that*

$$\frac{\gamma_{j,i}}{\gamma_{i,j}}\left(\prod_{k\in I-\{i\}}\frac{(i-k+x+y)^{a_{k,-j}}}{(i-k-x+y)^{a_{k,j}}}\right)\left(\prod_{k\in I-\{j\}}\frac{(j-k+x-y)^{a_{k,i}}}{(j-k+x+y)^{a_{k,-i}}}\right)=r_{i,-j}r_{-i,j}r_{j,i}r_{-j,-i}.$$

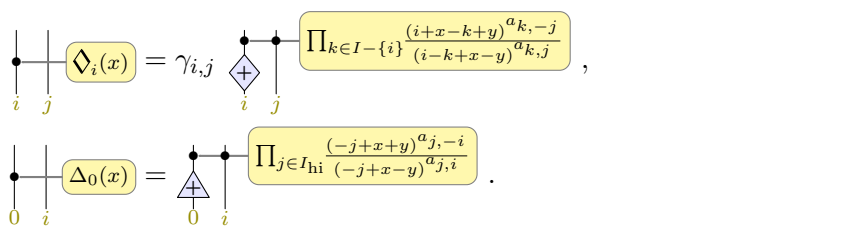
Proof. Let $\phi(z) := \frac{z^2}{z^2-1}$. Some elementary calculation gives that

$$\prod_{k\in I-\{i\}} \frac{(i-k+x+y)^{a_{k,-j}}}{(i-k-x+y)^{a_{k,j}}} = \frac{\phi(i+j+x+y)}{\phi(i-j-x+y)} \cdot \frac{(y-x)^{a_{i,j}}}{(x+y)^{a_{i,-j}}},$$

$$\prod_{k\in I-\{j\}} \frac{(j-k+x-y)^{a_{k,i}}}{(j-k+x+y)^{a_{k,-i}}} = \frac{\phi(i-j-x+y)}{\phi(i+j+x+y)} \cdot \frac{(x+y)^{a_{j,-i}}}{(x-y)^{a_{j,i}}}.$$

The product of these is $(-1)^{a_{i,j}}$. The lemma now follows using Lemma 3.4. □

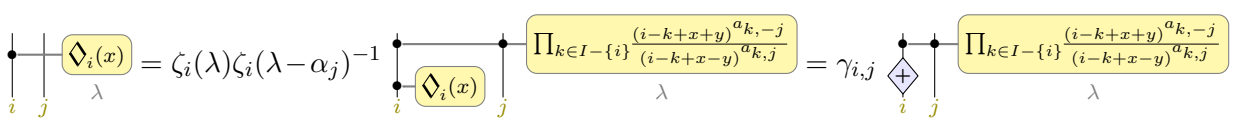
Lemma 7.12. *We have that*



(7.40)

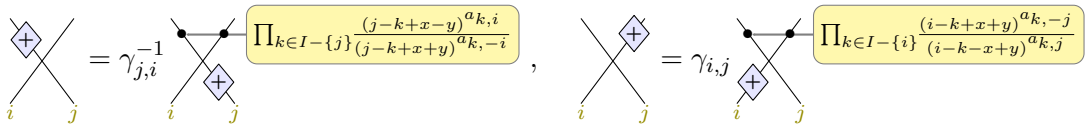
(7.41)

Proof. To prove the first identity, we use (7.17) for each of the pins in the product of internal bubbles from the definition (7.32):



The proof of the second identity is similar. □

Lemma 7.13. (i) *For $i \neq \pm j$, we have that*

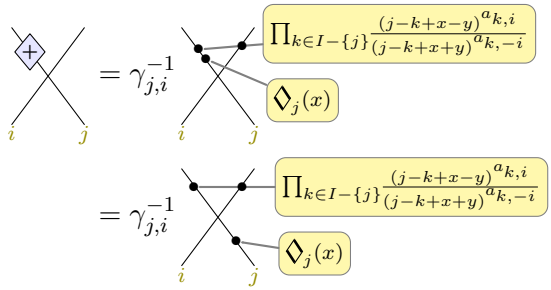


(7.42)

(ii) *If $i \neq 0$ then*

(7.43)

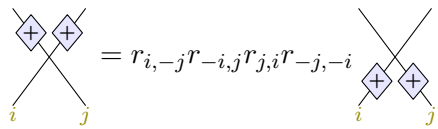
Proof. (i) By Lemma 7.12, we have that



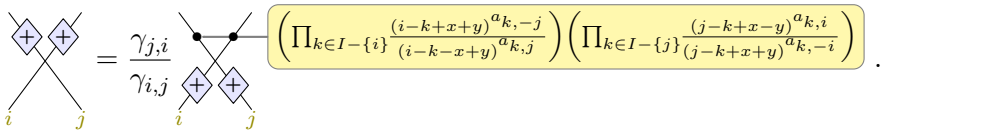
This proves the first equality. The proof of the second equality is similar.

(ii) Similar. □

Lemma 7.14. *For $i, j \in I$ with $(i, j) \neq (0, 0)$, we have that*

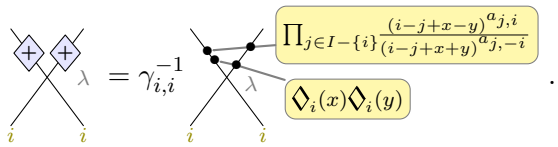


Proof. We first prove this in the case $i \neq \pm j$. By Lemma 7.13(i), we have that



It remains to apply Lemma 7.11.

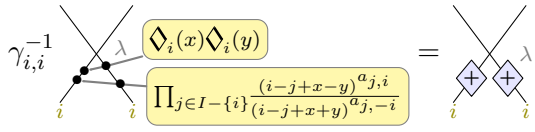
Next, we prove the lemma in the case $i = j \neq 0$. Expanding the definitions and using Lemma 7.12 gives that



Now we observe that both $\Diamond_{\lambda;i}(x)\Diamond_{\lambda;i}(y)$ and $\prod_{j\in I-\{i\}} \frac{(i-j+x-y)^{a_{j,i}}}{(i-j+x+y)^{a_{j,-i}}}$ are symmetric in x and y. To see this for the latter expression, the denominator is evidently symmetric. The numerator is symmetric because the only non-trivial factors arise from $j = i \pm 1$, these being

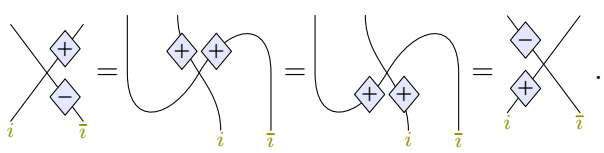
$$(i - (i + 1) + x - y)^{-1}(i - (i - 1) + x - y)^{-1} = (x - y - 1)^{-1}(x - y + 1)^{-1} = ((x - y)^2 - 1)^{-1},$$

which is symmetric in x and y. In view of this symmetry, (3.36) implies that both of the double pins can be commuted past the crossing to obtain



Since $r_{i,-i} r_{-i,i} r_{i,i} r_{-i,-i} = 1$, this completes the proof in this case.

Finally, to prove it when $i = -j \neq 0$, we take the $i = j \neq 0$ case just proved and rotate to get that



The desired relation follows easily from this using Lemma 7.10. □

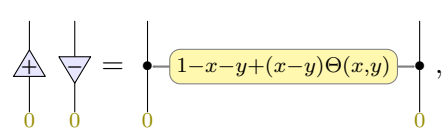
To prove further results about the glyphs on a 0-string, we let

$$D_\lambda(x, y) := \partial_{x,y}\Delta_{\lambda;0}(x) = \frac{\Delta_{\lambda;0}(x) - \Delta_{\lambda;0}(y)}{x - y}, \tag{7.44}$$

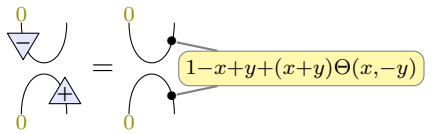
$$\Theta_\lambda(x, y) := 1 + D_\lambda(x, y)\nabla_{\lambda;0}(-y). \tag{7.45}$$

Both are elements of $\widehat{Z}_\lambda[\![x, y]\!]$, and $D_\lambda(x, y) = D_\lambda(y, x)$. Again, we drop the subscript λ when these are pinned into a string diagram.

Lemma 7.15. *For $\lambda \in X^\imath$, we have that*



(7.46)

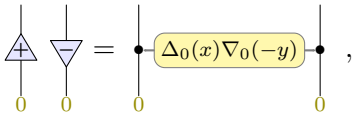


(7.47)

Proof. The identity (7.47) follows from (7.46) by attaching a cap to the northwest of the diagram and a cup to the southeast then simplifying in an obvious way. To prove (7.46), rearranging (7.44) and using (7.28) gives

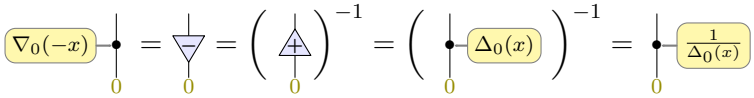
$$\Delta_{\lambda;0}(x)\nabla_{\lambda;0}(-y) = 1 - x - y + (x-y)\Theta_\lambda(x,y). \tag{7.48}$$

Since

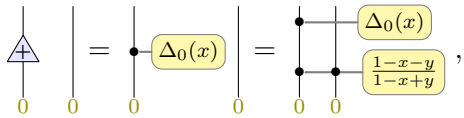


the identity (7.46) follows from (7.48). □

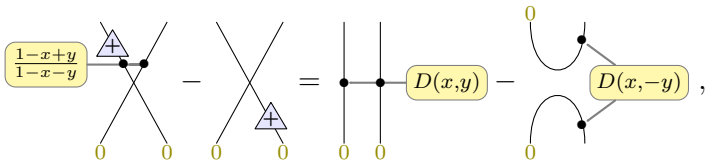
Some more relations for the glyphs on a 0-string:



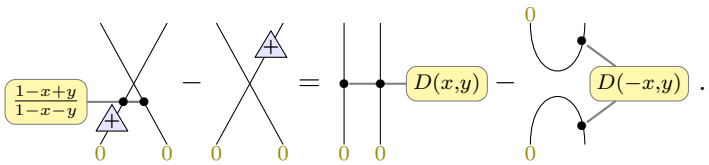
(7.49)



(7.50)



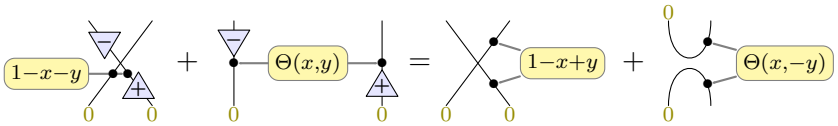
(7.51)



(7.52)

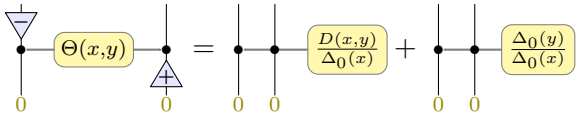
For the first of these, we applied Lemma 7.10. The second follows from the definitions and (7.17), noting that the only $i \in I_{\mathrm{hi}}$ with $a_{i,0} \neq 0$ is $i = 1$. The third follows by first applying (7.50) then using (3.33). The fourth is proved in a similar way to the third. These relations are needed to prove the following:

Lemma 7.16. *For any $\lambda \in X^\imath$, we have that*



(7.53)

Proof. We first show that

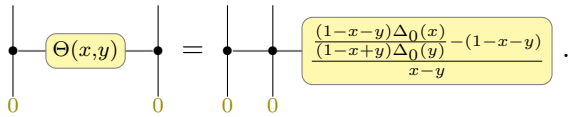


(7.54)

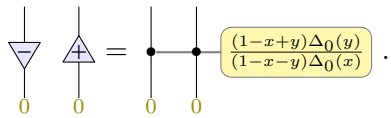
To see this, we have by its definition that

$$\Theta_\lambda(x,y) = 1 + \frac{\Delta_{\lambda;0}(x)\nabla_{\lambda;0}(-y) - \Delta_{\lambda;0}(y)\nabla_{\lambda;0}(-y)}{x-y} \overset{(7.28)}{=} \frac{\Delta_{\lambda;0}(x)\nabla_{\lambda;0}(-y) - (1-x-y)}{x-y}.$$

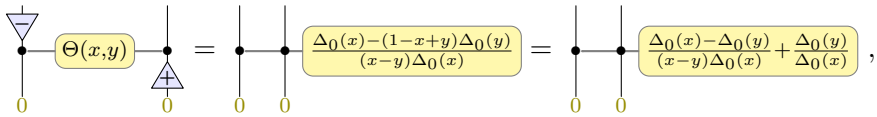
Using this, (7.49) and (7.50), it follows that



Also (7.49) and (7.50) imply that

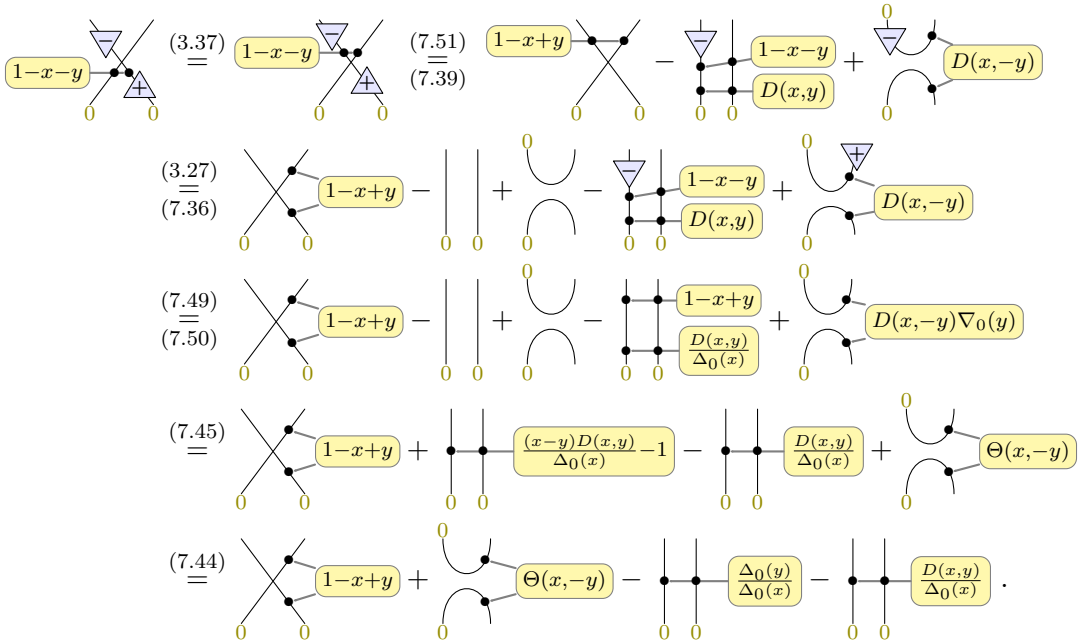


Composing these two equations gives



which is (7.54).

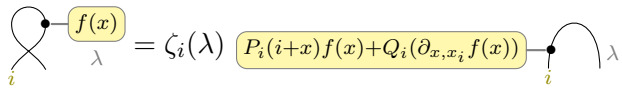
Now consider the first term on the left-hand side of (7.53):



Adding this equation to (7.54) completes the proof of the lemma. □

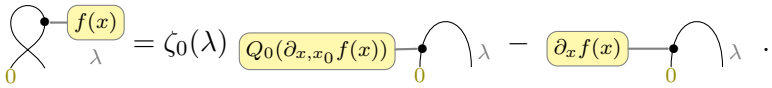
Lemma 7.17. *Suppose that $\lambda \in X^\imath$ and $f(x) \in \widehat{Z}_\lambda[\![x]\!]$.*

(i) *For any $i \in I - \{0\}$, we have that*



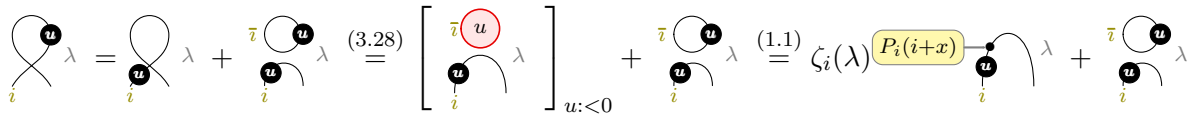
(7.55)

(ii) *We have that*



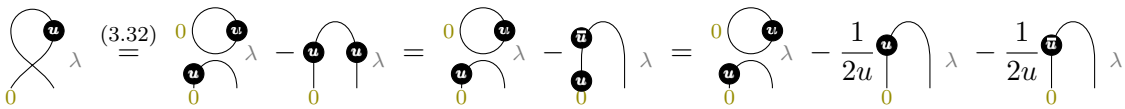
(7.56)

Proof. (i) By (3.32) (actually, its mirror image), we have that



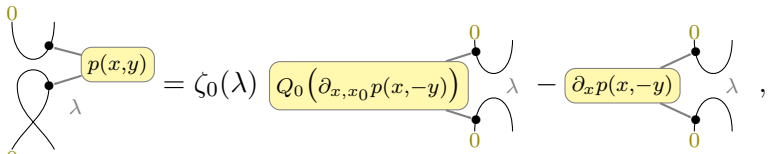
Now we multiply this by $f(u)$ then take the u^{-1}-coefficients using (1.1) and (1.4), extended continuously to formal power series.

(ii) We again start by sliding the dot generating function past the crossing. Now the undotted curl is 0 due to (3.28), but we get an extra error term:

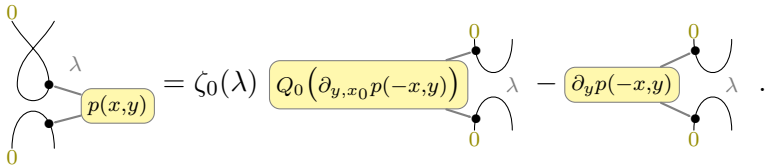


The final equality here used $\frac{1}{(u-x)(u+x)} = \frac{1}{2u}\left(\frac{1}{u-x} + \frac{1}{u+x}\right)$. Then once again we multiply by $f(u)$ and take u^{-1}-coefficients. □

Corollary 7.18. *For any $\lambda \in X^\imath$ and $p(x,y) \in \widehat{Z}_\lambda[\![x,y]\!]$, we have that*



(7.57)



(7.58)

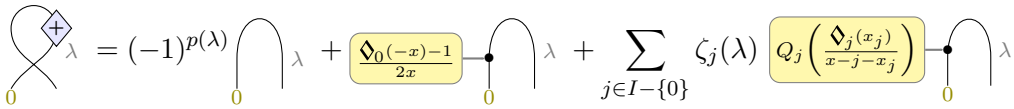
Proof. The first equality follows from Lemma 7.17(ii) and (3.25). The second follows by rotating the first through 180°. □

Corollary 7.19. *Suppose that $\lambda \in X^\imath$.*

(i) *For $i \in I - \{0\}$, we have that*

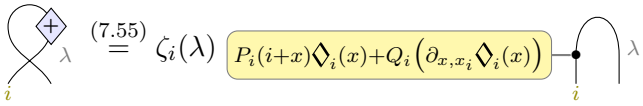
(7.59)

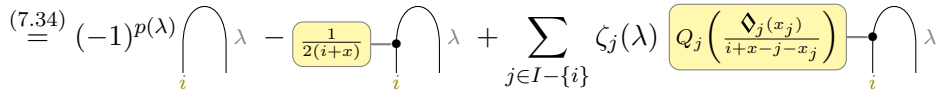
(ii) *We have that*



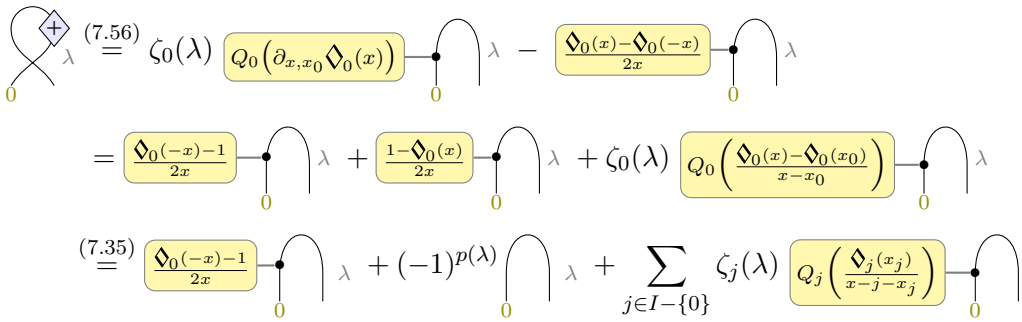
(7.60)

Proof. (i) We have that



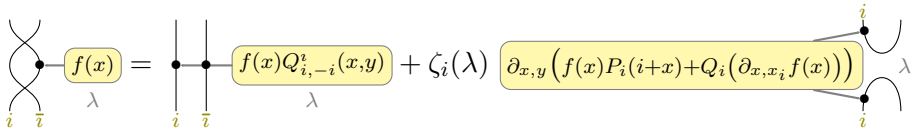


(ii) We have that



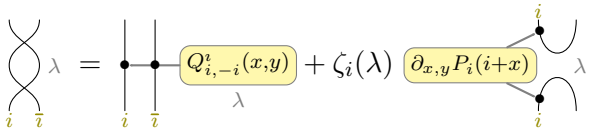
The second equality here makes sense because of (7.29). □

Lemma 7.20. *For $i \in I - \{0\}$, $\lambda \in X^\imath$ and $f(x) \in \widehat{Z}_\lambda[\![x]\!]$, we have that*

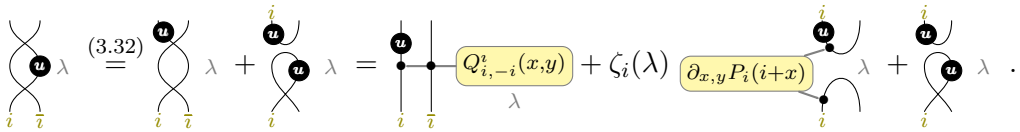


(7.61)

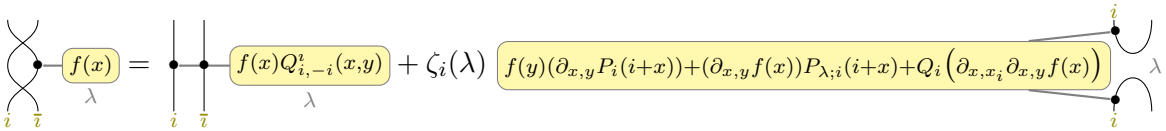
Proof. It suffices to prove this in the case that $f(x) \in \widehat{Z}_\lambda[x]$, since it then extends by continuity to all power series. We note to start with that



as follows from (3.29) using (1.4). Using this for the second equality, it follows that



Now we multiply this by $f(u)$ and take the u^{-1}-coefficients using (1.1), (1.4) and Lemma 7.17 to deduce that

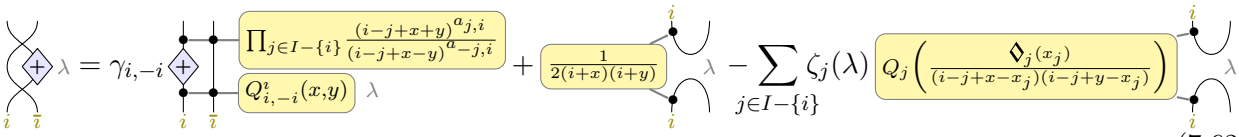


It remains to observe that

$$f(y)\big(\partial_{x,y}P_{\lambda;i}(i+x)\big) + \big(\partial_{x,y}f(x)\big)P_{\lambda;i}(i+x) = \partial_{x,y}\big(f(x)P_{\lambda;i}(i+x)\big)$$

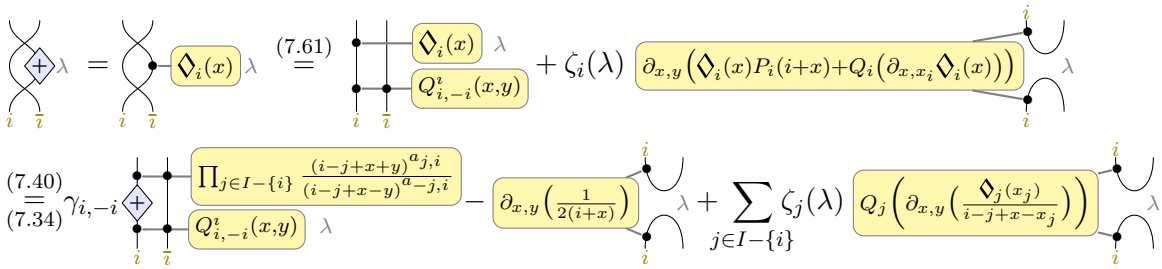
by the twisted Leibniz rule for Demazure operators, and $Q_{\lambda;i}(\partial_{x,x_i}\partial_{x,y}f(x)) = \partial_{x,y}Q_{\lambda;i}(\partial_{x,x_i}f(x))$. □

Corollary 7.21. *For $i \in I - \{0\}$ and any λ, we have that*



(7.62)

Proof. We have that



It remains to note that

$$\partial_{x,y}\left(\frac{1}{2(i+x)}\right) = -\frac{1}{2(i+x)(i+y)},$$
$$\partial_{x,y}\left(\frac{\diamondsuit_j(x_j)}{i-j+x-x_j}\right) = -\frac{\diamondsuit_j(x_j)}{(i-j+x-x_j)(i-j+y-x_j)}. \qquad \square$$

8. Iquantum to Brauer

In the rest of the article, we are going to show that the notions of "Brauer categorification" and "even nilpotent iquantum categorification" are equivalent in such a way that the following hold:

- The canonical decomposition $\mathbf{R} = \bigoplus_{\lambda \in X^\imath} \mathbf{R}_\lambda$ of an iquantum categorification is the same as the "block" decomposition of $\mathbf{R}$ viewed as a Brauer categorification from (4.11) and (4.12), interpreting $\mathbf{R}_\lambda$ as zero for $\lambda \in X^\imath_{\bar{1}}$.
- The action of the functor B_i arising from the $\mathfrak{U}^\imath$-action (whose identity endomorphism is a black string labelled by i) is the same as the eigenfunctor B_i arising from the $\mathbf{AB}$-action (whose identity endomorphism is a magenta string labelled by i).
- The natural transformations defined by the dots on the two sides are related by

(8.1)

From this, it follows that the β-numbers and β-degree of an irreducible object of $\mathbf{R}^{\mathrm{fg}}$ defined via the Brauer action are the same as its β-numbers and β-degree defined via the iquantum action.

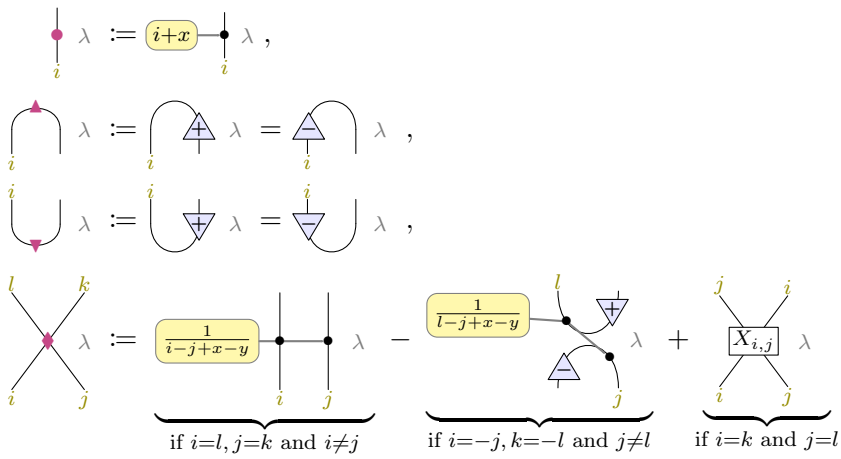
In this section, we start from the iquantum side, proving Theorem A from the introduction. In fact, we prove a stronger result formulated in terms of the topological monoidal category $\widehat{\mathbf{AB}}$ from Definition 6.6: we will show that there is a strict $\Bbbk$-linear monoidal functor $\Psi\colon \mathbf{AB} \to \widehat{\mathbf{AB}}$ which sends the generating object B of $\mathbf{AB}$ to the object B of $\widehat{\mathbf{AB}}$. Any even nilpotent iquantum categorification can be made into a strict left $\widehat{\mathbf{AB}}$-module category in an obvious way, so Theorem A will then follow simply by restricting the action of $\widehat{\mathbf{AB}}$ to $\mathbf{AB}$ along Ψ.

To prepare for the statement of the main theorem, we first introduce some notation which will be used to denote the images of the generating morphisms of $\mathbf{AB}$ under the functor Ψ. Recall $\Theta_\lambda(x,y) \in \widehat{Z}_\lambda[\![x,y]\!]$ from (7.45) (which is denoted simply $\Theta(x,y)$ when drawn in a 2-cell labelled λ in some string diagram). For $i,j \in I$ and $\varepsilon,\delta \in \{+,-\}$ such that ($\varepsilon i \in I_+ \cup \{0\}$ and $\delta j \in I_+$) or

$\varepsilon i = \delta j = 0$, let

$$G^{\varepsilon\delta}_{i,j}(x,y) := \begin{cases} 1 + \dfrac{1}{\varepsilon(i+x) - \delta(j+y)} & \text{if } \varepsilon i \notin \{0, \delta j, \delta j - 1\} \\ \dfrac{t^{-1}_{\varepsilon i, \delta j}}{\varepsilon(i+x) - \delta(j+y)} & \text{if } \varepsilon i = \delta j - 1 \neq 0 \\ 1 + \varepsilon(i+x) - \delta(j+y) & \text{if } \varepsilon i = \delta j \\ \displaystyle\prod_{k \in \delta I_{\mathrm{lo}}} \left(\sqrt{-1}(\varepsilon x - \delta(k+y))\right)^{-a_{k,j}} & \text{if } \varepsilon i = 0 \neq \delta j, \end{cases} \tag{8.2}$$

a power series in $\Bbbk[\![x,y]\!]^\times$. For $\lambda \in X^\imath_0$ and $i,j,k,l \in I$ such that $\lambda - \alpha_i - \alpha_j = \lambda - \alpha_k - \alpha_l$, let



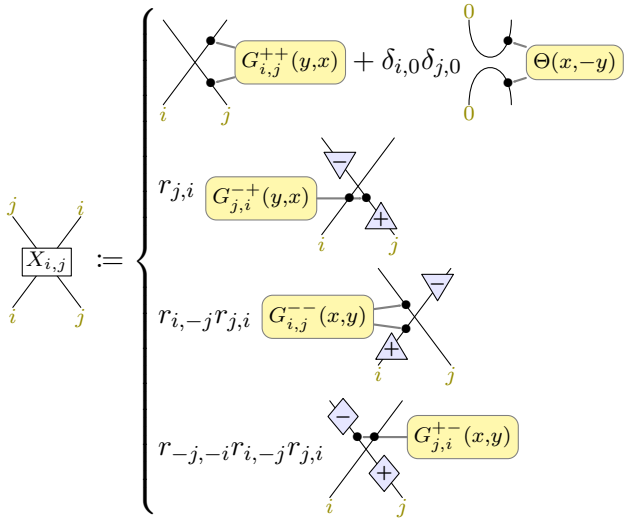
(8.3)

(8.4)

(8.5)

(8.6)

omitting the terms on the right-hand side of (8.6) if the indicated conditions are not satisfied, where



if $(i \in I_+ \cup \{0\}$ and $j \in I_+)$ or $i = j = 0$

if $i \in I_+$ and $j \in I_- \cup \{0\}$

if $i \in I_- \cup \{0\}$ and $j \in I_-$

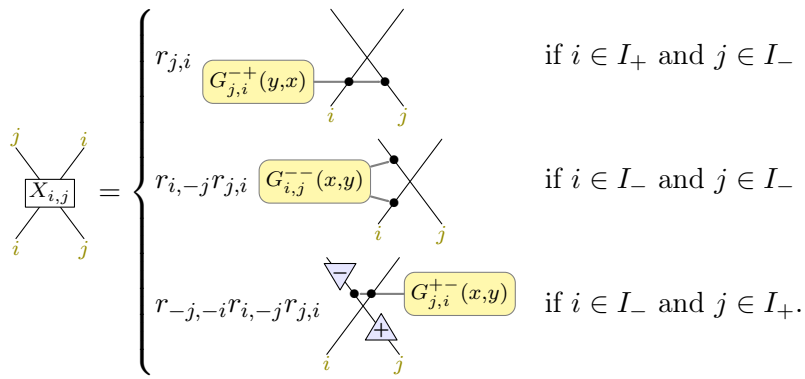
if $i \in I_-$ and $j \in I_+ \cup \{0\}$.

(8.7)

These are all 2-morphisms in $\widehat{\mathfrak{U}}^\imath$. In (8.3) to (8.6), we have been careful to include the required iweight λ in the rightmost 2-cell. We have not done that in (8.7), but we are following our usual convention that when no 2-cell label is present we are writing an equation which is true for all possible labels.

Remark 8.1. We have written the definition of $X_{i,j}$ in the most case-free way that we can. When i and j are non-zero, some of the glyphs in the last three cases in (8.7) can be omitted since they

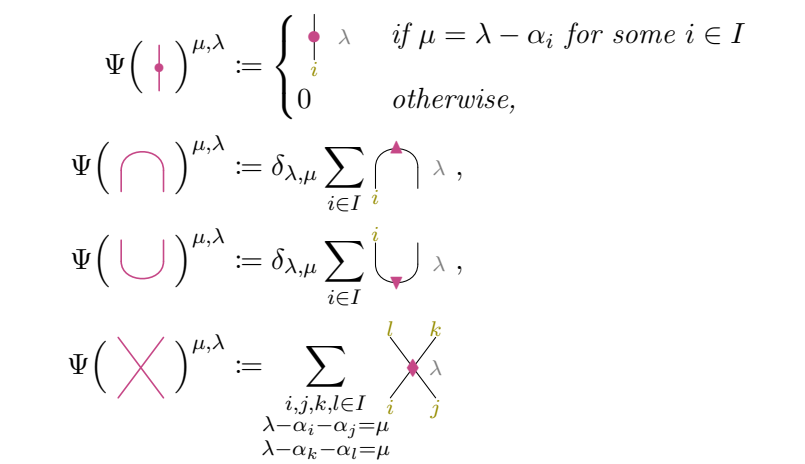
are identities according to the definitions (7.30) and (7.31): we have that



(8.8)

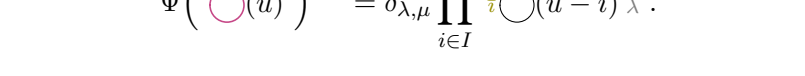
Recall from Definition 6.6 that a morphism $\theta\colon B^{\otimes n} \to B^{\otimes m}$ in $\widehat{\mathbf{AB}}$ is a matrix $\theta = (\theta^{\mu,\lambda})_{\mu,\lambda \in X_0^{\imath}}$ with (μ,λ)-entry $\theta^{\mu,\lambda} \in \mathrm{Hom}_{\widehat{\mathfrak{U}}^{\imath}}(B_{\boldsymbol{J}}\mathbb{1}_\lambda, B_{\boldsymbol{I}}\mathbb{1}_\lambda)$ for $\boldsymbol{I} := \{\boldsymbol{i} \in I^m \mid \lambda - \alpha_{\boldsymbol{i}} = \mu\}$ and $\boldsymbol{J} := \{\boldsymbol{j} \in I^n \mid \lambda - \alpha_{\boldsymbol{j}} = \mu\}$. Each $\theta^{\mu,\lambda}$ is itself a matrix $\theta^{\mu,\lambda} = (\theta^{\mu,\lambda}_{\boldsymbol{i},\boldsymbol{j}})_{\boldsymbol{i}\in\boldsymbol{I},\boldsymbol{j}\in\boldsymbol{J}}$ with $\theta^{\mu,\lambda}_{\boldsymbol{i},\boldsymbol{j}} \in \widehat{H}_\lambda(\boldsymbol{j},\boldsymbol{i})$. In the following theorem, we will denote such a matrix also by the formal sum $\sum_{\boldsymbol{i}\in\boldsymbol{I},\boldsymbol{j}\in\boldsymbol{J}} \theta^{\mu,\lambda}_{\boldsymbol{i},\boldsymbol{j}}$.

Theorem 8.2. *There is a strict $\Bbbk$-linear monoidal functor $\Psi\colon \mathbf{AB} \to \widehat{\mathbf{AB}}$ mapping the generating object B of $\mathbf{AB}$ to the object B of $\widehat{\mathbf{AB}}$, and defined on generating morphisms of $\mathbf{AB}$ so that*



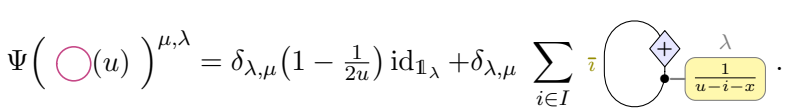
(8.9) (8.10) (8.11) (8.12)

for $\lambda, \mu \in X_0^{\imath}$. Moreover,



(8.13)

Proof. We first explain how to deduce (8.13). By the definition (2.10), we have that $(u) = \left(1 - \frac{1}{2u}\right)\mathrm{id}_{\mathbb{1}} +$ $\frac{1}{u-x}$, so by (8.3) to (8.5) we have that



Since λ is even, the right-hand side here is the same as the right-hand side of (7.33) multiplied by $\delta_{\lambda,\mu}$. So (8.13) follows by Theorem 7.8. To complete the proof of Theorem 8.2, it remains to show that all of the affine Brauer relations are satisfied. This will be done in a series of lemmas in the remainder of the section. $\square$

Corollary 8.3. *Any even nilpotent Abelian (resp., Karoubian) iquantum categorification* $\mathbf{R}$ *can be made into an Abelian (resp., Karoubian) Brauer categorification in a canonical way.*

Proof. First apply the construction of Corollary 6.9 to make $\mathbf{R}$ into a strict left $\widehat{\mathbf{AB}}$-module category, then restrict the categorical action from $\widehat{\mathbf{AB}}$ to $\mathbf{AB}$ using the monoidal functor Ψ from Theorem 8.2. □

The remainder of the section will consist of a sequence of lemmas required to complete the proof of Theorem 8.2. We begin by establishing some properties of the power series $G^{\varepsilon\delta}_{i,j}(x,y)$. Note that $G^{\varepsilon\delta}_{i,j}(x,y)$ depends only on $\varepsilon i, \varepsilon x, \delta j$ and δy, hence, simultaneously replacing (ε, i, x) by $(-\varepsilon, -i, -x)$ or (δ, j, y) by $(-\delta, -j, -y)$ leaves $G^{\varepsilon\delta}_{i,j}(x,y)$ unchanged. Also, if $\varepsilon \neq \delta$ then $G^{\varepsilon\delta}_{i,j}(x,y) = G^{\varepsilon\delta}_{i,j}(y,x)$.

Lemma 8.4. (i) *For $i, j \in I_+$ with $i \neq j$, we have that*

$$Q^{\imath}_{i,j}(x,y)G^{++}_{i,j}(x,y)G^{++}_{j,i}(y,x) = 1 - \frac{1}{(i-j+x-y)^2}.$$

(ii) *For $i \in I_+$ and $j \in I_-$ with $i \neq -j$, we have that*

$$r_{j,i}r_{-i,-j}r_{j,-i}r_{i,j}Q^{\imath}_{i,j}(x,y)G^{+-}_{i,j}(x,y)G^{-+}_{j,i}(y,x)\prod_{k\in I-\{i\}}\frac{(i-k+x+y)^{a_{k,-j}}}{(i-k+x-y)^{a_{k,j}}} = 1 - \frac{1}{(i-j+x-y)^2}.$$

(iii) *For $i \in I_+$, we have that*

$$Q^{\imath}_{i,-i}(x,y)G^{+-}_{i,-i}(x,y)G^{-+}_{-i,i}(y,x)\prod_{j\in I-\{i\}}\frac{(i-j+x+y)^{a_{j,i}}}{(i-j+x-y)^{a_{j,-i}}} = 1 - \frac{1}{(2i+x-y)^2}.$$

(iv) *For $j \in I_+$, we have that*

$$r_{0,j}Q^{\imath}_{0,j}(x,y)G^{++}_{0,j}(x,y)G^{-+}_{0,j}(y,x)\prod_{k\in I_{\mathrm{hi}}}\frac{(-k+x+y)^{a_{k,-j}}}{(-k+x-y)^{a_{k,j}}} = 1 - \frac{1}{(-j+x-y)^2}.$$

Proof. (i) This follows on expanding the definition (8.2). It is easiest to consider the three cases $i = j-1, i = j+1$, or $i \notin \{j \pm 1\}$ separately.

(ii) The Cartan integer $a_{k,j}$ is zero unless $k \in \{j, j \pm 1\}$ and $a_{k,-j}$ is zero unless $k \in \{-j, -j \pm 1\}$. Using this, the product term can be rewritten by first including the terms for all of these k then excluding ones in which $k = i$ by multiplying by its inverse, to show that

$$\prod_{k\in I-\{i\}}\frac{(i-k+x+y)^{a_{k,-j}}}{(i-k+x-y)^{a_{k,j}}} = \Big(1 - \frac{1}{(i-j+x-y)^2}\Big)\Big(1 - \frac{1}{(i+j+x+y)^2}\Big)^{-1}\frac{(x-y)^{a_{i,j}}}{(x+y)^{a_{i,-j}}}.$$

Also (8.2) gives that

$$G^{+-}_{i,j}(x,y)G^{-+}_{j,i}(y,x) = t^{-1}_{i,-j}\Big(1 - \frac{1}{(i+j+x+y)^2}\Big)(x+y)^{a_{i,-j}}.$$

Finally, by Lemma 3.3 and the definition (3.11), we have that

$$r_{j,i}r_{-i,-j}r_{j,-i}r_{i,j}Q^{\imath}_{i,j}(x,y) = t_{i,-j}(x-y)^{-a_{i,j}}.$$

The result follows on multiplying these three equations together.

(iii) Arguing as in (ii), the product term is

$$\prod_{j\in I-\{i\}}\frac{(i-j+x+y)^{a_{j,i}}}{(i-j+x-y)^{a_{j,-i}}} = \Big(1 - \frac{1}{(2i+x-y)^2}\Big)\Big(1 - \frac{1}{(x+y)^2}\Big)^{-1}\frac{(x-y)^{a_{i,-i}}}{(x+y)^2}.$$

Also (8.2) gives that

$$G^{+-}_{i,-i}(x,y)G^{-+}_{-i,i}(y,x) = (1+x+y)(1-x-y) = -(x+y)^2\Big(1-\frac{1}{(x+y)^2}\Big).$$

Finally, $Q^\imath_{i,-i}(x,y) = -t_{i,-i}(x-y)^{-a_{i,-i}} = -(x-y)^{-a_{i,-i}}$, since $t_{i,-i}=1$ for $i \in I_+$ (the only case in which i and $-i$ are joined by an edge with $i \in I_+$ is $i=h$, and $t_{h,-h}=1$). The result follows on multiplying these three equations together.

(iv) Since $I_{\mathrm{hi}} = -I_{\mathrm{lo}}$, we have that

$$\begin{aligned} G^{++}_{0,j}(x,y)G^{-+}_{0,j}(y,x) &= \prod_{k\in I_{\mathrm{lo}}} \big(\sqrt{-1}(-k+x-y)\big)^{-a_{k,j}}\big(\sqrt{-1}(-k-x-y)\big)^{-a_{k,j}} \\ &= \prod_{k\in I_{\mathrm{lo}}} (-k+x-y)^{-a_{k,j}}(k+x+y)^{-a_{k,j}} \\ &= \prod_{k\in I_{\mathrm{lo}}} (-k+x-y)^{-a_{k,j}} \times \prod_{k\in I_{\mathrm{hi}}} (-k+x+y)^{-a_{k,-j}}. \end{aligned}$$

Also, $r_{0,j}Q^\imath_{0,j}(x,y) = (x-y)^{-a_{0,j}}$ and $I = I_{\mathrm{hi}} \sqcup \{0\} \sqcup I_{\mathrm{lo}}$, so

$$\begin{aligned} r_{0,j}Q^\imath_{0,j}(x,y)G^{++}_{0,j}(x,y)G^{-+}_{0,j}(y,x) \prod_{k\in I_{\mathrm{hi}}} \frac{(-k+x+y)^{a_{k,-j}}}{(-k+x-y)^{a_{k,j}}} &= \prod_{k\in I}(-k+x-y)^{-a_{k,j}} \\ = \frac{(-(j+1)+x-y)(-(j-1)+x-y)}{(-j+x-y)^2} &= 1-\frac{1}{(-j+x-y)^2}. \end{aligned}$$

□

Lemma 8.5. *In the setup of Theorem 8.2, the zig-zag identities hold:*

Proof. In view of the definitions (8.10) and (8.11), we must show that

for $i \in I$. These identities are true by (3.24) and Lemma 7.10. □

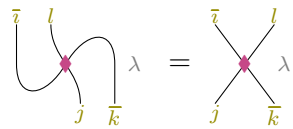
Lemma 8.6. *In the setup of Theorem 8.2, the dot slide relations hold:*

Proof. This follows using (3.25). □

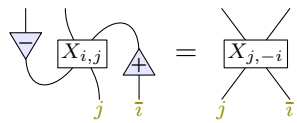
Lemma 8.7. *In the setup of Theorem 8.2, the pitchfork relations hold:*

Proof. Since we have already checked the zig-zag relations, the proof reduces just to checking the first one of these relations, and this is equivalent to showing that

Using the definitions (8.10) to (8.12), to verify this relation, we must show that



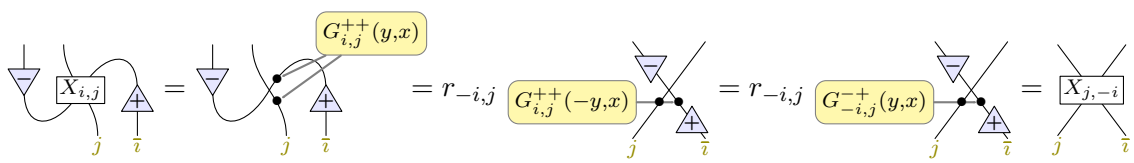
for all $\lambda \in X_{\bar{0}}^{\imath}$ and $i, j, k, l \in I$ such that $\lambda + \alpha_k - \alpha_j = \lambda - \alpha_l + \alpha_i$. The first two terms on the right-hand side of (8.6) have the necessary cyclic symmetry, so the proof reduces to checking that



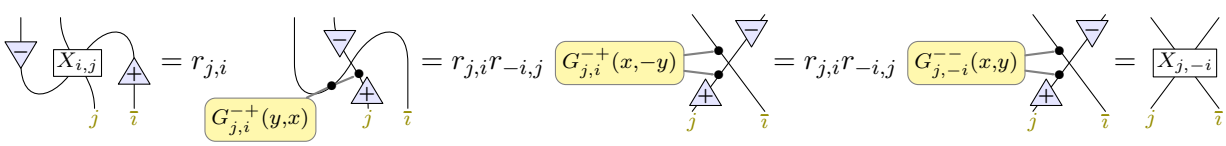
(8.14)

for all $i, j \in I$. It remains to check (8.14) for the various cases in the definition (8.7):

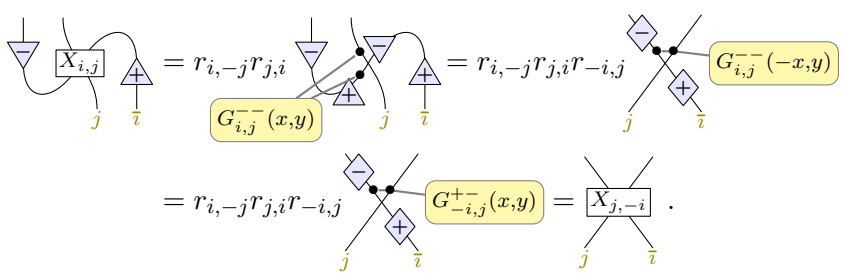
- When $i \in I_+ \cup \{0\}$ and $j \in I_+$, using the pitchfork relation (3.26), we have that



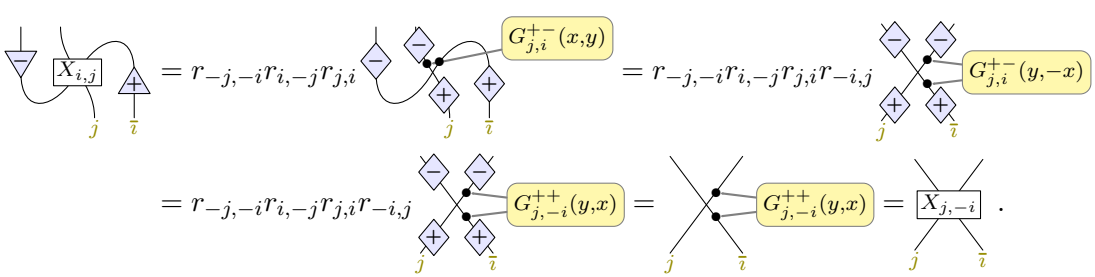
- If $i \in I_+$ and $j \in I_- \cup \{0\}$ then both glyphs in (8.14) are identities, so we get



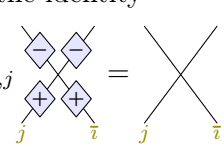
- If $i \in I_- \cup \{0\}$ and $j \in I_-$ then



- If $i \in I_-$ and $j \in I_+ \cup \{0\}$ then the glyphs in (8.14) can both be replaced by double glyphs (the other half being trivial), so we get

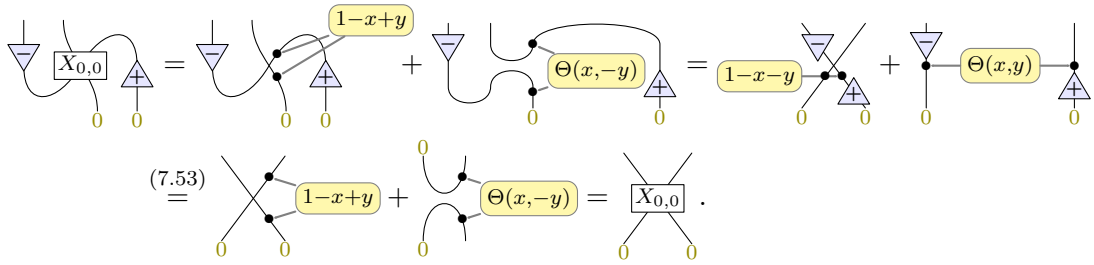


For the penultimate equality here, we used the identity



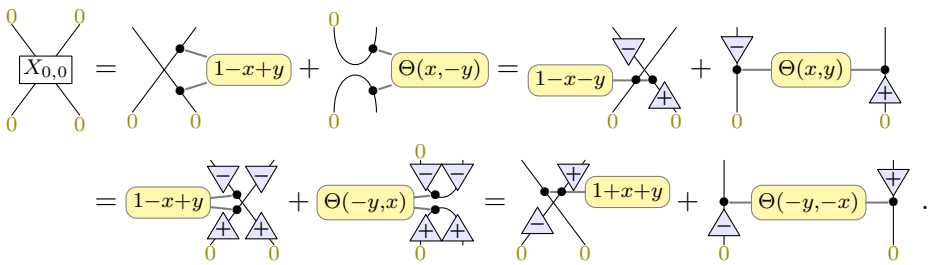
which is true thanks to Lemmas 7.10 and 7.14.

- Finally, for the exceptional case $i = j = 0$, we have that $G^{++}_{0,0}(y, x) = 1 - x + y$, so



□

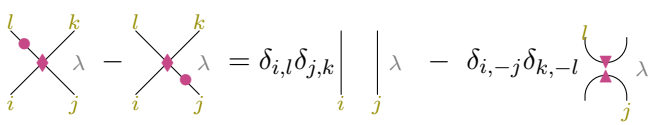
Remark 8.8. In the case $i = j = 0$, the identity (8.14) established in the proof of Lemma 8.7 implies that there are *four* different ways to write $X_{0,0}$: we have that



(8.15)

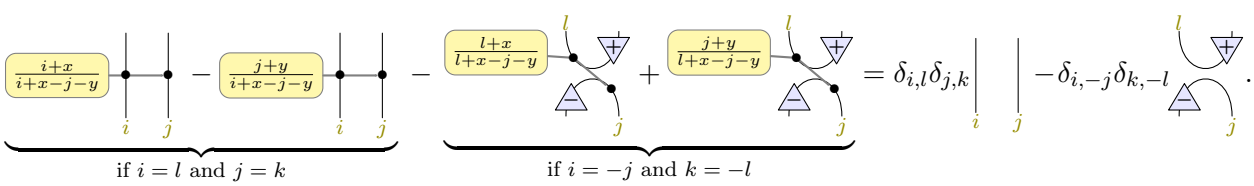
Lemma 8.9. *In the setup of Theorem 8.2, the Demazure relations hold:*

Proof. The second equality follows from the first using the zig-zag and pitchfork relations already proved. For the first, we must show that



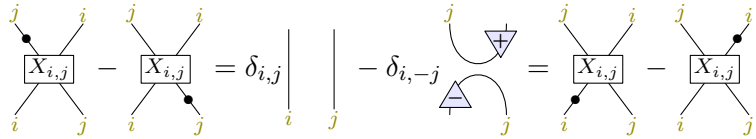
for all $\lambda \in X_{\bar{0}}^{\imath}$ and $i, j, k, l \in I$ such that $\lambda - \alpha_i - \alpha_j = \lambda - \alpha_k - \alpha_l$.

If $i \neq k$ or $j \neq l$, by the definition of the dot and crossing, this reduces to showing that



This is clear.

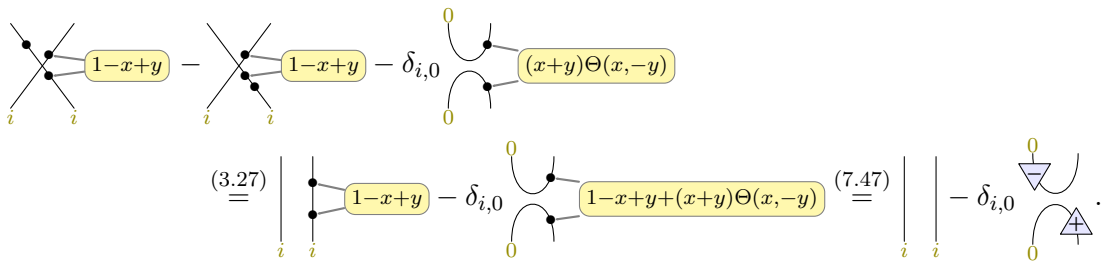
Now assume that $i = k$ and $j = l$. Then we must show that



(8.16)

Actually, as already noted, we only need to prove the first equality for all $i, j \in I$, then the second follows by cyclic rotation. However, we will prove both equalities, preferring to use cyclic rotation to reduce to proving both equalities just in the case that both i and j are in $I_+ \cup \{0\}$. Assuming this from now on, if $i \neq j$ then $i \neq -j$, and it is clear from the definition (8.7) and (3.27) that all three sides of (8.16) are 0.

It remains to check (8.16) when $i = j \in I_+ \cup \{0\}$. We have that $G^{++}_{i,i}(y,x) = 1 - x + y$, so the left-hand side of (8.16) is

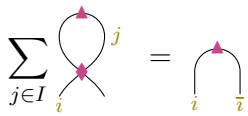


In view of (7.36), this proves the first equality in (8.16). The proof of the second equality is similar. □

Lemma 8.10. *In the setup of Theorem 8.2, the Reidemeister I relations hold:*

(8.17)

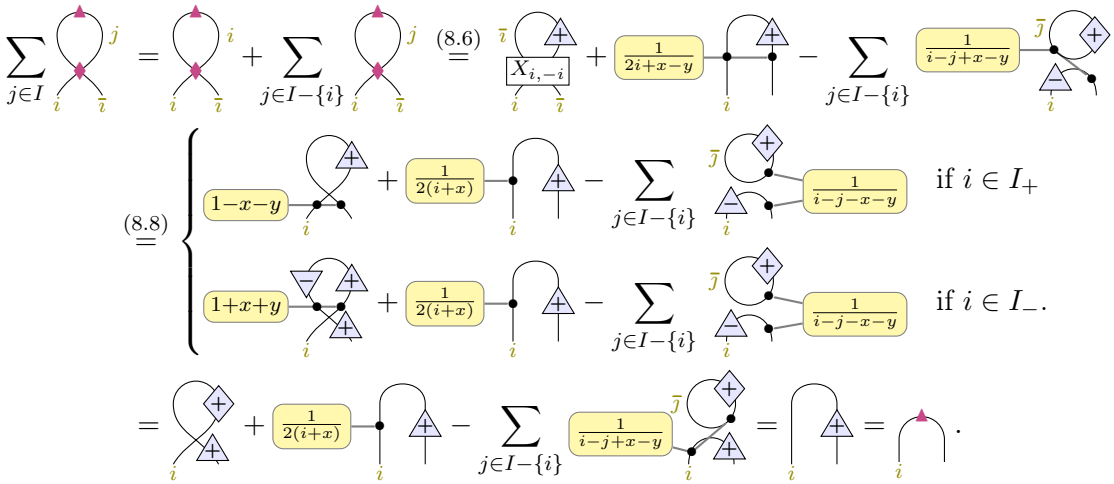
Proof. It suffices to prove the first equality. The second one then follows using the zig-zag and pitchfork relations already established. Applying (8.10) and (8.12), we need to show that



(8.18)

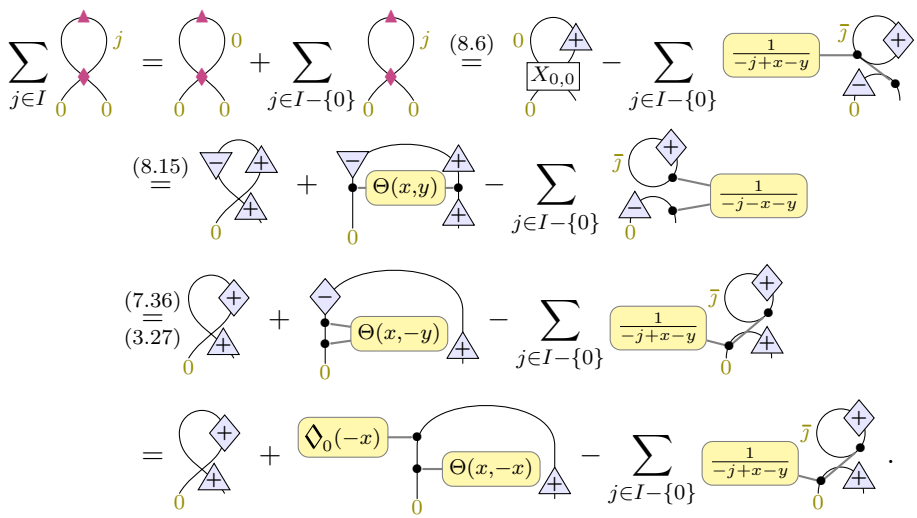
for all $i \in I$. Now we consider two cases.

The following calculation proves (8.18) when $i \neq 0$:



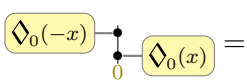
In the third equality here, we omitted some factors $r_{\pm i,\pm i}$ in the first term since they all equal 1. For the fourth equality, we simplified the first term, thereby reunifying the two cases: the factor $1 - x - y$ commutes past the crossing, and both of $x + y$ and $-x - y$ are zero when at the top due to the dot slide relation, so the pins $1 + x + y$ and $1 - x - y$ can be removed. We also moved some glyphs using (7.36), and added a couple of extra glyphs which are identities. For the penultimate equality, we applied the identity from (7.59) composed on the bottom right with a glyph labelled $+$, noting that $(-1)^{p(\lambda)} = 1$ as $\lambda \in X^{\imath}_0$, and the final equality is the definition (8.4).

In the $i = 0$ case, we instead argue as follows:

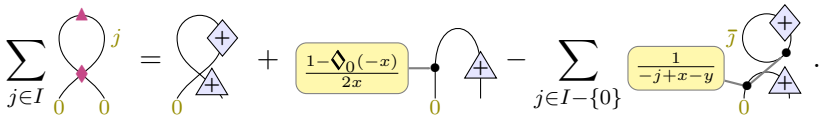


Now we note that

$$\Theta_\lambda(x,-x) = 1 + \frac{\Delta_{\lambda;0}(x) - \Delta_{\lambda;0}(-x)}{2x}\nabla_{\lambda;0}(x) \overset{(7.28)}{=} 1 + \frac{\Diamond_{\lambda;0}(x) - 1 - 2x}{2x} = \frac{\Diamond_{\lambda;0}(x) - 1}{2x}$$

for any λ. As 

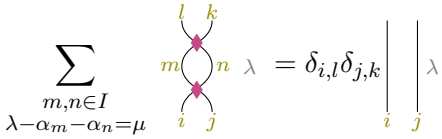
$=$ by Lemma 7.10, it follows that



It just remains to apply (7.60) like in the previous paragraph. □

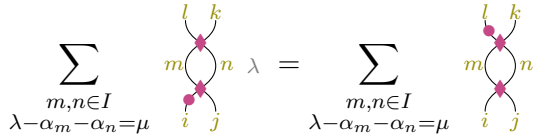
Lemma 8.11. *In the setup of Theorem 8.2, the Reidemeister II relations hold:*

Proof. Expanding the definition (8.12), we need to prove that



$$\sum_{\substack{m,n\in I\\ \lambda-\alpha_m-\alpha_n=\mu}} \quad = \delta_{i,l}\delta_{j,k} \quad (8.19)$$

for all $\lambda, \mu \in X_0^\imath$ and $i, j, k, l \in I$ such that $\lambda - \alpha_i - \alpha_j = \mu = \lambda - \alpha_k - \alpha_l$. Suppose first that $i \neq l$, so that the right-hand side of (8.19) is 0. We need to show that the left-hand side is 0 too. Since all of the defining relations of the category $\mathbf{AB}_{\mathrm{II}}$ from Remark 2.5 have been verified, we can appeal to the images in $\widehat{\mathbf{AB}}$ of the relations (2.17). The first of these implies that

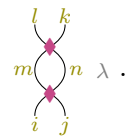


$$\sum_{\substack{m,n\in I\\ \lambda-\alpha_m-\alpha_n=\mu}} \quad = \sum_{\substack{m,n\in I\\ \lambda-\alpha_m-\alpha_n=\mu}} \quad . \quad (8.20)$$

Hence, letting b (resp., t) be the endomorphism of $\mathrm{Hom}_{\widehat{\mathfrak{U}}^\imath}(B_iB_j\mathbb{1}_\lambda, B_lB_k\mathbb{1}_\lambda)$ defined by vertical composition on the bottom with (resp., vertical composition on top with), we have that

$((l-i)\,\mathrm{id}+t-b)(\theta)=0$ where

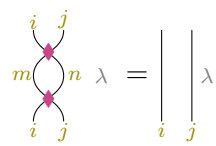
$$\theta := \sum_{\substack{m,n\in I\\ \lambda-\alpha_m-\alpha_n=\mu}}$$



Since $t-b$ is of positive degree and $l\neq i$, $(l-i)\,\mathrm{id}+t-b$ is invertible, so this implies that $\theta=0$. This proves (8.19) if $i\neq l$. A similar argument using the second relation from (2.17) proves it if $j\neq k$. (The argument just explained is essentially the same as the argument in the proof of [GRS26, Lem. 7.22].)

It remains to prove that

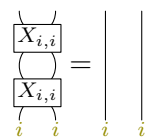
$$\sum_{\substack{m,n\in I\\ \lambda-\alpha_m-\alpha_n=\mu}}$$



(8.21)

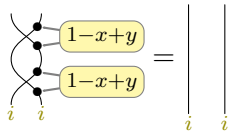
for all $\lambda,\mu\in X_0^\imath$ and $i,j\in I$ such that $\lambda-\alpha_i-\alpha_j=\mu$. We prove this by considering various cases.

- Suppose first that $i=j\neq 0$ (the easiest case). The summation on the left-hand side of (8.21) has at most one non-zero term, the one in which $m=n=i$, hence, the proof reduces to showing that

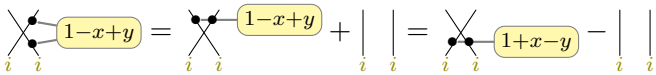


(8.22)

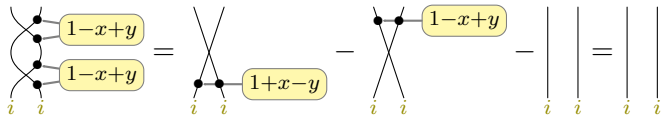
Rotating through 180°, using the relation (8.14) twice, shows that this is true for i if and only if it is true for $-i$. So it suffices to prove (8.22) in the case that $i\in I_+$. By the definitions (8.2) and (8.7), we therefore need to show that



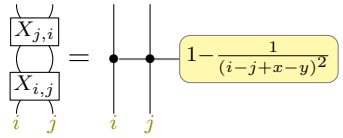
for $i\in I_+$. To prove this, note by (3.27) that



Since the square of a crossing of two strings of the same color is 0 by (3.29), it follows that



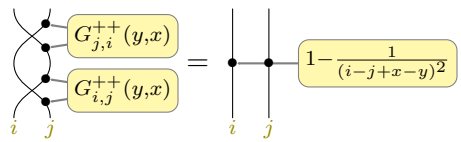
- Suppose next that $i\neq\pm j$. Then the left-hand side of (8.21) has at most two non-zero terms, namely, the ones with $(m,n)=(i,j)$ or (j,i). Expanding according to (8.6) and also moving the term with $(m,n)=(i,j)$ to the other side of the equality, we need to show that



(8.23)

Rotating through 180° using (8.14) shows that this is true for (i,j) if and only if it is true for $(-j,-i)$. Using this, the proof of (8.23) reduces to five sub-cases: $(i,j)\in I_+\times I_+, (i,j)\in I_+\times I_-, (i,j)\in I_-\times I_+, (i,j)\in\{0\}\times I_+$ and $(i,j)\in I_+\times\{0\}$. Now we go through these cases in turn, expanding the definitions of $X_{i,j}$ and $X_{j,i}$ using (8.7) and (8.8) in each one:

- If $(i,j) \in I_+ \times I_+$, we need to show that

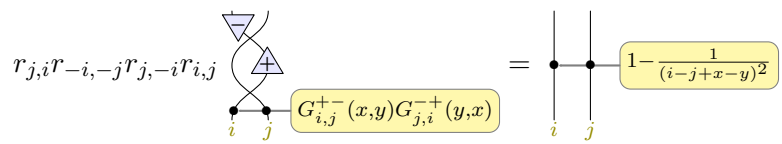


Since $i \neq j$, dots commute past crossings, so we can simplify the left-hand side by moving the pins to the bottom, then removing the double crossing using (3.29). In this way, the proof reduces to checking the algebraic identity

$$Q_{i,j}(x,y)G^{++}_{i,j}(x,y)G^{++}_{j,i}(y,x) = 1 - \frac{1}{(i-j+x-y)^2}.$$

This is true by Lemma 8.4(i).

- If $(i,j) \in I_+ \times I_-$, we need to show

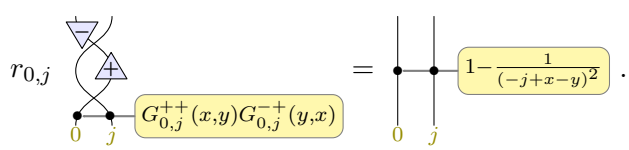


Here, we already moved the pins (which, in this case, are symmetric in x and y) to the bottom. Next we use Lemma 7.13(i) to slide the glyph labelled $+$ to the top, where it cancels with the glyph labelled $-$ thanks to Lemma 7.10, then expand the double crossing like in the previous sub-case. In this way, the proof reduces to showing that

$$r_{j,i}r_{-i,-j}r_{j,-i}r_{i,j}Q_{i,j}(x,y)G^{+-}_{i,j}(x,y)G^{-+}_{j,i}(y,x)\gamma_{i,j}\prod_{k\in I-\{i\}}\frac{(i-k+x+y)^{a_{k,-j}}}{(i-k+x-y)^{a_{k,j}}} = 1-\frac{1}{(i-j+x-y)^2}.$$

Since $i \in I_+$, we have that $\gamma_{i,j} = 1$, so this follows from Lemma 8.4(ii).

- The case $(i,j) \in I_- \times I_+$ is similar to the previous case. It reduces to checking the same identity but with i and j interchanged. Of course, this is also true by Lemma 8.4(ii).
- If $(i,j) \in \{0\} \times I_+$, we need to show



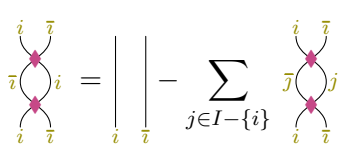
Using Lemma 7.13(ii), this reduces to the identity

$$r_{0,j}Q_{0,j}(x,y)G^{++}_{0,j}(x,y)G^{-+}_{0,j}(y,x)\prod_{k\in I_{\mathrm{hi}}}\frac{(-k+x+y)^{a_{k,-j}}}{(-k+x-y)^{a_{k,j}}} = 1-\frac{1}{(-j+x-y)^2}.$$

This is Lemma 8.4(iv).

- The fifth sub-case $(i,j) \in I_+ \times \{0\}$ is similar to the fourth sub-case.

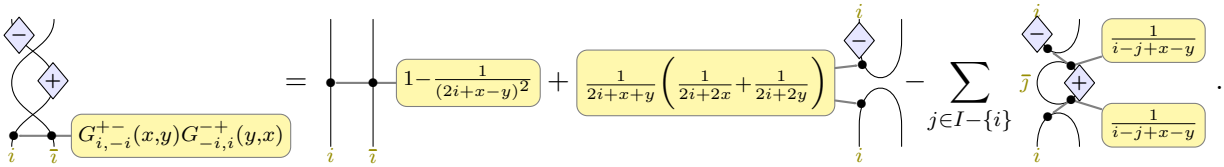
- Finally suppose that $i = -j$ (possibly 0). On the left-hand side of (8.21), we need to allow all terms in the summation with $m = -n$. Keeping the one with $n = i$ on the left-hand side and moving the others to the other side, we must show that



(8.24)

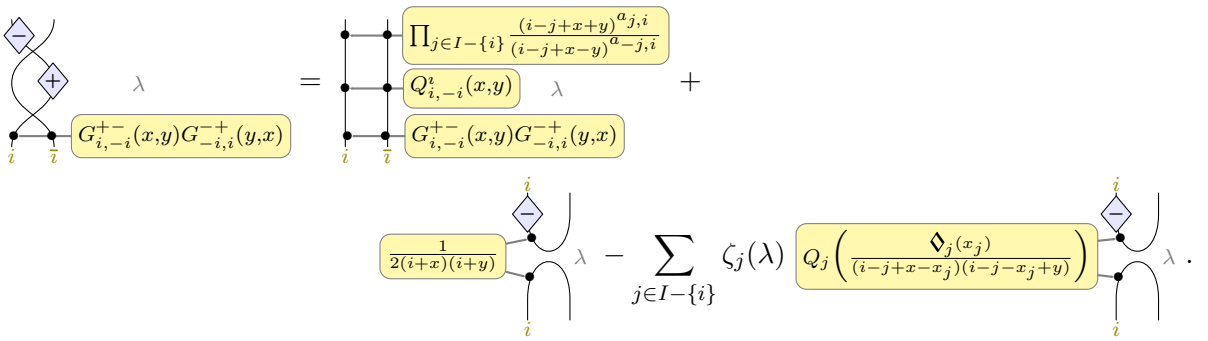
We consider three sub-cases $i \in I_+$, $i \in I_-$ and $i = 0$:

- If $i \in I_+$ then, using (8.6) and (8.8) and rearranging slightly, the equation (8.24) expands to



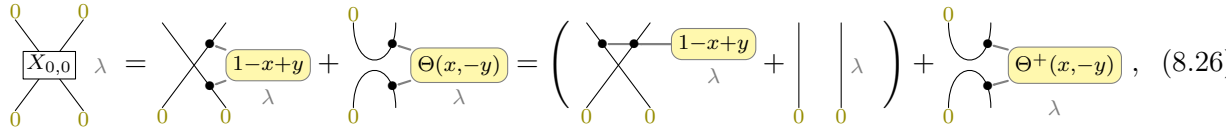
(8.25)

In this equation, we also added and removed some trivial glyphs so that we could draw double glyphs everywhere. To prove it, we apply the identity from Corollary 7.21 composed on the top left by a double glyph labelled $-$ and on the bottom by $G^{+-}_{i,-i}(x,y)G^{-+}_{-i,i}(y,x)$, noting that $\gamma_{i,-i} = 1$ as $i \in I_+$. It follows for any $\lambda \in X_0^\imath$ that



In the second and third terms on the right-hand side here, we omitted the pin labelled $G^{+-}_{i,-i}(x,y)G^{-+}_{-i,i}(y,x)$ because this equals $1-(x+y)^2$, so it is the identity on the bottom cap due to the dot slide relation. The first term is the first term on the right-hand side of (8.25) by Lemma 8.4(iii), the second term is easily seen to be the second term in (8.25) by adding fractions, and the third term is the desired third term too.

- The sub-case when $i \in I_-$ is similar.
- In the final sub-case $i = 0$, fix a choice of $\lambda \in X_0^\imath$. We begin by noting that

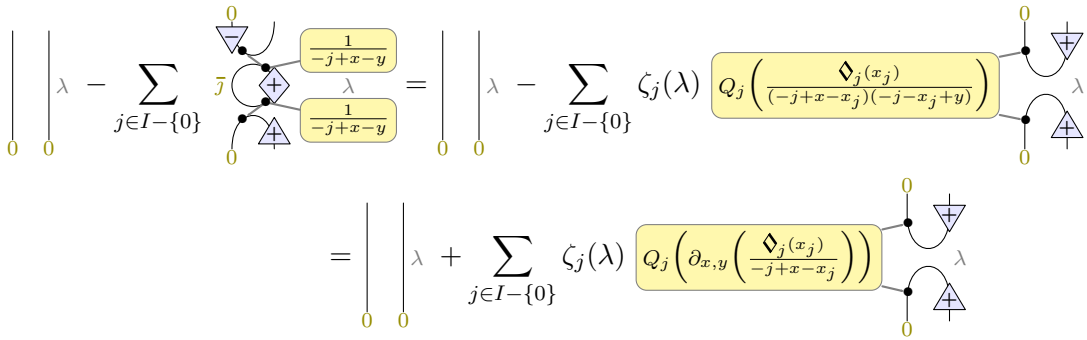


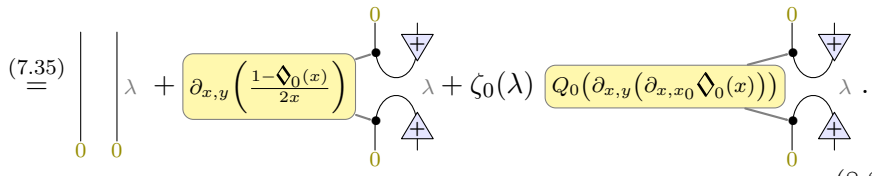
(8.26)

where

$$\Theta^+(x,y) := \Theta(x,y) - 1 \overset{(7.28)}{=} \frac{\Delta_0(x)\nabla_0(-y) + 2y - 1}{x-y}.$$

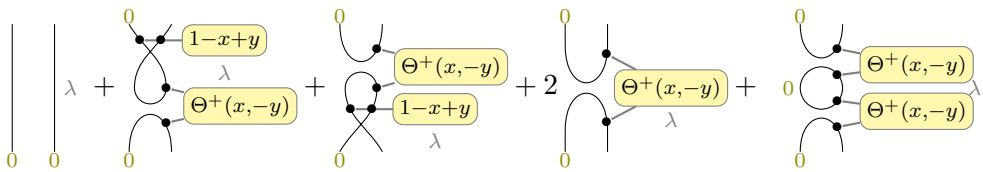
According to (8.24), we need to show that the square of this equals





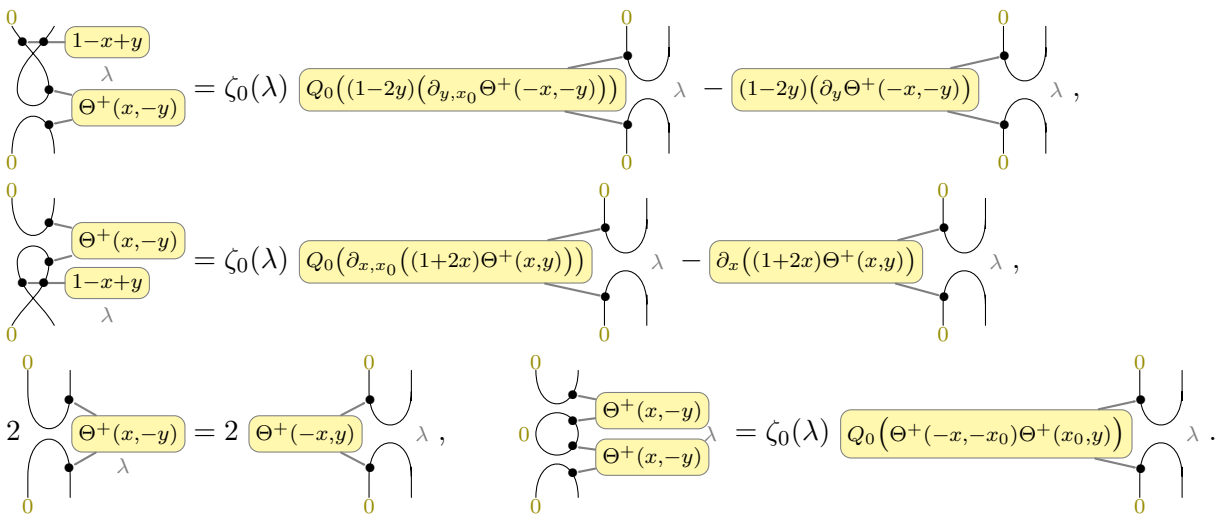
(8.27)

Now we actually compute the square of (8.26). It is easy to check that the bracketed term on its right-hand side squares to the identity, so we obtain



(8.28)

We need to show this is equal to (8.27). The first terms are both identities. For the remaining four terms on the right-hand side of (8.28), we use (7.57) and (7.58) to eliminate the curls and simplify to see that the four terms are



On the right-hand sides here, there are two types of term—ones with Q_0 and ones without. Comparing the two types with (8.27), it follows that the sum of these four terms is equal to the sum of the last two terms in (8.27) providing

$$\Delta_0(-x)\nabla_0(-y)\partial_{x,y}\Big(\tfrac{1-\diamondsuit_0(x)}{2x}\Big) = -(1-2y)\left(\partial_y\Theta^+(-x,-y)\right) - \partial_x\left((1+2x)\Theta^+(x,y)\right) + 2\Theta^+(-x,y),$$

$$\Delta_0(-x)\nabla_0(-y)\partial_{x,y}\left(\partial_{x,x_0}\diamondsuit_0(x)\right) = (1-2y)\left(\partial_{y,x_0}\Theta^+(-x,-y)\right) + \partial_{x,x_0}\left((1+2x)\Theta^+(x,y)\right) + \Theta^+(-x,-x_0)\Theta^+(x_0,y).$$

Both of these equations may be checked by some explicit expansion using the identity $\Delta_0(-x)\nabla_0(x) = 1+2x$ from (7.28).

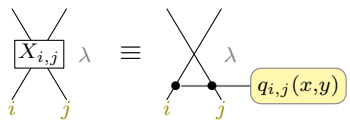
□

For the final Reidemeister III relation, we need one more preparation. Consider a morphism space $\widehat{H}_\lambda(\boldsymbol{j},\boldsymbol{i}) = \mathrm{Hom}_{\widehat{\mathfrak{U}}^\imath}(B_{\boldsymbol{j}}\mathbb{1}_\lambda, B_{\boldsymbol{i}}\mathbb{1}_\lambda)$ in the completed 2-iquantum group. By Lemma 6.2, it has topological basis $\mathrm{B}_\lambda(\boldsymbol{i}\times\boldsymbol{j})$ as a topological right $\widehat{R}$-module. Let n be the greatest number of crossings in any $D \in \mathrm{S}(\boldsymbol{i}\times\boldsymbol{j})$. Then there is a filtration

$$\{0\} = F_{-1}\widehat{H}_\lambda(\boldsymbol{j},\boldsymbol{i}) \subseteq F_0\widehat{H}_\lambda(\boldsymbol{j},\boldsymbol{i}) \subseteq F_1\widehat{H}_\lambda(\boldsymbol{j},\boldsymbol{i}) \subseteq \cdots \subseteq F_n\widehat{H}_\lambda(\boldsymbol{j},\boldsymbol{i}) = \widehat{H}_\lambda(\boldsymbol{j},\boldsymbol{i}) \tag{8.29}$$

such that $F_d\widehat{H}_\lambda(\boldsymbol{j},\boldsymbol{i})$ is the closure of the $\widehat{R}$-submodule spanned by the vectors in the basis for all reduced shapes $D \in \mathrm{S}(\boldsymbol{i}\times\boldsymbol{j})$ with at most d crossings. Let $\operatorname{gr}\widehat{H}_\lambda(\boldsymbol{j},\boldsymbol{i})$ be the associated graded $\widehat{R}$-module with $\operatorname{gr}_d\widehat{H}_\lambda(\boldsymbol{j},\boldsymbol{i}) := F_d\widehat{H}_\lambda(\boldsymbol{j},\boldsymbol{i})/F_{d-1}\widehat{H}_\lambda(\boldsymbol{j},\boldsymbol{i})$. We remind the reader that dots in string diagrams can be commuted freely past crossings modulo terms with strictly fewer crossings. Moreover, as the proof of the spanning part of Theorem 3.6 in [BWW25, Lem. 5.9] proceeds by induction on the number of crossings, any (not necessarily reduced) string diagram with at most d crossings lies in F_dH. In other words, in the associated graded, dots commute past crossings.

Lemma 8.12. *For any $i,j\in I$ and $\lambda\in X_0^\imath$, we have that*

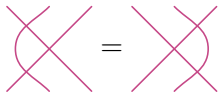


$$(\text{mod } F_0\widehat{H}_\lambda(ij,ji)) \tag{8.30}$$

for some $q_{i,j}(x,y)\in\Bbbk[\![x,y]\!]^\times$.

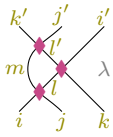
Proof. This follows from the definition (8.7). In the last three cases, one needs to commute a glyph past the crossing, where it cancels with the other glyph. This may be done using the formulae in Lemma 7.13, which are valid also when $i=\pm j$ modulo a crossing-free term. Explicit formulae for $q_{i,j}(x,y)$ are not hard to work out, but we will not need to know them later. For example, if $i\in I_+\cup\{0\}$ and $j\in I_+$ then $q_{i,j}(x,y)=G^{++}_{i,j}(x,y)$, but it is more complicated in other cases as there is an extra product term arising from the glyph slide. □

Lemma 8.13. *In the setup of Theorem 8.2, the Reidemeister III relations hold:*

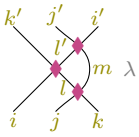


Proof. Fix a choice of $\lambda\in X_0^\imath$. Expanding the definition (8.12), we need to show that

$$\sum_{\substack{l,l',m\in I\\ \lambda-\alpha_k-\alpha_l-\alpha_m=\mu\\ \lambda-\alpha_{i'}-\alpha_{l'}-\alpha_m=\mu}}$$



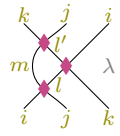
$$=\sum_{\substack{l,l',m\in I\\ \lambda-\alpha_m-\alpha_l-\alpha_i=\mu\\ \lambda-\alpha_m-\alpha_{l'}-\alpha_{k'}=\mu}}$$



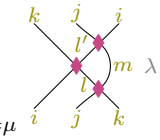
$$\tag{8.31}$$

for all $\lambda,\mu\in X_0^\imath$ and $i,j,k,i',k',j'\in I$ such that $\lambda-\alpha_i-\alpha_j-\alpha_k=\mu=\lambda-\alpha_{i'}-\alpha_{j'}-\alpha_{k'}$. We have proved already that all of the defining relations of $\mathbf{AB}_{\mathrm{III}}$ are satisfied. So, like at the start of Lemma 8.11, we can use (2.18) to (2.20) to prove the equality if $i\neq i', j\neq j'$ or $k\neq k'$; see also [GRS26, Lem. 7.23]. Thus, the problem is reduced to showing for $\lambda,\mu\in X_0^\imath$ and $i,j,k\in I$ with $\lambda-\alpha_i-\alpha_j-\alpha_k=\mu$ that

$$\Theta_{i,j,k} := \sum_{\substack{l,l',m\in I\\ \lambda-\alpha_k-\alpha_l-\alpha_m=\mu\\ \lambda-\alpha_i-\alpha_{l'}-\alpha_m=\mu}}$$



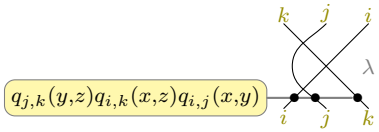
$$-\sum_{\substack{l,l',m\in I\\ \lambda-\alpha_m-\alpha_l-\alpha_i=\mu\\ \lambda-\alpha_m-\alpha_{l'}-\alpha_k=\mu}}$$



equals 0.

Suppose for a contradiction that $\Theta_{i,j,k}\neq 0$ in $\widehat{H}_\lambda(ijk,kji)$. We filter $\widehat{H}_\lambda(ijk,kji)$ by number of crossings according to (8.29). Since dots commute with crossings modulo terms with fewer crossings, the definition (8.6) and Lemma 8.12 imply that

$\Theta_{i,j,k}\equiv$



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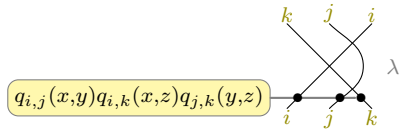


$$(\text{mod } F_2\widehat{H}_\lambda(ijk,kji)).$$

The relation (3.30) implies that the braid relation holds modulo $F_2\widehat{H}_\lambda(ijk,kji)$, so this is 0. Thus, we see that $\Theta_{i,j,k} \in F_2\widehat{H}_\lambda(ijk,kji)$, so it is an infinite $\widehat{R}$-linear combination

$$\Theta_{i,j,k} = \sum_{D,e_1,e_2,e_3\geq 0} c_{D;e_1,e_2,e_3} D(x_1^{e_1}x_2^{e_2}x_3^{e_3})\ \lambda$$

summing over reduced shapes $D \in \mathrm{S}((k,j,i)\times(i,j,k))$ with at most two crossings. Let $D \in \mathrm{S}((k,j,i)\times(i,j,k))$ be a reduced shape with the greatest number of crossings such that $c_{D;e_1,e_2,e_3} \neq 0$ for some $e_1,e_2,e_3 \geq 0$. Let d be the number of crossings in D. Since $d<3$, we can find $r \in \{1,2,3\}$ such that the rth point on the bottom boundary lies on a different string of D to the $(4-r)$th point on the top boundary. Let b be multiplication by a black dot at the rth point on the bottom boundary and t be multiplication by a black dot at the $(4-r)$th point on the top boundary. The linear map $b-t\colon \widehat{H}_\lambda(ijk,kji) \to \widehat{H}_\lambda(ijk,kji)$ induces a linear map

$$\mathrm{gr}_d(b-t)\colon\ \mathrm{gr}_d\,\widehat{H}_\lambda(ijk,kji) \to \mathrm{gr}_d\,\widehat{H}_\lambda(ijk,kji)$$

which preserves each shape component. For the shape D, the operators b and t act up to signs by multiplication by two different string variables x_s and $x_{s'}$. So $\mathrm{gr}_d(b-t)$ is multiplication by $\pm x_s \pm x_{s'}$ for some signs, which is injective on the completed polynomial algebra $\widehat{R}[\![x_1,x_2,x_3]\!]$ in variables corresponding to the strings of the shape D. Hence the non-zero D-component of $\Theta_{i,j,k}$ remains non-zero. So we have that $(b-t)(\Theta_{i,j,k}) \neq 0$. However, the relations (2.18) to (2.20) imply that $(b-t)(\Theta_{i,j,k}) = 0$; the scalars from (8.3) cancel since the label at the rth point on the bottom boundary equals the label at the $(4-r)$th point on the top boundary. This is a contradiction. □

9. Brauer to iquantum

In Theorem 8.2, we constructed a strict $\Bbbk$-linear monoidal functor

$$\Psi\colon \mathbf{AB} \to \widehat{\mathbf{AB}}. \tag{9.1}$$

In particular, this defines linear maps

$$\psi_{m,n}\colon\ \mathrm{Hom}_{\mathbf{AB}}(B^{\otimes n},B^{\otimes m}) \to \mathrm{Hom}_{\widehat{\mathbf{AB}}}(B^{\otimes n},B^{\otimes m}) \tag{9.2}$$

for each $m,n \geq 0$. In this section, we study the functor Ψ further, first showing that it is faithful. Since $\widehat{\mathbf{AB}}$ is enriched in topological vector spaces, there is an induced topology on the morphism spaces of $\mathbf{AB}$ with respect to which Ψ is continuous, i.e., each $\psi_{m,n}$ is continuous. Then we show that Ψ is topologically dense, i.e., each $\psi_{m,n}$ has image that is a dense subset of the topological space $\mathrm{Hom}_{\widehat{\mathbf{AB}}}(B^{\otimes n},B^{\otimes m})$. Hence, $\widehat{\mathbf{AB}}$ can be interpreted as a completion of $\mathbf{AB}$, justifying our notation. Finally, we use this general framework to prove Theorem B from the introduction.

We begin with some notation that will be used throughout the section. Recall that

$$Z = \mathrm{End}_{\mathbf{AB}}(\mathbb{1}), \qquad \widehat{Z} = \mathrm{End}_{\widehat{\mathbf{AB}}}(\mathbb{1}), \qquad Z_\lambda = \mathrm{End}_{\mathfrak{U}^\imath}(\mathbb{1}_\lambda), \qquad \widehat{Z}_\lambda = \mathrm{End}_{\widehat{\mathfrak{U}}^\imath}(\mathbb{1}_\lambda).$$

Also, by (6.13), we have that $\widehat{Z} = \prod_{\lambda\in X_{\bar{0}}^\imath} \widehat{Z}_\lambda$ with the λth projection of $z \in \widehat{Z}$ being $z^{\lambda,\lambda} \in \widehat{Z}_\lambda$. For $\lambda \in X_{\bar{0}}^\imath$, let

$$\psi_\lambda\colon Z \to \widehat{Z}_\lambda, \qquad z \mapsto \psi_{0,0}(z)^{\lambda,\lambda}. \tag{9.3}$$

We denote the Brauer bubble generating function $\bigcirc(u)$ from (2.10) by $O(u) \in \mathrm{id}_{\mathbb{1}} + u^{-1}Z[\![u^{-1}]\!]$. As in (3.41), the iquantum bubble generating function is $O_{\lambda;i}(u) \in \widehat{Z}_\lambda(\!(u^{-1})\!)$. In this section, it will also be helpful to have available the renormalized generating function

$$z_{\lambda;i}(u) := \varphi_\lambda(z_i(u)) = u^{\lambda_i-\varsigma_i}O_{\lambda;i}(u) \tag{9.4}$$

for φ as in (3.45). This has the property that

$$\prod_{i\in I} O_{\lambda;i}(u-i) = \chi_\lambda(u)\prod_{i\in I} z_{\lambda;i}(u-i) \tag{9.5}$$

where $\chi_\lambda(u)$ is the rational function from (4.7). Hence, using also (8.13), we have that

$$\psi_\lambda(O(u)) = \prod_{i\in I} O_{\lambda;i}(u-i) = \chi_\lambda(u)\prod_{i\in I} z_{\lambda;i}(u-i) \tag{9.6}$$

in $\widehat{Z}_\lambda[\![u^{-1}]\!]$, where $\chi_\lambda(u)\in\Bbbk(u)$ is the rational function from (4.7).

Now fix $m,n\geq 0$ with $m\equiv n \pmod 2$ (required for $\mathrm{Hom}_{\mathbf{AB}}(B^{\otimes n},B^{\otimes m})$ to be non-zero), and set $l := \frac{m+n}{2}$. Recall the set $\mathrm{S}(m\times n)$ of reduced shapes fixed before the statement of Theorem 2.3. Also there, for $D\in\mathrm{S}(m\times n)$, we introduced the notation $D(f)$ for the morphism in $\mathrm{Hom}_{\mathbf{AB}}(B^{\otimes n},B^{\otimes m})$ obtained by pinning $f\in\Bbbk[x_1,\dots,x_l]$ to the distinguished points $p_1,\dots,p_l$ of D. We extend this notation to define $D(f)$ for any $f\in Z[x_1,\dots,x_l]$ by declaring that the pinhead labelled f is located in the rightmost 2-cell. By Theorem 2.3, any $\theta\in\mathrm{Hom}_{\mathbf{AB}}(B^{\otimes n},B^{\otimes m})$ can be written as

$$\theta = \sum_{D\in\mathrm{S}(m\times n)} D(\mathrm{pr}_D(\theta)) \tag{9.7}$$

for unique $\mathrm{pr}_D(\theta)\in Z[x_1,\dots,x_l]$. This defines linear maps

$$\mathrm{pr}_D : \mathrm{Hom}_{\mathbf{AB}}(B^{\otimes n},B^{\otimes m}) \to Z[x_1,\dots,x_l] \tag{9.8}$$

for each $D\in\mathrm{S}(m\times n)$.

Similarly, on the iquantum side, we upgrade the notation (3.46) to allow any $f\in\widehat{Z}_\lambda[\![x_1,\dots,x_l]\!]$: for $D\in\mathrm{S}(\boldsymbol{i}\times\boldsymbol{j})$ and $\lambda\in X^\imath$ as there, $D(f)\ \lambda$ means the 2-morphism in $\widehat{H}_\lambda(\boldsymbol{j},\boldsymbol{i})$ obtained by pinning f to the distinguished points of $D\ \lambda$ in such a way that the pinhead is located in the rightmost 2-cell. By Lemma 6.2, any $\theta\in\widehat{H}_\lambda(\boldsymbol{j},\boldsymbol{i})$ can be written as

$$\theta = \sum_{D\in\mathrm{S}(\boldsymbol{i}\times\boldsymbol{j})} D(\mathrm{pr}_{\lambda;D}(\theta))\ \lambda \tag{9.9}$$

for unique $\mathrm{pr}_{\lambda;D}(\theta)\in\widehat{Z}_\lambda[\![x_1,\dots,x_l]\!]$. This defines right $\widehat{Z}_\lambda$-module homomorphisms

$$\mathrm{pr}_{\lambda;D}\colon \widehat{H}_\lambda(\boldsymbol{j},\boldsymbol{i}) \to \widehat{Z}_\lambda[\![x_1,\dots,x_l]\!]. \tag{9.10}$$

Moreover, by Lemma 6.2 and (6.3), $\mathrm{pr}_{\lambda;D}$ is continuous with respect to the GC topology on the domain, and the topology on $\widehat{Z}_\lambda[\![x_1,\dots,x_l]\!] = \prod_{e_1,\dots,e_l\geq 0} x_1^{e_1}\cdots x_l^{e_l}\widehat{Z}_\lambda$ that is the product of the GC topologies on each factor.

Returning to the situation that $D\in\mathrm{S}(m\times n)$ and $l := \frac{m+n}{2}$, a *coloring* of D simply means a tuple $\boldsymbol{c}\in I^l$. We write $D_{\boldsymbol{c}}$ for the string diagram obtained from D by changing all of its magenta strings to black strings, then labelling the strings by $c_1,\dots,c_l$ so that the string label at the kth distinguished point p_k is c_k, using our usual convention that string labels change sign at critical points. Let $\boldsymbol{i}^{D_{\boldsymbol{c}}}\in I^m$ and $\boldsymbol{j}^{D_{\boldsymbol{c}}}\in I^n$ be the words labelling the top and bottom boundary points of the strings in $D_{\boldsymbol{c}}$. For any $\lambda\in X_0^\imath$, we have that

$$\mu := \lambda - \alpha_{\boldsymbol{i}^{D_{\boldsymbol{c}}}} = \lambda - \alpha_{\boldsymbol{j}^{D_{\boldsymbol{c}}}}. \tag{9.11}$$

The diagram $D_{\boldsymbol{c}}$ is a reduced $\boldsymbol{i}^{D_{\boldsymbol{c}}}\times\boldsymbol{j}^{D_{\boldsymbol{c}}}$-shape in the sense explained just before (3.46). Since Theorem 3.6 holds for any choice of $\mathrm{S}(\boldsymbol{i}^{D_{\boldsymbol{c}}}\times\boldsymbol{j}^{D_{\boldsymbol{c}}})$ and any distinguished points, we can assume that $D_{\boldsymbol{c}}$ is one of the fixed choices of reduced shapes in $\mathrm{S}\left(\boldsymbol{i}^{D_{\boldsymbol{c}}}\times\boldsymbol{j}^{D_{\boldsymbol{c}}}\right)$, and that the distinguished points $p_1,\dots,p_l$ of $D_{\boldsymbol{c}}$ on the iquantum side are at exactly the same positions as the distinguished points of D on the Brauer side. Let $\#(D)$ denote the number of crossings in D, and recall the filtration of $\widehat{H}_\lambda(\boldsymbol{j}^{D_{\boldsymbol{c}}},\boldsymbol{i}^{D_{\boldsymbol{c}}})$ by number of crossings from (8.29).

Lemma 9.1. *Let $D \in \mathrm{S}(m \times n)$ and $l := \frac{m+n}{2}$, $\boldsymbol{c} \in I^l$ be a coloring, and $\lambda, \mu \in X_0^l$ be as in (9.11). For any $f \in Z[x_1, \dots, x_l]$, we have that $\psi_{m,n}(D(f))^{\mu,\lambda}_{\boldsymbol{i}^{D_{\boldsymbol{c}}},\boldsymbol{j}^{D_{\boldsymbol{c}}}} \in F_{\#(D)}\widehat{H}_\lambda(\boldsymbol{j}^{D_{\boldsymbol{c}}}, \boldsymbol{i}^{D_{\boldsymbol{c}}})$, and*

$$\mathrm{pr}_{\lambda;D_{\boldsymbol{c}}}\big(\psi_{m,n}(D(f))^{\mu,\lambda}_{\boldsymbol{i}^{D_{\boldsymbol{c}}},\boldsymbol{j}^{D_{\boldsymbol{c}}}}\big) = u\psi_\lambda\big(f(c_1 + x_1, \dots, c_l + x_l)\big)$$

for a unit $u \in \widehat{Z}_\lambda[\![x_1, \dots, x_l]\!]^\times$ depending only on $D, \boldsymbol{c}$ and λ, where $\psi_\lambda(f(c_1+x_1, \dots, c_l+x_l))$ means the element of $\widehat{Z}_\lambda[\![x_1, \dots, x_l]\!]$ obtained from the shifted polynomial $f(c_1+x_1, \dots, c_l+x_l)$ by applying $\psi_\lambda \colon Z \to \widehat{Z}_\lambda$ to each of its coefficients. Moreover,

$$\mathrm{pr}_{\lambda;D_{\boldsymbol{c}}}\big(\psi_{m,n}(D'(f'))^{\mu,\lambda}_{\boldsymbol{i}^{D_{\boldsymbol{c}}},\boldsymbol{j}^{D_{\boldsymbol{c}}}}\big) = 0$$

for $f' \in Z[x_1, \dots, x_l]$ and $D' \in \mathrm{S}(m \times n)$ with $D' \neq D$ and $\#(D') \leq \#(D)$.

Proof. By linearity, it suffices to consider $f = x_1^{e_1} \cdots x_l^{e_l} z$ for $e_1, \dots, e_l \geq 0$ and $z \in Z$. Since Ψ is monoidal, we have that $\psi_{m,n}\big(D(x_1^{e_1} \cdots x_l^{e_l} z)\big) = \psi_{m,n}\big(D(x_1^{e_1} \cdots x_l^{e_l})\big) \otimes \psi_{0,0}(z)$, so

$$\psi_{m,n}\big(D(x_1^{e_1} \cdots x_l^{e_l} z)\big)^{\mu,\lambda}_{\boldsymbol{i}^{D_{\boldsymbol{c}}},\boldsymbol{j}^{D_{\boldsymbol{c}}}} = \psi_{m,n}\big(D(x_1^{e_1} \cdots x_l^{e_l})\big)^{\mu,\lambda}_{\boldsymbol{i}^{D_{\boldsymbol{c}}},\boldsymbol{j}^{D_{\boldsymbol{c}}}}\psi_\lambda(z).$$

In view of this, we may assume that $z = 1$.

The morphism $D(x_1^{e_1} \cdots x_l^{e_l})$ can be written as a vertical composition of the generators—dots, caps, cups and crossings—of **AB** tensored on the left and right by some identity morphisms. To compute $\psi_{m,n}\big(D(x_1^{e_1} \cdots x_l^{e_l})\big)^{\mu,\lambda}_{\boldsymbol{i}^{D_{\boldsymbol{c}}},\boldsymbol{j}^{D_{\boldsymbol{c}}}}$, we sum over all labellings of the segments of D (the pieces of strings between consecutive generators) by elements of I which are compatible with the boundary labellings $\boldsymbol{i}^{D_{\boldsymbol{c}}}$ at the top and $\boldsymbol{j}^{D_{\boldsymbol{c}}}$ at the bottom, with the generators themselves replaced by their images under Ψ from (8.9) to (8.12) (see (8.31) for an example of such an expansion). The secondary definitions (8.9) to (8.12) are then further expanded using the primary definitions (8.3) to (8.6), the last producing a sum of at most two terms for each crossing.

Consider first the terms in this expansion in which the last term of (8.6), the one involving $X_{i,j}$, is chosen at every crossing. The image of a cap or cup has the same label, up to sign, at its endpoints, and $X_{i,j}$ only appears when the labels at the two ends of each strand of the crossing agree. So in such a term the labelling $\boldsymbol{c}$ is constant along each string of D, up to sign changes at critical points; that is, the labelling of segments is determined completely by $\boldsymbol{c}$, and there is exactly one such term. In it, by (8.3), the e_k dots at the point p_k have been replaced by the pin $(c_k + x)^{e_k}$; each cap and each cup carries the glyph from (8.4) or (8.5), which is a pin labelled by an invertible power series due to (7.28); and each crossing has been replaced by some $X_{i,j}$, which by Lemma 8.12 is a black crossing with the invertible power series $q_{i,j}(x, y)$ pinned below the crossing, plus some terms without the crossing.

In all of the other terms, at least one crossing of D has been removed, so these terms have at most $\#(D)-1$ crossings. Since $F_{\#(D)-1}\widehat{H}_\lambda\big(\boldsymbol{j}^{D_{\boldsymbol{c}}}, \boldsymbol{i}^{D_{\boldsymbol{c}}}\big)$ is closed and vertical composition is continuous, we deduce that $\psi_{m,n}\big(D(x_1^{e_1} \cdots x_l^{e_l})\big)^{\mu,\lambda}_{\boldsymbol{i}^{D_{\boldsymbol{c}}},\boldsymbol{j}^{D_{\boldsymbol{c}}}}$ lies in $F_{\#(D)}\widehat{H}_\lambda\big(\boldsymbol{j}^{D_{\boldsymbol{c}}}, \boldsymbol{i}^{D_{\boldsymbol{c}}}\big)$. Moreover, modulo lower terms in the crossing filtration, i.e., terms in $F_{\#(D)-1}\widehat{H}_\lambda\big(\boldsymbol{j}^{D_{\boldsymbol{c}}}, \boldsymbol{i}^{D_{\boldsymbol{c}}}\big)$, it is congruent to $D_{\boldsymbol{c}}\ \lambda$ decorated with $(c_k + x)^{e_k}$ pinned at p_k, an invertible glyph near each cup and cap, and an invertible power series $q_{i,j}(x, y)$ near each crossing.

Now we move all glyph coefficients from interior 2-cells to the rightmost 2-cell using (7.17) and (7.18) and the definitions (7.25) and (7.26). This produces additional pins of invertible power series. Then we move the pointy ends of the pins to the distinguished points, which produces some sign changes over critical points but, otherwise, does not change the diagram modulo lower terms in the crossing filtration. In this way, we obtain $D_{\boldsymbol{c}}\big(u(c_1+x_1)^{e_1} \cdots (c_l+x_l)^{e_l}\big)\ \lambda$ modulo lower terms in the crossing filtration for some unit $u \in \widehat{Z}_\lambda[\![x_1, \dots, x_l]\!]^\times$. Applying $\mathrm{pr}_{\lambda;D_{\boldsymbol{c}}}$ proves the first assertion.

For the second assertion, take $D' \in \mathrm{S}(m \times n)$ with $D' \neq D$ and $\#(D') \leq \#(D)$ and $e'_1, \dots, e'_l \geq 0$. Then we expand $\psi_{m,n}\big(D'(x_1^{e'_1} \cdots x_l^{e'_l})\big)^{\mu,\lambda}_{\boldsymbol{i}^{D_{\boldsymbol{c}}},\boldsymbol{j}^{D_{\boldsymbol{c}}}}$ in the same way as above. All of the terms lie in

$F_{\#(D)-1}\widehat{H}_\lambda\big(\boldsymbol{j}^{D_{\boldsymbol{c}}},\boldsymbol{i}^{D_{\boldsymbol{c}}}\big)$ except if D' has exactly $\#(D)$ crossings, when there could be a term in which $X_{i,j}$ is chosen at every crossing. This term is present only if there is a coloring $\boldsymbol{c}'$ of D' such that the boundary colors of $D'_{\boldsymbol{c}'}$ agree with $\boldsymbol{i}^{D_{\boldsymbol{c}}}$ at the top and $\boldsymbol{j}^{D_{\boldsymbol{c}}}$ at the bottom. By the same sliding argument, if present, this term is congruent to $D'_{\boldsymbol{c}'}(g)\,\lambda$ modulo lower terms in the crossing filtration for some $g\in\widehat{Z}_\lambda[\![x_1,\dots,x_l]\!]$. The image of this under $\mathrm{pr}_{\lambda;D_{\boldsymbol{c}}}$ is zero as $D'\neq D$. □

Lemma 9.2. *The homomorphism $\psi_0\colon Z\to\widehat{Z}_0$ (that is, $\psi_\lambda\colon Z\to\widehat{Z}_\lambda$ in the case $\lambda=0$) is injective.*

Proof. Let A be the graded algebra that is the quotient of Z_0 by the homogeneous ideal generated by all $z^{(r)}_{0;i}$ for $i\neq 0$ and $r\geq 1$, and let a_r be the image of $z^{(r)}_{0;0}$ in A. Setting $a(u)\coloneqq 1+\sum_{r\geq 1}a_ru^{-r}$, we have that $a(u)a(-u)=1$. Since $Z_0\cong R$, A is freely generated by $a_1,a_3,a_5,\dots$; cf. the sentence after (3.45). Equating u^{-2s}-coefficients in $a(u)a(-u)=1$ gives $2a_{2s}=-\sum_{t=1}^{2s-1}(-1)^ta_ta_{2s-t}$. As $p\neq 2$, it follows by induction on s that $a_{2s}\in\Bbbk[a_1,a_3,\dots,a_{2s-1}]$.

The quotient map $Z_0\twoheadrightarrow A$ maps the GC neighborhood $U_{0;d,I-\{0\}}$ into $A_{\geq d}$. Hence, it extends to a continuous homomorphism $\pi\colon\widehat{Z}_0\to\widehat{A}$, where $\widehat{A}$ is the completion of A with respect to the grading. We have that $\pi(O_{0;0}(u))=u^{-1}a(u)$, $\pi(O_{0;h}(u-h))=u-h$, and $\pi(O_{0;i}(u-i))=1$ for $i\in I-\{0,h\}$. So (9.6) gives that

$$\pi\big(\psi_0(O(u))\big)=(1-hu^{-1})a(u)=1+(a_1-h)u^{-1}+(a_2-ha_1)u^{-2}+\cdots.$$

On the other hand, by (2.10),

$$O(u)=\frac{u-h}{u}+\sum_{r\geq 0}z_ru^{-r-1}=1+(z_0-h)u^{-1}+z_1u^{-2}+z_2u^{-3}+\cdots$$

where z_r is the bubble with r dots from (2.14). Comparing coefficients, we deduce that

$$\pi(\psi_0(z_0))=a_1,\qquad\qquad \pi(\psi_0(z_r))=a_{r+1}-ha_r\quad(r\geq 1).$$

Hence, $\pi(\psi_0(z_{2s}))\in a_{2s+1}+\Bbbk[a_1,a_3,\dots,a_{2s-1}]$ for all $s\geq 0$. Since Z is spanned by the monomials in $z_0,z_2,z_4,\dots$ (see Remark 2.4), this triangularity shows that $\pi\circ\psi_0$ maps these monomials to a basis of A. So it maps Z isomorphically onto A. In particular, ψ_0 is injective. □

Lemma 9.3. *The functor Ψ is faithful.*

Proof. Take $m,n\geq 0$ with $m\equiv n\pmod 2$ and set $l\coloneqq\frac{m+n}{2}$. Let $\theta\in\mathrm{Hom}_{\mathbf{AB}}(B^{\otimes n},B^{\otimes m})$ be non-zero. Using the easy part of Theorem 2.3 (see Remark 2.4), we can write

$$\theta=\sum_{D\in\mathrm{S}(m\times n)}D(\theta_D(x_1,\dots,x_l))$$

for some $\theta_D(x_1,\dots,x_l)\in Z[x_1,\dots,x_l]$. Choose $D\in\mathrm{S}(m\times n)$ with the greatest number of crossings among all shapes such that $\theta_D(x_1,\dots,x_l)\neq 0$. Then we apply Lemma 9.1, taking the coloring $\boldsymbol{c}$ to be $\mathbf{0}\coloneqq(0,\dots,0)$ and the iweight λ to be 0. As every $D'\neq D$ with $\theta_{D'}(x_1,\dots,x_l)\neq 0$ has at most $\#(D)$ crossings, we get that

$$\mathrm{pr}_{0;D_{\mathbf{0}}}\left(\psi_{m,n}(\theta)^{0,0}_{\boldsymbol{i}^{D_{\mathbf{0}}},\boldsymbol{j}^{D_{\mathbf{0}}}}\right)=u\psi_0\big(\theta_D(x_1,\dots,x_l)\big)$$

for some unit $u\in\widehat{Z}_0[\![x_1,\dots,x_l]\!]^\times$. It remains to show that $\psi_0(\theta_D(x_1,\dots,x_l))\neq 0$. As this is obtained from $\theta_D(x_1,\dots,x_l)$ simply by applying ψ_0 to its coefficients, this follows from Lemma 9.2. □

Henceforth, we identify $\mathrm{Hom}_{\mathbf{AB}}(B^{\otimes n},B^{\otimes m})$ with a subspace of the complete topological vector space $\mathrm{Hom}_{\widehat{\mathbf{AB}}}(B^{\otimes n},B^{\otimes m})$ using the injective map $\psi_{m,n}$. In particular, this identifies Z with a subalgebra of $\widehat{Z}$.

Lemma 9.4. *The subalgebra Z is dense in the complete topological algebra $\widehat{Z}$.*

Proof. The topology on $\widehat{Z} = \prod_{\lambda \in X_{\bar{0}}^{\imath}} \widehat{Z}_\lambda$ is the product of the GC topologies on the factors. Since $Z_\lambda \cong R$ as topological algebras, Lemma 6.1 implies that the neighborhoods $U_{\lambda;d,J}$ with $J = -J$ form a system of neighborhoods of 0 in Z_λ. In view of this, it suffices to show that the homomorphism

$$\pi : Z \to \prod_{\lambda \in \Lambda} \widehat{Z}_\lambda / \overline{U}_{\lambda;d_\lambda,J_\lambda}, \qquad z \mapsto \big(\psi_\lambda(z) + \overline{U}_{\lambda;d_\lambda,J_\lambda}\big)_{\lambda \in \Lambda} \tag{9.12}$$

is surjective for any finite set Λ of even iweights, $d_\lambda \geq 1$ and cofinite subsets $J_\lambda = -J_\lambda \subseteq I$ for each $\lambda \in \Lambda$. This follows by the two claims below:

<u>Claim 1</u>: *The homomorphism π is surjective when $\Lambda = \{\lambda\}$ is a singleton.*

Proof: Write d and J for d_λ and J_λ, and let $A := \widehat{Z}_\lambda / \overline{U}_{\lambda;d,J}$. Let $K := I - J$, which is finite and closed under $\pm$, and choose $I' \subset I$ such that $I = I' \sqcup \{0\} \sqcup (-I')$. Let $K' := K \cap I'$. As $Z_\lambda \cong R$, the description of R after (3.45) implies that $Z_\lambda / U_{\lambda;J}$ is a polynomial algebra freely generated by the images of $z_{\lambda;0}^{(2r-1)}$ (if $0 \in K$) and $z_{\lambda;i}^{(r)}$ ($i \in K'$) for $r \geq 1$, which are all of positive degree. Hence, $A = \widehat{Z}_\lambda / \overline{U}_{\lambda;d,J} \cong Z_\lambda / U_{\lambda;d,J}$ is finite-dimensional and local, its unique maximal ideal $\mathfrak{m}$ is the image of $Z_{\lambda;\geq 1}$, which is nilpotent, and $\mathfrak{m}/\mathfrak{m}^2$ has a basis given by the images of the elements of the finite set

$$\Omega := \big\{ z_{\lambda;0}^{(2r-1)} \,\big|\, 0 \in K, 2(2r-1) < d \big\} \cup \big\{ z_{\lambda;i}^{(r)} \,\big|\, i \in K', 2r < d \big\}.$$

Let $S \subseteq A$ be the image of Z. To show that $S = A$, it suffices to show that $S + \mathfrak{m}^2 \supseteq \mathfrak{m}$. Indeed, assuming this, we get that $\mathfrak{m} \subseteq S + \mathfrak{m}^2 \subseteq S + (S \cap \mathfrak{m} + \mathfrak{m}^2)^2 \subseteq S + \mathfrak{m}^3 \subseteq \cdots$, and $\mathfrak{m}^r = 0$ for $r \gg 0$.

Let $z(u) \in A(\!(u^{-1})\!)$ be the image of $\psi_\lambda(O(u))$. Its coefficients belong to S. By (9.6), $z(u)$ is the image of $\chi_\lambda(u) \prod_{i \in K} z_{\lambda;i}(u-i)$, a finite product. To compute this modulo $\mathfrak{m}^2$, note that $\prod_{i \in K} z_{\lambda;i}(u-i) \equiv 1 + \sum_{i \in K} \big(z_{\lambda;i}(u-i) - 1 \big)$, and that $z_{\lambda;-i}(u) = z_{\lambda;i}(-u)^{-1} \equiv 1 - \big(z_{\lambda;i}(-u) - 1 \big)$; in particular, $2 z_{\lambda;0}^{(2r)} \in \mathfrak{m}^2$, hence $z_{\lambda;0}^{(2r)} \in \mathfrak{m}^2$ as $p \neq 2$. Remembering that generators of degree at least d are zero in A, we deduce that

$$z(u) \equiv \chi_\lambda(u) \Big(1 + \sum_{z \in \Omega} z\, R_z(u) \Big) \pmod{\mathfrak{m}^2(\!(u^{-1})\!)},$$

where $R_z(u) := u^{-r}$ if $z = z_{\lambda;0}^{(r)}$ and $R_z(u) := (u-i)^{-r} - (-1)^r (u+i)^{-r}$ if $z = z_{\lambda;i}^{(r)}$ for $i \in K'$. The rational functions $R_z(u)$ for $z \in \Omega$ are linearly independent: as K', $\{0\}$ and $(-K')$ are pairwise disjoint, the principal parts of $\sum_{z \in \Omega} c_z R_z(u)$ at 0 and at $i \in K'$ are $\sum_r c_{z_{\lambda;0}^{(r)}} u^{-r}$ and $\sum_r c_{z_{\lambda;i}^{(r)}} (u-i)^{-r}$, respectively. Hence the rational functions $\chi_\lambda(u) R_z(u)$ for $z \in \Omega$ are linearly independent too, and so are their expansions in $\Bbbk(\!(u^{-1})\!)$. So there are integers $n_1, \dots, n_{|\Omega|}$ such that the matrix $\big([\chi_\lambda(u) R_z(u)]_{u:n_t} \big)_{z \in \Omega, 1 \leq t \leq |\Omega|}$ is invertible. Since $[z(u)]_{u:n_t} - [\chi_\lambda(u)]_{u:n_t} \equiv \sum_{z \in \Omega} [\chi_\lambda(u) R_z(u)]_{u:n_t}\, z \pmod{\mathfrak{m}^2}$, it follows that the elements $[z(u)]_{u:n_t} - [\chi_\lambda(u)]_{u:n_t}$ of S span $\mathfrak{m}$ modulo $\mathfrak{m}^2$.

<u>Claim 2</u>: *The homomorphism π is surjective in general.*

Proof: For $\lambda \in \Lambda$, let K_λ be the kernel of the homomorphism $\pi_\lambda : Z \to A_\lambda := \widehat{Z}_\lambda / \overline{U}_{\lambda;d_\lambda,J_\lambda}$ obtained by composing π with the λth projection. This is surjective by Claim 1. Let $M_\lambda \supseteq K_\lambda$ be the kernel of the composition of π_λ with the projection of the local algebra A_λ onto its residue field $\Bbbk$. As all $z_{\lambda;i}^{(n)}$ with $n \geq 1$ map to zero in $\Bbbk$, (9.6) shows that this composition maps $O(u)$ to $\chi_\lambda(u)$. The rational functions $\chi_\lambda(u)$ for $\lambda \in \Lambda$ are distinct by Lemma 4.6, so the maximal ideals M_λ ($\lambda \in \Lambda$) of Z are distinct. As the unique maximal ideal of A_λ is nilpotent, we have that $M_\lambda^r \subseteq K_\lambda$ for $r \gg 0$, so the ideals K_λ are pairwise comaximal too. The Chinese Remainder Theorem now gives that $Z \to \prod_{\lambda \in \Lambda} Z/K_\lambda \cong \prod_{\lambda \in \Lambda} A_\lambda$ is surjective. $\square$

Since vertical and horizontal composition in $\widehat{\mathbf{AB}}$ are continuous (Lemmas 6.4 and 6.5), there is a monoidal subcategory $\overline{\mathbf{AB}}$ of $\widehat{\mathbf{AB}}$ with objects $B^{\otimes n}$ for all $n \geq 0$, and morphism spaces

$$\mathrm{Hom}_{\overline{\mathbf{AB}}}(B^{\otimes n}, B^{\otimes m}) := \overline{\mathrm{Hom}_{\mathbf{AB}}(B^{\otimes n}, B^{\otimes m})}$$

for $m, n \geq 0$, i.e., the closure of $\mathrm{Hom}_{\mathbf{AB}}(B^{\otimes n}, B^{\otimes m})$ in $\mathrm{Hom}_{\widehat{\mathbf{AB}}}(B^{\otimes n}, B^{\otimes m})$.

Theorem 9.5. $\overline{\mathbf{AB}} = \widehat{\mathbf{AB}}$.

Proof. For $\lambda, \mu \in X_{\bar{0}}^{\imath}$, $\boldsymbol{i} \in I^m, \boldsymbol{j} \in I^n$ with $\mu := \lambda - \alpha_{\boldsymbol{i}} = \lambda - \alpha_{\boldsymbol{j}}$, and any $\theta \in \mathrm{Hom}_{\widehat{\mathfrak{U}}^{\imath}}(B_{\boldsymbol{j}}\mathbb{1}_\lambda, B_{\boldsymbol{i}}\mathbb{1}_\lambda)$, let $\iota(\theta) \in \mathrm{Hom}_{\widehat{\mathbf{AB}}}(B^{\otimes n}, B^{\otimes m})$ be the morphism with $\iota(\theta)^{\mu,\lambda}_{\boldsymbol{i},\boldsymbol{j}} = \theta$, all other components being 0. It follows from Definition 6.6 that $\iota(\theta) \circ \iota(\phi) = \iota(\theta \circ \phi)$ when the vertical composition $\theta \circ \phi$ makes sense, and $\iota(\theta) \otimes \iota(\phi) = \iota(\theta\phi)$ when the horizontal composition $\theta\phi$ makes sense. We denote $\iota(\mathrm{id}_{\mathbb{1}_\lambda}) \in \widehat{Z}$ by e_λ, and $\iota(\mathrm{id}_{B_i\mathbb{1}_\lambda}) \in \mathrm{End}_{\widehat{\mathbf{AB}}}(B)$ by $e_{\lambda;i}$. By Remark 6.7, the subspace of $\mathrm{Hom}_{\widehat{\mathbf{AB}}}(B^{\otimes n}, B^{\otimes m})$ spanned by $\iota(\theta)$ for all $\theta \in \mathrm{Hom}_{\mathfrak{U}^{\imath}}(B_{\boldsymbol{j}}\mathbb{1}_\lambda, B_{\boldsymbol{i}}\mathbb{1}_\lambda)$ and all $\lambda, \mu, \boldsymbol{i}, \boldsymbol{j}$ is dense. Also Lemma 9.4 shows that $\mathrm{End}_{\overline{\mathbf{AB}}}(\mathbb{1}) = \widehat{Z}$, hence, it contains e_λ and $\iota(z)$ for all $z \in \widehat{Z}_\lambda$. These observations reduce the proof to showing that the images under ι of the generating 2-morphisms of $\mathfrak{U}^{\imath}$ belong to the appropriate morphism space of $\overline{\mathbf{AB}}$. We go through these shortly, after checking that $\overline{\mathbf{AB}}$ contains the idempotents $e_{\lambda;i}$.

<u>Idempotents</u>: *For $i \in I$ and $\lambda \in X_{\bar{0}}^{\imath}$, $\mathrm{End}_{\overline{\mathbf{AB}}}(B)$ contains the idempotent $e_{\lambda;i}$.*
Proof: By the definition of tensor product in Definition 6.6, we have that $\mathrm{id}_B \otimes e_\lambda = \sum_{j\in I} e_{\lambda;j}$. By (6.14), the closed subalgebra $(\mathrm{id}_B \otimes e_\lambda) \circ \mathrm{End}_{\widehat{\mathbf{AB}}}(B) \circ (\mathrm{id}_B \otimes e_\lambda)$ is identified with $\prod_{j\in I} \mathrm{End}_{\widehat{\mathfrak{U}}^{\imath}}(B_j\mathbb{1}_\lambda)$ with the product topology. By (8.3) and (8.9), we have that

$$\Psi\Big(\ \bullet \ \Big) \circ (\mathrm{id}_B \otimes e_\lambda) = \Big(\ \boxed{j+x} \text{—} \underset{j}{\bullet} \ \lambda \ \Big)_{j\in I}. \tag{9.13}$$

Now fix $i \in I$. By the discussion before (6.14), any neighborhood of 0 in $\prod_{j\in I} \mathrm{End}_{\widehat{\mathfrak{U}}^{\imath}}(B_j\mathbb{1}_\lambda)$ contains

$$N_{F,d} := \prod_{j\in I} N_j \qquad \text{where} \qquad N_j := \begin{cases} \overline{\mathrm{End}_{\mathfrak{U}^{\imath}}(B_j\mathbb{1}_\lambda)_{\geq 2d}} & \text{if } j \in F \\ \mathrm{End}_{\widehat{\mathfrak{U}}^{\imath}}(B_j\mathbb{1}_\lambda) & \text{if } j \notin F, \end{cases}$$

for suitable finite $F \subset I$ and $d \geq 0$. By the Chinese Remainder Theorem, there is a polynomial $f(x) \in \Bbbk[x]$ with $f(x) \equiv \delta_{i,j} \pmod{(x-j)^d}$ for every $j \in F$. Then $\Psi\Big(\ \bullet \text{—} \boxed{f(x)} \ \Big) \circ (\mathrm{id}_B \otimes e_\lambda)$ has jth component $\boxed{f(j+x)} \text{—} \underset{j}{\bullet} \ \lambda \ \equiv \delta_{i,j} \underset{j}{|} \ \lambda \ = \delta_{i,j} e_{\lambda;j} \pmod{N_j}$ for $j \in F$, so $\boxed{f(j+x)} \text{—} \underset{j}{\bullet} \ \lambda \ \equiv e_{\lambda;i} \pmod{N_{F,d}}$. As we have already established that $e_\lambda \in \mathrm{End}_{\overline{\mathbf{AB}}}(\mathbb{1})$, this shows that $e_{\lambda;i} \in \mathrm{End}_{\overline{\mathbf{AB}}}(B)$.

<u>Dots</u>: *For $i \in I$ and $\lambda \in X_{\bar{0}}^{\imath}$, $\mathrm{End}_{\overline{\mathbf{AB}}}(B)$ contains $\iota\Big(\ \underset{i}{\bullet} \ \lambda \ \Big)$. More generally, it contains $\iota\Big(\ \underset{i}{\bullet} \text{—} \boxed{f(x)} \ \lambda \ \Big)$ and $\iota\Big(\ \lambda \ \boxed{f(x)} \text{—} \underset{i}{\bullet} \ \Big)$ for any $f(x) \in \widehat{Z}_\lambda[\![x]\!]$. Even more generally, the analogous statement is true for power series in more than one variable pinned to dots on several parallel strings.*
Proof: By (9.13), we have that

$$\iota\Big(\ \underset{i}{\bullet} \ \lambda \ \Big) = \Psi\Big(\ \boxed{x-i} \text{—} \bullet \ \Big) \circ e_{\lambda;i}, \tag{9.14}$$

which is in $\mathrm{End}_{\overline{\mathbf{AB}}}(B)$ as we have already shown that $e_{\lambda;i} \in \mathrm{End}_{\overline{\mathbf{AB}}}(B)$. The power series $f(x) = \sum_{r\geq 0} f_r x^r \in \widehat{Z}_\lambda[\![x]\!]$ pinned to the right of the string is in $\mathrm{End}_{\overline{\mathbf{AB}}}(B)$ because we have that

$$\iota\Big(\ \underset{i}{\bullet} \text{—} \boxed{f(x)} \ \lambda \ \Big) = \sum_{r\geq 0} \iota\Big(\ r \ \underset{i}{\bullet} \ \lambda \ \Big) \circ (\mathrm{id}_B \otimes \iota(f_r)) \circ e_{\lambda;i}, \tag{9.15}$$

which converges because the dot with multiplicity r is of degree $2r$. Pins with the pinhead on the left are treated similarly.

<u>Caps and cups</u>: *For $i \in I$ and $\lambda \in X_{\bar{0}}^{\imath}$, $\mathrm{Hom}_{\overline{\mathbf{AB}}}(B^{\otimes 2}, \mathbb{1})$ contains $\iota\Big($* *$\Big)$ and $\mathrm{Hom}_{\overline{\mathbf{AB}}}(\mathbb{1}, B^{\otimes 2})$ contains $\iota\Big($* *$\Big)$.*

Proof: By (7.30), (8.4) and (8.5) and the invertibility established in (7.28), we have that

(9.16)

By (8.10), $\iota\Big($ $\Big) = e_\lambda \circ \Psi\Big($ $\Big) \circ \big(e_{\lambda-\alpha_{-i};i} \otimes e_{\lambda;-i}\big)$ which is in $\mathrm{Hom}_{\overline{\mathbf{AB}}}(B^{\otimes 2}, \mathbb{1})$, and $\iota\Big($

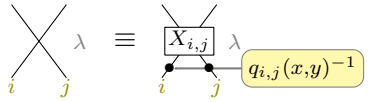
$\Big)$ is in $\mathrm{End}_{\overline{\mathbf{AB}}}(B^{\otimes 2})$, by the results established already. The cup is treated in the same way using (8.11).

<u>Crossings</u>: *For $i, j \in I$ and $\lambda \in X_{\bar{0}}^{\imath}$, $\mathrm{End}_{\overline{\mathbf{AB}}}(B^{\otimes 2})$ contains $\iota\Big($* *$\Big)$.*

Proof: Take $i, j \in I$. The only term on the right-hand side of (8.6) which survives when the bottom labels are i and j and the top labels are j and i is the one involving $X_{i,j}$. So, by (8.12),

$$\iota\Big(\boxed{X_{i,j}}\ \lambda\Big) = \big(e_{\lambda-\alpha_i;j} \otimes e_{\lambda;i}\big) \circ \Psi\Big(\ \Big) \circ \big(e_{\lambda-\alpha_j;i} \otimes e_{\lambda;j}\big), \tag{9.17}$$

which is in $\mathrm{End}_{\overline{\mathbf{AB}}}(B^{\otimes 2})$ by the results established already. By Lemma 8.12, we have that



$$(\mathrm{mod}\ F_0 \mathrm{Hom}_{\widehat{\mathfrak{U}}^\imath}(B_i B_j \mathbb{1}_\lambda, B_j B_i \mathbb{1}_\lambda)).$$

The right-hand side is in $\mathrm{End}_{\overline{\mathbf{AB}}}(B^{\otimes 2})$ by the results established already. Also

$$\iota\big(F_0 \mathrm{Hom}_{\widehat{\mathfrak{U}}^\imath}(B_i B_j \mathbb{1}_\lambda, B_j B_i \mathbb{1}_\lambda)\big) \subseteq \mathrm{End}_{\overline{\mathbf{AB}}}(B^{\otimes 2})$$

because $F_0 \mathrm{Hom}_{\widehat{\mathfrak{U}}^\imath}(B_i B_j \mathbb{1}_\lambda, B_j B_i \mathbb{1}_\lambda)$ is contained in the closure of the $\widehat{Z}_\lambda$-span of crossing-free diagrams with dots, and the images under ι of all such diagrams are in the closed subspace $\mathrm{End}_{\overline{\mathbf{AB}}}(B^{\otimes 2})$. □

For the remainder of the section, we assume that

$$\boxed{\mathbf{R} \text{ is an Abelian or Karoubian Brauer categorification.}}$$

Recall the notation of section 4. In particular, for $V \in \mathbf{R}$, we write Z_V for the center of $\mathrm{End}_{\mathbf{R}}(V)$, a finite-dimensional commutative algebra, and $z_V \in Z_V$ for the evaluation on V of $z \in Z$. In order to prove Theorem B from the introduction, the key additional ingredient is that the action of $\mathbf{AB}$ on $\mathbf{R}$ is continuous for the topology induced by Ψ and the discrete topology on $\mathrm{Hom}_{\mathbf{R}}(B^{\otimes n}V, B^{\otimes m}V)$ for each $V \in \mathbf{R}$ and $m, n \geq 0$. We prove this in the next few lemmas.

Lemma 9.6. *Let $V \in \mathbf{R}$ be an indecomposable object and $\lambda \in X_{\bar{0}}^{\imath}$ be the unique even iweight such that $V \in \mathbf{R}_\lambda$. Then there is a unique algebra homomorphism $\omega\colon \widehat{Z}_\lambda \to Z_V$ such that*

(i) *$\omega(O_{\lambda;i}(u)) = O_{V;i}(u)$ for all $i \in I$, where $O_{V;i}(u)$ is the local factor of $O_V(u)$ at $u = i$;*

(ii) *ω is continuous for the GC topology on $\widehat{Z}_\lambda$ and the discrete topology on Z_V.*

Moreover:

(iii) $\omega(\psi_\lambda(z)) = z_V$ for any $z \in Z$.

Proof. The commutative algebra Z_λ is generated by the elements $z_{\lambda;i}^{(r)}$ subject only to the relations $O_{\lambda;-i}(u)O_{\lambda;i}(-u) = (-1)^{\varsigma_i+\lambda_i}u^{\varsigma_i+\varsigma_{-i}}\,\mathrm{id}_{\mathbb{1}_\lambda}$. Thus, to show that there is an algebra homomorphism $Z_\lambda \to Z_V$ satisfying (i), it suffices to show that

$$O_{V;-i}(u)O_{V;i}(-u) = (-1)^{\varsigma_i+\lambda_i}u^{\varsigma_i+\varsigma_{-i}}\,\mathrm{id}_V\,. \tag{9.18}$$

As V is indecomposable, Z_V is a finite-dimensional local commutative algebra. Let $\mathfrak{m}$ be its unique maximal ideal. As explained in Remark 4.8, the image $\bar{O}_V(u)$ of $O_V(u)$ in $\Bbbk(u) = (Z_V/\mathfrak{m})(u)$ is the rational function $\chi_\lambda(u)$ from (4.7). Hence, the local factors $O_{V;i}(u)$ of $O_V(u)$ satisfy

$$O_{V;i}(u) \in u^{\varsigma_i-\lambda_i}\,\mathrm{id}_V + u^{\varsigma_i-\lambda_i-1}\mathfrak{m}[u^{-1}].$$

Since $\sum_{i\in I}(\varsigma_i-\lambda_i) = 0$, we have that

$$O_V(-u) = \prod_{i\in I} O_{V;i}(-u-i) = \prod_{i\in I}\widetilde{O}_{V;-i}(u-i)$$

where $\widetilde{O}_{V;i}(u) \coloneqq (-1)^{\varsigma_i+\lambda_i}O_{V;i}(-u)$. Also, the local factor of $\frac{(u-h)(u+h)}{u^2}$ at $u=i$ is $u^{-a_{i,-i}} = u^{\varsigma_i+\varsigma_{-i}}$. By (2.11), $O_V(u)O_V(-u) = \frac{(u-h)(u+h)}{u^2}$ so, using the uniqueness of local factors from Corollary 1.2, we get that $O_{V;i}(u)\widetilde{O}_{V;-i}(u) = u^{\varsigma_i+\varsigma_{-i}}$. The identity (9.18) follows from this on replacing i with $-i$ and using the definition of $\widetilde{O}_{V;i}(u)$.

To construct ω, it remains to show that the homomorphism just constructed is continuous for the topology on Z_λ induced by the GC topology on $\widehat{Z}_\lambda$ and the discrete topology on Z_V, for then it extends uniquely to the desired continuous homomorphism ω. To prove this, Corollary 1.2 implies that the renormalized local factor $u^{\lambda_i-\varsigma_i}O_{V;i}(u)$ is a polynomial in $\mathrm{id}_V + u^{-1}\mathfrak{m}[u^{-1}]$, and it equals id_V for all but finitely many i. Consequently, only finitely many generators $z_{\lambda;i}^{(r)}$ have nonzero images, and those images lie in the nilpotent ideal $\mathfrak{m}$. The degree argument used to prove Claim 1 in the proof of Theorem 6.8 then shows that $\ker\omega$ contains a GC neighborhood.

Finally, (iii) holds from the local factorization $O_V(u) = \prod_{i\in I} O_{V;i}(u-i)$ and (9.6). □

Lemma 9.7. *Suppose that $m,n \geq 0$ with $m \equiv n \pmod 2$ and $l \coloneqq \frac{m+n}{2}$. Recalling (9.8), for any $D \in \mathrm{S}(m\times n), \boldsymbol{c} \in I^l$ and $\lambda \in X_{\bar{0}}^{\imath}$, the function*

$$\mathrm{Hom}_{\mathbf{AB}}(B^{\otimes n}, B^{\otimes m}) \to \widehat{Z}_\lambda[\![x_1,\dots,x_l]\!], \qquad \theta \mapsto \psi_\lambda\big(\mathrm{pr}_D(\theta)(c_1+x_1,\dots,c_l+x_l)\big)$$

is continuous for the subspace topology on the domain $\mathrm{Hom}_{\mathbf{AB}}(B^{\otimes n}, B^{\otimes m})$ induced by the GC topology on $\mathrm{Hom}_{\widehat{\mathbf{AB}}}(B^{\otimes n}, B^{\otimes m})$, and the topology on $\widehat{Z}_\lambda[\![x_1,\dots,x_l]\!] = \prod_{e_1,\dots,e_l\geq 0} x_1^{e_1}\cdots x_l^{e_l}\widehat{Z}_\lambda$ that is the product of the GC topologies on each factor.

Proof. We proceed by downward induction on $\#(D)$. Let μ be as in (9.11). We apply the linear map $\mathrm{pr}_{\lambda;D_{\boldsymbol{c}}}\big(\psi_{m,n}(-)^{\mu,\lambda}_{\boldsymbol{i}^{D_{\boldsymbol{c}}},\boldsymbol{j}^{D_{\boldsymbol{c}}}}\big)$ to $\theta = \sum_{D'\in\mathrm{S}(m\times n)} D'(\mathrm{pr}_{D'}(\theta)(x_1,\dots,x_l))$ and use Lemma 9.1 to deduce that there is a unit $u \in \widehat{Z}_\lambda[\![x_1,\dots,x_l]\!]^\times$ such that

$$\mathrm{pr}_{\lambda;D_{\boldsymbol{c}}}\big(\psi_{m,n}(\theta)^{\mu,\lambda}_{\boldsymbol{i}^{D_{\boldsymbol{c}}},\boldsymbol{j}^{D_{\boldsymbol{c}}}}\big) = u\psi_\lambda\big(\mathrm{pr}_D(\theta)(c_1+x_1,\dots,c_l+x_l)\big) + \sum_{D'}\mathrm{pr}_{\lambda;D_{\boldsymbol{c}}}\Big(\psi_{m,n}\big(D'(\mathrm{pr}_{D'}(\theta))\big)^{\mu,\lambda}_{\boldsymbol{i}^{D_{\boldsymbol{c}}},\boldsymbol{j}^{D_{\boldsymbol{c}}}}\Big),$$

summing over $D' \in \mathrm{S}(m\times n)$ with $\#(D') > \#(D)$. As noted after (9.10), $\mathrm{pr}_{\lambda;D_{\boldsymbol{c}}}$ is continuous, hence, the function on the left-hand side is a continuous function of θ. Also multiplication by u^{-1} is continuous. Thus, we are reduced to showing that the function

$$\mathrm{Hom}_{\mathbf{AB}}(B^{\otimes n}, B^{\otimes m}) \to \widehat{H}_\lambda\big(\boldsymbol{j}^{D_{\boldsymbol{c}}}, \boldsymbol{i}^{D_{\boldsymbol{c}}}\big), \qquad \theta \mapsto \psi_{m,n}\big(D'(\mathrm{pr}_{D'}(\theta))\big)^{\mu,\lambda}_{\boldsymbol{i}^{D_{\boldsymbol{c}}},\boldsymbol{j}^{D_{\boldsymbol{c}}}}$$

is continuous for $D' \in \mathrm{S}(m\times n)$ with $\#(D') > \#(D)$.

As in the proof of Lemma 9.1, for $f \in Z[x_1, \dots, x_l]$ we have that

$$\psi_{m,n}(D'(f))^{\mu,\lambda}_{\boldsymbol{i}^{D_{\boldsymbol{c}}},\boldsymbol{j}^{D_{\boldsymbol{c}}}} = \sum_{\kappa} E_\kappa\big(\psi_\lambda(f(\kappa_1 + x_1, \dots, \kappa_l + x_l))\big).$$

Here, the sum is over all labellings κ of the segments of D' by elements of I which are compatible with the boundary labels $\boldsymbol{i}^{D_{\boldsymbol{c}}}$ at the top and $\boldsymbol{j}^{D_{\boldsymbol{c}}}$ at the bottom, with $\kappa_1, \dots, \kappa_l$ being the labels at the distinguished points. To define E_κ, let D'_κ be obtained from D' by replacing magenta strings with black strings, labelling segments according to κ, and replacing crossings, cups and caps by the decorations in (8.10) to (8.12). Then $E_\kappa \colon \widehat{Z}_\lambda[\![x_1, \dots, x_l]\!] \to \widehat{H}_\lambda\big(\boldsymbol{j}^{D_{\boldsymbol{c}}}, \boldsymbol{i}^{D_{\boldsymbol{c}}}\big)$ is the linear map taking $g \in \widehat{Z}_\lambda[\![x_1, \dots, x_l]\!]$ to the morphism obtained by pinning g to D'_κ so that its pinhead is in the rightmost 2-cell, labelling this 2-cell by λ. Each E_κ is continuous, since horizontal and vertical composition are continuous, as is pinning a formal power series at the specified distinguished point.

Let N be a GC neighbourhood of zero in $\widehat{H}_\lambda(\boldsymbol{j}^{D_{\boldsymbol{c}}}, \boldsymbol{i}^{D_{\boldsymbol{c}}})$. Choose a cofinite subset $J \subseteq I$ such that $\overline{H_\lambda(\boldsymbol{j}^{D_{\boldsymbol{c}}}, \boldsymbol{i}^{D_{\boldsymbol{c}}})U_{\lambda;J}} \subseteq N$, and finitely many horizontal slices of the string diagram D' so that every segment meets one of them. At each slice, we apply the claim in the proof of Lemma 6.4(ii), and take the union of the resulting finite sets of labels. This produces a finite $K \subset I$ such that $E_\kappa(g) \in \overline{H_\lambda(\mathbf{j}^{D_c}, \mathbf{i}^{D_c})U_{\lambda;J}} \subseteq N$ for every $g \in \widehat{Z}_\lambda[\![x_1, \dots, x_l]\!]$ whenever κ involves a label outside K. Thus, modulo N, the expansion reduces to the finite sum over labellings taking values in K. For each such labelling, the map

$$\theta \mapsto E_\kappa\big(\psi_\lambda\big(\mathrm{pr}_{D'}(\theta)(\kappa_1 + x_1, \dots, \kappa_l + x_l)\big)\big)$$

is continuous by the induction hypothesis and the continuity of E_κ. Consequently, the required map is continuous modulo every GC neighbourhood of zero, and hence is continuous. This completes the induction. □

Lemma 9.8. *For $V \in \mathbf{R}$ and $m, n \geq 0$ with $m \equiv n \pmod 2$, the evaluation map*

$$\mathrm{Hom}_{\mathbf{AB}}(B^{\otimes n}, B^{\otimes m}) \to \mathrm{Hom}_{\mathbf{R}}(B^n V, B^m V), \qquad \theta \mapsto \theta_V$$

is continuous with respect to the topology on the domain induced by $\psi_{m,n}$ and the discrete topology on the codomain.

Proof. We may assume that V is indecomposable. Let λ and $\omega \colon \widehat{Z}_\lambda \to Z_V$ be as in Lemma 9.6. Set $U := \ker \omega$, which is an open subspace of $\widehat{Z}_\lambda$. Assume that the shapes $D \in \mathrm{S}(m \times n)$ have been chosen so that they are composites of caps, cups and crossings, with the distinguished points at distinct horizontal levels avoiding any of the generators. There is a finite set $F \subset I$ and $M \geq 1$ such that for each distinguished point the morphism x obtained by evaluating the dot at that point on the object obtained by applying the appropriate power of B (i.e., the number of strings to the right) to V has spectrum contained in F, and its restriction to the generalized i-eigenspace of x is annihilated by $(x - i)^M$.

For $D \in \mathrm{S}(m \times n)$, $\boldsymbol{c} \in F^l$, and $\boldsymbol{d} \in \mathbb{N}^l$, define $T_{D,\boldsymbol{c},\boldsymbol{d}} \in \mathrm{Hom}_{\mathbf{R}}(B^n V, B^m V)$ by evaluating D on V, inserting the projection onto B_{c_k} at the distinguished point p_k, then placing $(x - c_k)^{d_k}$ there, for each $1 \leq k \leq l$. Then $T_{D,\boldsymbol{c},\boldsymbol{d}} = 0$ if $d_k \geq M$ for some k. Define a linear map $\Theta_{D,\boldsymbol{c}} : \widehat{Z}_\lambda[\![x_1, \dots, x_l]\!] \to \mathrm{Hom}_{\mathbf{R}}(B^n V, B^m V)$ by the finite sum

$$\Theta_{D,\boldsymbol{c}} \colon \sum_{\boldsymbol{d}} g_{\boldsymbol{d}} x_1^{d_1} \cdots x_l^{d_l} \mapsto \sum_{\substack{\boldsymbol{d} \in \mathbb{N}^l \\ d_k < M \text{ for all } k}} T_{D,\boldsymbol{c},\boldsymbol{d}} \circ B^n(\omega(g_{\boldsymbol{d}})).$$

This vanishes on the open subspace $\Big\{\sum_{\boldsymbol{d}} g_{\boldsymbol{d}} x_1^{d_1} \cdots x_l^{d_l} \;\Big|\; g_{\boldsymbol{d}} \in U \text{ whenever } d_1, \dots, d_l < M\Big\}$. So $\Theta_{D,\boldsymbol{c}}$ is continuous for the discrete topology on its codomain. For every $f \in Z[x_1, \dots, x_l]$, we have that

$$D(f)_V = \sum_{\boldsymbol{c} \in F^l} \Theta_{D,\boldsymbol{c}}\big(\psi_\lambda\big(f(c_1 + x_1, \dots, c_l + x_l)\big)\big).$$

Consequently,

$$\theta_V = \sum_{D\in S(m\times n)} \sum_{\boldsymbol{c}\in F^l} \Theta_{D,\boldsymbol{c}}\big(\psi_\lambda\big(\mathrm{pr}_D(\theta)(c_1+x_1,\dots,c_l+x_l)\big)\big)\,.$$

Each summand is continuous by Lemma 9.7 and the continuity of $\Theta_{D,\boldsymbol{c}}$. Both indexing sets are finite, proving the lemma. □

Theorem 9.9. *Any Brauer categorification* $\mathbf{R}$ *can be made into an even nilpotent iquantum categorification via a construction which is the inverse of the one in Corollary 8.3.*

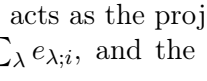
Proof. Recall the "block" decomposition of $\mathbf{R}$ from (4.11) and (4.12) and the eigenfunctor decomposition $B = \bigoplus_{i\in I} B_i$ of the endofunctor $B : \mathbf{R} \to \mathbf{R}$ from (4.13). We also define $\mathbf{R}_\lambda$ to be a trivial (zero) $\Bbbk$-linear category in case $\lambda \in X^\imath_{\bar{1}}$, so $\mathbf{R} = \bigoplus_{\lambda\in X^\imath} \mathbf{R}_\lambda$. The subcategories $\mathbf{R}_\lambda$ are the required iweight categories in order to have an even iquantum categorification, and Corollary 4.10 shows that the endofunctors B_i take $\mathbf{R}_\lambda$ to $\mathbf{R}_{\lambda-\alpha_i}$. To prove the theorem, the idea is to show that the action of $\mathbf{AB}$ on $\mathbf{R}$ extends uniquely to a continuous action of $\widehat{\mathbf{AB}}$, then to pass from there to the action of $\mathfrak{U}^\imath$. First, we establish two claims:

<u>Claim 1</u>: *The action of* $\mathbf{AB}$ *on* $\mathbf{R}$ *extends uniquely to an action of* $\widehat{\mathbf{AB}}$ *which is continuous in the sense that the evaluation maps* $\mathrm{Hom}_{\widehat{\mathbf{AB}}}(B^{\otimes n}, B^{\otimes m}) \to \mathrm{Hom}_{\mathbf{R}}(B^nV, B^mV)$ *are continuous with respect to the discrete topology on* $\mathrm{Hom}_{\mathbf{R}}(B^nV, B^mV)$.

Proof: To extend the action of $\mathbf{AB}$ to $\widehat{\mathbf{AB}}$, we first need to define $\theta_V : B^nV \to B^mV$ for any $\theta \in \mathrm{Hom}_{\widehat{\mathbf{AB}}}(B^{\otimes n}, B^{\otimes m})$ and $V \in \mathbf{R}$, then we need to check that $(\theta_V)_{V\in\mathbf{R}}$ is a natural transformation, and that the assignments $B^{\otimes n} \mapsto B^n$ and $\theta \mapsto (\theta_V)_{V\in\mathbf{R}}$ define a strict $\Bbbk$-linear monoidal functor $\widehat{\mathbf{AB}} \to \mathbf{End}_\Bbbk(\mathbf{R})$. To define $(\theta_V)_{V\in\mathbf{R}}$, take some object $V \in \mathbf{R}$. By Lemma 9.8, there is an open subspace N of $\mathrm{Hom}_{\widehat{\mathbf{AB}}}(B^{\otimes n}, B^{\otimes m})$ such that all elements of $\mathrm{Hom}_{\mathbf{AB}}(B^{\otimes n}, B^{\otimes m}) \cap N$ evaluate to 0 on V. By Theorem 9.5, there is $\tilde\theta \in \mathrm{Hom}_{\mathbf{AB}}(B^{\otimes n}, B^{\otimes m})$ with $\theta - \tilde\theta \in N$. Then we let $\theta_V := \tilde\theta_V$ for this $\tilde\theta$; this is well defined independent of the choice of $\tilde\theta$. To prove naturality and compatibility with horizontal and vertical composition, any identity that needs to be checked involves only finitely many objects of $\mathbf{R}$. As horizontal and vertical composition in $\widehat{\mathbf{AB}}$ are continuous, the identity follows from the corresponding identity for morphisms of $\mathbf{AB}$ approximating the morphisms of $\widehat{\mathbf{AB}}$ involved closely enough. This makes $\mathbf{R}$ into a strict left $\widehat{\mathbf{AB}}$-module category whose restriction along Ψ is the given $\mathbf{AB}$-module structure. Uniqueness is clear by density.

<u>Claim 2</u>: *The idempotents* $e_\lambda = \iota(\mathrm{id}_{\mathbb{1}_\lambda})$ *and* $e_{\lambda;i} = \iota(\mathrm{id}_{B_i\mathbb{1}_\lambda})$ *from the proof of Theorem 9.5 act as the projections onto the "blocks"* $\mathbf{R}_\lambda$ *and the eigenfunctors* B_i *from (4.13), respectively.*

Proof: For an indecomposable $V \in \mathbf{R}_\mu$, consider the composition $\widehat{Z} \to \widehat{Z}_\mu \xrightarrow{\omega} Z_V$ for ω as in Lemma 9.6. By Lemma 9.6(iii) this continuous homomorphism maps $z \mapsto z_V$ for z in the dense subalgebra Z, hence, it is equal to the evaluation map from Claim 1. In particular, $(e_\lambda)_V = \delta_{\lambda,\mu}\,\mathrm{id}_V$. Taking finite direct sums proves that e_λ acts as the projection onto $\mathbf{R}_\lambda$.

Next, $\mathrm{id}_B = \sum_{i\in I} e_i$ where $e_i := \sum_\lambda e_{\lambda;i}$, and the discussion of idempotents in the proof of Theorem 9.5 shows that e_i is the limit over all finite $F \subset I$ and $r \geq 0$ of $\psi_{1,1}\Big($  $\Big)$ for polynomials $f_{F,r}(x)$ chosen so that $f_{F,r}(x) \equiv \delta_{i,j} \pmod{(x-j)^r}$ for all $j \in F$. Then e_i acts on BV as the projection onto B_iV. Since $e_{\lambda;i} = e_i \circ (\mathrm{id}_B \otimes e_\lambda)$, it acts on BV ($V \in \mathbf{R}_\lambda$) as the projection onto B_iV, and it acts as 0 on BV for $V \in \mathbf{R}_\mu$ with $\mu \neq \lambda$.

Now we can complete the proof of the theorem. For $\lambda \in X^\imath_{\bar0}$, let $B_i\mathbb{1}_\lambda$ act as the restriction $B_i|_{\mathbf{R}_\lambda} : \mathbf{R}_\lambda \to \mathbf{R}_{\lambda-\alpha_i}$. Any 2-morphism $\theta \in \mathrm{Hom}_{\mathfrak{U}^\imath}(B_{\boldsymbol{j}}\mathbb{1}_\lambda, B_{\boldsymbol{i}}\mathbb{1}_\lambda)$ acts on $V \in \mathbf{R}_\lambda$ as the restriction of $\iota(\theta)_V : B^nV \to B^mV$ to the summand $B_{\boldsymbol{j}}V$ of B^nV; by Claim 2, this maps it into the summand $B_{\boldsymbol{i}}V$ of B^mV. As ι is compatible with vertical and horizontal composition and the action of $\widehat{\mathbf{AB}}$ on $\mathbf{R}$ is a strict monoidal action, all of the relations of $\mathfrak{U}^\imath$ are satisfied. Also $B_iV \neq 0$ for only

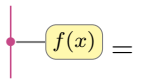
finitely many $i \in I$. So this makes $\mathbf{R} = \bigoplus_{\lambda \in X^\imath} \mathbf{R}_\lambda$ into an even iquantum categorification. By (8.3) and (8.9), acts as , as it should by (8.1), and this is nilpotent on $B_i V$ since $B_i V$ is a generalized eigenspace. Hence, $\mathbf{R}$ is nilpotent.

Finally, we indicate why this construction and the one from Corollary 8.3 are inverse to each other. If we start from a Brauer categorification, apply the construction just explained, and then apply the construction of Corollary 8.3, we obtain an action of $\widehat{\mathbf{AB}}$ on $\mathbf{R}$ via Theorem 6.8. It is continuous, because the action of a morphism of $\widehat{\mathbf{AB}}$ on $V \in \mathbf{R}$ only involves the finitely many entries of the matrix indexed by the iweights of V and the words $\boldsymbol{j}$ with $B_{\boldsymbol{j}} V \neq 0$, and each of these acts continuously by Claim 2 in the proof of Theorem 6.8. It agrees with the action from Claim 1 on all $\iota(\theta)$ by construction, so the two actions agree by Remark 6.7, and their restrictions along Ψ are both the original action of $\mathbf{AB}$. Conversely, if we start from an even nilpotent iquantum categorification, apply Corollary 8.3 and then the construction just explained, the action of $\widehat{\mathbf{AB}}$ obtained in Claim 1 is the continuous one coming from Corollary 8.3, by the uniqueness in Claim 1. So e_λ and $e_{\lambda;i}$ act as the projections onto $\mathbf{R}_\lambda$ and $B_i \mathbb{1}_\lambda$, and $\iota(\theta)$ acts as θ, i.e., we do indeed recover the original iquantum categorification. □

10. Cyclotomic quotients

In this section, we prove that cyclotomic quotients of $\mathbf{AB}$ and cyclotomic quotients of $\mathfrak{U}^\imath$ are equivalent categories. The former are Karoubian Brauer categorifications, and the latter are even Karoubian iquantum categorifications. In the case of $\mathbf{AB}$, the definition of these cyclotomic quotients appeared already in [RS19]; in fact, it can be traced back to [AMR06]. In the case of $\mathfrak{U}^\imath$, the definition was written down already in [GRS26] in the $\mathrm{AIII}_{2\infty}$ case. Recall for a monic polynomial $f(u)$ that $\widetilde{f}(u) = (-1)^{\deg f(u)} f(-u)$.

Definition 10.1. Let A be a finite-dimensional commutative $\Bbbk$-algebra and $f(u) \in A[u]$ be a monic polynomial of degree $\beta \geq 0$. Let $\mathbf{AB} \otimes_\Bbbk A$ be the strict A-linear monoidal category obtained from $\mathbf{AB}$ by base change and $\mathbf{I}(A, f)$ be the A-linear left tensor ideal of $\mathbf{AB} \otimes_\Bbbk A$ generated by the relations



$$= 0, \qquad \bigcirc(u) = \begin{cases} \dfrac{u-h}{u} \cdot \dfrac{\widetilde{f}(u)}{f(u)} \mathrm{id}_{\mathbb{1}} & \text{if } \beta \text{ is even} \\ \dfrac{u+h}{u} \cdot \dfrac{\widetilde{f}(u)}{f(u)} \mathrm{id}_{\mathbb{1}} & \text{if } \beta \text{ is odd.} \end{cases} \tag{10.1}$$

The *deformed cyclotomic Brauer category* of level β associated to A and f is

$$\mathbf{cB}(A, f) \coloneqq \mathrm{Kar}\big(\mathbf{AB} \otimes_\Bbbk A \,/\, \mathbf{I}(A, f)\big).$$

By definition, $\mathbf{cB}(A, f)$ is a strict A-linear left $\mathbf{AB}$-module category. Lemma 10.2 below shows that it is a Karoubian Brauer categorification. In the case that $A = \Bbbk$, we denote $\mathbf{cB}(A, f)$ simply by $\mathbf{cB}(f)$, calling it the *cyclotomic Brauer category* of level β associated to f.

Lemma 10.2 (Rui-Song)**.** *Morphism spaces in the deformed cyclotomic quotient $\mathbf{cB}(A, f)$ are finite-dimensional for every finite-dimensional commutative algebra A and monic $f \in A[x]$. Hence, $\mathbf{cB}(A, f)$ is a Karoubian Brauer categorification.*

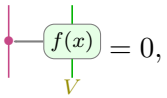
Proof. The proof of [RS19, Prop. 3.7(b)–(c)] extends unchanged to the deformed setting to show that $\mathrm{Hom}_{\mathbf{cB}(A,f)}(B^{\otimes n}, B^{\otimes m})$ is finite-dimensional for every $m, n \geq 0$. □

Remark 10.3. There is an explicit basis theorem for the morphism spaces $\mathrm{Hom}_{\mathbf{cB}(A,f)}(B^{\otimes n}, B^{\otimes m})$. For the cyclotomic Brauer category $\mathbf{cB}(f)$, this follows immediately from [RS19, Th. C]. Rui and Song prove their basis theorem working over an arbitrary ground ring, not just a field, so that the basis theorem in our deformed setting can be deduced from their result by making a flat base change

from $\Bbbk$ to a larger ring containing the bubble parameters, then applying [RS19, Th. C] there. In particular, $\mathrm{End}_{\mathbf{cB}(A,f)}(\mathbb{1}) \cong A$, hence, $\mathbf{cB}(f)$ is non-zero, something which is by no means obvious from the definition.

The following lemma is the universal property of deformed cyclotomic Brauer categories:

Lemma 10.4. *Let V be an indecomposable (hence, non-zero) object of an Abelian or Karoubian Brauer categorification $\mathbf{R}$, and Z_V be the center of $\mathrm{End}_{\mathbf{R}}(V)$ as usual. There exists a monic polynomial $f(u) \in Z_V[u]$ of degree $\beta \geq 0$ such that*



$$= 0, \qquad O_V(u) = \begin{cases} \dfrac{u-h}{u} \cdot \dfrac{\widetilde{f}(u)}{f(u)} & \text{if } \beta \text{ is even} \\[2ex] \dfrac{u+h}{u} \cdot \dfrac{\widetilde{f}(u)}{f(u)} & \text{if } \beta \text{ is odd.} \end{cases} \tag{10.2}$$

Moreover, there is a canonical morphism of strict left $\mathbf{AB}$-module categories $\Pi\colon \mathbf{cB}(Z_V, f) \to \mathbf{R}$ mapping the generating object $\mathbb{1}$ of $\mathbf{cB}(Z_V, f)$ to V.

Proof. In view of (4.3), the first assertion follows immediately from Theorem 4.3. There is a canonical morphism of strict left $\mathbf{AB}$-module categories $\mathbf{AB} \otimes_{\Bbbk} Z_V \to \mathbf{R}$ mapping $\mathbb{1}$ to V and $\mathrm{id}_{\mathbb{1}} \otimes z \in \mathrm{End}_{\mathbf{AB}}(\mathbb{1}) \otimes_{\Bbbk} Z_V$ to $z \in \mathrm{End}_{\mathbf{R}}(V)$. This sends morphisms in $\mathbf{I}(Z_V, f)$ to 0, as follows by comparing (4.4) and (10.1), so it factors through the quotient to induce Π. Finally, it extends to the additive Karoubi envelope because $\mathbf{R}$ is additive and idempotent complete. $\square$

Next, we formulate the definition of a cyclotomic quotient of $\mathfrak{U}^\imath$. For any $\lambda \in X^\imath$, the corresponding *regular 2-representation* of $\mathfrak{U}^\imath$ is $(\mathbb{1}_\mu \mathfrak{U}^\imath \mathbb{1}_\lambda)_{\mu \in X^\imath}$ with μth iweight subcategory

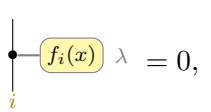
$$\mathbb{1}_\mu \mathfrak{U}^\imath \mathbb{1}_\lambda \coloneqq \mathrm{Hom}_{\mathfrak{U}^\imath}(\lambda, \mu)$$

for each $\mu \in X^\imath$, and categorical action defined by horizontal composition on the left. It is an even 2-representation if λ is even.

Definition 10.5. Let A be a finite-dimensional commutative $\Bbbk$-algebra and $f_i(u) \in A[u]$ be monic polynomials for all $i \in I$ such that $f_i(u) = 1$ for all but finitely many $i \in I$. Let $\beta_i \coloneqq \deg f_i(u)$, so that $\beta_i = 0$ for all but finitely many $i \in I$, and $\beta \coloneqq \sum_{i \in I} \beta_i$. Let $\lambda \in X_{\bar{0}}^\imath$ be the canonical image of the even weight

$$\hat{\lambda} \coloneqq \begin{cases} \displaystyle\sum_{i \in I} \beta_i \varpi_i & \text{if } \beta \text{ is even} \\[2ex] \displaystyle\varpi_h + \sum_{i \in I} \beta_i \varpi_i & \text{if } \beta \text{ is odd.} \end{cases} \tag{10.3}$$

Let $\mathbf{I}(A, f_i \,|\, i \in I)$ be the sub-2-representation[3] of the A-linear 2-representation $\big(\mathbb{1}_\mu \mathfrak{U}^\imath \mathbb{1}_\lambda \otimes_{\Bbbk} A\big)_{\mu \in X^\imath}$ generated by the relations



$$= 0, \qquad \bar{\imath}\bigcirc(u)\,\lambda = \begin{cases} u^{\delta_{i,h} - \delta_{i,0}} \dfrac{\widetilde{f}_{-i}(u)}{f_i(u)} \mathrm{id}_{\mathbb{1}_\lambda} & \text{if } \beta \text{ is even} \\[2ex] u^{\delta_{i,-h} - \delta_{i,0}} \dfrac{\widetilde{f}_{-i}(u)}{f_i(u)} \mathrm{id}_{\mathbb{1}_\lambda} & \text{if } \beta \text{ is odd.} \end{cases} \tag{10.4}$$

Then the *deformed cyclotomic iquantum category* $\mathbf{cU}^\imath(A, f_i \,|\, i \in I)$ is the iquantum categorification with μth iweight subcategory

$$\mathbb{1}_\mu \mathbf{cU}^\imath(A, f_i \,|\, i \in I) \coloneqq \mathrm{Kar}\left(\mathbb{1}_\mu \mathfrak{U}^\imath \mathbb{1}_\lambda \otimes_{\Bbbk} A \;/\; \mathbb{1}_\mu \mathbf{I}(A, f_i \,|\, i \in I)\right)$$

[3]We mean an A-linear ideal of the category which is closed under the left action of $\mathfrak{U}^\imath$.

for each $\mu \in X^\imath$. By Lemma 10.6 below, it is a Karoubian iquantum categorification. In the case that $A = \Bbbk$, we denote $\mathbf{cU}^\imath(A, f_i \,|\, i \in I)$ simply by $\mathbf{cU}^\imath(f_i \,|\, i \in I)$, calling it the *cyclotomic iquantum category.*

The following standard lemma referenced in Definition 10.5 extends [GRS26, Prop. 8.5]:

Lemma 10.6. *In the setup of Definition 10.5,* $\mathbf{cU}^\imath(A, f_i \,|\, i \in I)$ *is a Karoubian iquantum categorification. Moreover, if the coefficients of each* $f_i(u)$ *apart from the leading coefficient 1 belong to the Jacobson radical of* A*, then* $\mathbf{cU}^\imath(A, f_i \,|\, i \in I)$ *is nilpotent.*

Proof. Let $\mathbf{cU}^\imath := \mathbf{cU}^\imath(A, f_i \,|\, i \in I)$, and $\lambda = \pi(\hat{\lambda})$ as in (10.3), so that the generating object of $\mathbf{cU}^\imath$ is $\mathbb{1}_\lambda$. We first show by induction on n that $\mathrm{Hom}_{\mathbf{cU}^\imath}(B_{\boldsymbol{j}}\mathbb{1}_\lambda, B_{\boldsymbol{i}}\mathbb{1}_\lambda)$ is finite-dimensional for words $\boldsymbol{i} \approx \boldsymbol{j}$ of total length $2n$. For the induction base $n = 0$, Theorem 3.6 and the bubble relations in (10.4) show that $\mathrm{End}_{\mathbf{cU}^\imath}(\mathbb{1}_\lambda)$ is a quotient of A, so it is finite-dimensional.

Now consider the induction step. By adjunction, we may assume that both of the words $\boldsymbol{i}$ and $\boldsymbol{j}$ are of the same length $n > 0$. Fix a finite subset $S \subset I$ closed under $\pm$ and containing the letters of the words $\boldsymbol{i}$ and $\boldsymbol{j}$. In the additive envelope, let

$$T_n := \bigoplus_{\boldsymbol{i} \in S^n} B_{\boldsymbol{i}}\mathbb{1}_\lambda, \qquad\qquad E_n := \mathrm{End}_{\mathbf{cU}^\imath}(T_n).$$

Let J_n be the two-sided ideal of E_n consisting of sums of morphisms which factor through objects of the form $B_{\boldsymbol{k}}\mathbb{1}_\lambda$ with $\boldsymbol{k} \in S^{n-2}$. In particular, $J_1 = 0$. For each such $\boldsymbol{k}$, both of the spaces $\mathrm{Hom}_{\mathbf{cU}^\imath}(T_n, B_{\boldsymbol{k}}\mathbb{1}_\lambda)$ and $\mathrm{Hom}_{\mathbf{cU}^\imath}(B_{\boldsymbol{k}}\mathbb{1}_\lambda, T_n)$ are finite-dimensional by induction. Since there are only finitely many $\boldsymbol{k}$, it follows that the ideal J_n is finite-dimensional.

By Theorem 3.6 (the easy spanning part), every morphism in E_n is an R-linear combination of string diagrams of reduced shapes decorated by some dots at the distinguished points. Elements of R are bubbles in the rightmost 2-cell, which can be replaced by scalars from the algebra A using the second of the relations (10.4). We assume that the reduced shapes are chosen so that all caps are below all crossings and all cups are above them. Then any reduced diagram with one or more cups (and, necessarily, the same number of caps) has $\leq (n-2)$ propagating strings, so it factors through an object $B_{\boldsymbol{k}}\mathbb{1}_\lambda$ for $\boldsymbol{k}$ of length $n-2$, i.e., it belongs to J_n. Consequently, the quotient E_n/J_n is generated by dots, crossings, and idempotents. Moreover, modulo J_n, defining relations of $\mathfrak{U}^\imath$ simplify to ordinary quiver Hecke algebra relations in the deformed quiver Hecke algebras $QH_n \otimes_\Bbbk A$ associated to the finite Cartan matrix $(a_{i,j})_{i,j \in S}$ with parameters $Q^\imath_{i,j}(x,y)$ $(i,j \in S)$, and the cyclotomic relation that is the first relation from (10.4) for dots on the rightmost string. Thus, E_n/J_n is a quotient of the corresponding deformed cyclotomic quiver Hecke algebra, which is well known to be finitely generated as a right A-module, an observation which goes back at least to [BK09, Cor. 2.2]. Since A is finite-dimensional, it follows that E_n/J_n is finite-dimensional, and the induction is complete.

To complete the proof that $\mathbf{cU}^\imath$ is a Karoubian iquantum categorification, we still need to show that $\big|\{j \in I \mid B_j V \neq 0\}\big| < \infty$ for every object V. It suffices to prove this in the case that $V = B_{\boldsymbol{i}}\mathbb{1}_\lambda$ for $\boldsymbol{i} = i_1 \cdots i_n$. Let $K := \{j \,|\, f_j(u) \neq 1\} \cup \bigcup_{r=1}^n \{\pm i_r,\ \pm i_r + 1,\ \pm i_r - 1\}$. This set is finite. If $j \notin K$, the crossings moving the j-string past the letters of $\boldsymbol{i}$ are invertible. Thus $B_j B_{\boldsymbol{i}}\mathbb{1}_\lambda \cong B_{\boldsymbol{i}} B_j \mathbb{1}_\lambda = 0$ since $f_j(u) = 1$.

It remains to prove the final assertion about nilpotency. This follows by modifying the above inductive argument. Under the nilpotency assumption on the polynomials $f_i(u)$, it is well known that dots are nilpotent in the deformed cyclotomic quiver Hecke algebra mapping onto E_n/J_n. Hence, in E_n/J_n, the dots are nilpotent.

In this paragraph, we prove by induction on n that every dot on every object $B_{\boldsymbol{i}}\mathbb{1}_\lambda$ with $\boldsymbol{i} \in S^n$ is nilpotent. The assertion is vacuous for $n = 0$, and follows from nilpotency modulo $J_1 = 0$ for n=1. Suppose that $n \geq 2$, and write $\boldsymbol{i} = i_1 \boldsymbol{i}'$. For $\boldsymbol{k} \in S^{n-2}$, adjunction gives

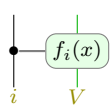
$$\mathrm{Hom}_{\mathbf{cU}^\imath}(B_{\boldsymbol{k}}\mathbb{1}_\lambda, B_{i_1} B_{\boldsymbol{i}'}\mathbb{1}_\lambda) \cong \mathrm{Hom}_{\mathbf{cU}^\imath}(B_{-i_1} B_{\boldsymbol{k}}\mathbb{1}_\lambda, B_{\boldsymbol{i}'}\mathbb{1}_\lambda).$$

Under this isomorphism, vertical composition on top by a dot on the first string in the left-hand space becomes vertical composition on the bottom by minus a dot on the first string in the right-hand space, and vertical composition on top by a dot on the rth string for $r > 1$ becomes vertical composition on top by a dot on the $(r-1)$th string. Both words on the right have length $n-1$, so each of these operators is nilpotent by induction.

Let $X_r \in E_n$ be the sum of the dots on the rth strings of the summands of T_n. Since S is finite, there is an exponent a such that vertical composition on top with X_r^a annihilates every $\mathrm{Hom}_{\mathbf{cU}^\imath}(B_{\boldsymbol{k}}\mathbb{1}_\lambda, T_n)$ with $\boldsymbol{k} \in S^{n-2}$. Hence $X_r^a J_n = 0$. Nilpotency modulo J_n gives $X_r^b \in J_n$ for some b, whence $X_r^{a+b} = 0$. This completes the induction. Every word is contained in S^n for some finite $S = -S$, and nilpotency passes to finite direct sums and direct summands, proving the assertion. □

The following lemma establishes the important universal property of $\mathbf{cU}^\imath(A, f_i \,|\, i \in I)$:

Lemma 10.7. *Let $\mathbf{R}$ be any even nilpotent iquantum categorification, $\lambda \in X_{\bar{0}}^\imath$ and $V \in \mathbf{R}_\lambda$ be an indecomposable object, so that Z_V is a finite-dimensional local commutative $\Bbbk$-algebra. Also fix a choice of a parity $\sigma \in \{\bar{0}, \bar{1}\}$. Then there exists a monic polynomial $f(u) \in Z_V[u]$ of degree $\beta \equiv \sigma \pmod 2$ with local factors $f_i(u)$ defined as in Lemma 1.1 such that*



$$= 0, \qquad O_{V;i}(u) = \begin{cases} u^{\delta_{i,h}-\delta_{i,0}} \dfrac{\widetilde{f}_{-i}(u)}{f_i(u)} & \text{if } \sigma = \bar{0} \\ u^{\delta_{i,-h}-\delta_{i,0}} \dfrac{\widetilde{f}_{-i}(u)}{f_i(u)} & \text{if } \sigma = \bar{1}. \end{cases} \tag{10.5}$$

Moreover, there is a canonical morphism of 2-representations $\Pi\colon \mathbf{cU}^\imath(Z_V, f_i \,|\, i \in I) \to \mathbf{R}$ mapping the generating object $\mathbb{1}_\lambda$ of $\mathbf{cU}^\imath(Z_V, f_i \,|\, i \in I)$ to V.

Proof. Let $\mathfrak{m}$ be the unique maximal ideal of Z_V. Also let $I = I' \cup \{0\} \cup (-I')$ be a decomposition chosen so that $-h \in I'$ if $\sigma = \bar{0}$ or $h \in I'$ if $\sigma = \bar{1}$. Let $f_0(u) \in Z_V[u]$ be a monic polynomial chosen as in the final assertion of Lemma 5.8. This lemma implies that (10.5) holds in the case $i = 0$. Next, for each $i \in I'$, we let $f_i(u) := m_{V;i}(u) \in \Bbbk[u] \subseteq Z_V[u]$, then define $f_{-i}(u)$ so that $\widetilde{f}_{-i}(u) = f_i(u) O_{V;i}(u)$. Both $f_i(u)$ and $f_{-i}(u)$ are then monic polynomials. Lemma 5.3 implies that both properties of (10.5) hold for $i \in I'$, and that the first property in (10.5) holds for $i \in -I'$. The second property in (10.5) is also true for $i \in -I'$ due to (3.22).

We have now constructed monic polynomials $f_i(u) \in Z_V[u]$ so that (10.5) holds for all $i \in I$. Moreover, each $f_i(u)$ is monic and the coefficients of its lower degree terms are nilpotent. This is true for $i = 0$ by Lemma 5.8, and it is clear for $i \in I'$ since those $f_i(u)$ are monomials. It follows for $i \in -I'$ since by the definition the lower degree terms of such $f_i(u)$ are linear combinations of bubbles of positive degree, which are nilpotent in Z_V by Lemma 5.7.

We have that $m_{V;i}(u) = 1$ for all but finitely many i, and $\lambda_i = 0$ for all but finitely many i. For $i \in I'$ avoiding these finite sets and $\{\pm h\}$, Lemma 5.3 shows that $f_{-i}(u)$ is monic of degree zero, hence is 1. Consequently, $f_i(u) = 1$ for all but finitely many $i \in I$. Hence, it makes sense to set $f(u) := \prod_{i \in I} f_i(u - i)$. Since the lower degree coefficients of all $f_i(u)$ are in the unique maximal ideal $\mathfrak{m}$, $f_i(u)$ is the local factor of $f(u)$ at $u = i$.

Let $\beta_i := \deg f_i(u)$ and $\beta := \deg f(u) = \sum_{i \in I} \beta_i$. By (3.21) and (10.5), for $i \neq 0$, we have that $\lambda_i = \beta_i - \beta_{-i}$ if $\sigma = \bar{0}$ and $\lambda_i = \beta_i - \beta_{-i} + \delta_{i,h} - \delta_{i,-h}$ if $\sigma = \bar{1}$. Also Lemma 5.8 gives that $\beta_0 \equiv t(\lambda) \pmod 2$. Since λ is even, it follows that $\beta \equiv \sigma \pmod 2$, and that $\lambda = \pi(\hat{\lambda})$ for $\hat{\lambda}$ defined by (10.3).

It just remains to prove the final assertion of the lemma. By the universal property of the regular 2-representation $\mathfrak{U}^\imath \mathbb{1}_\lambda \otimes_\Bbbk Z_V$, there is a canonical morphism of 2-representations $\mathfrak{U}^\imath \mathbb{1}_\lambda \otimes_\Bbbk Z_V \to \mathbf{R}$ mapping $\mathbb{1}_\lambda$ to V, and $\mathrm{id}_{\mathbb{1}_\lambda} \otimes z \in \mathrm{End}^\imath_{\mathfrak{U}}(\mathbb{1}_\lambda) \otimes_\Bbbk Z_V$ to $z \in \mathrm{End}_{\mathbf{R}}(V)$. This maps the generators of $\mathbf{I}(Z_V, f_i \,|\, i \in I)$ to 0, as follows by comparing (10.4) and (10.5), so it factors through the quotient

$\mathbf{cU}^\imath(Z_V, f_i \,|\, i \in I)$ to induce Π. Finally, this extends to the additive Karoubi envelope because $\mathbf{R}$ is additive and idempotent complete. □

Theorem 10.8. *Suppose that A is a finite-dimensional local commutative $\Bbbk$-algebra and $f(u) \in A[u]$ is a monic polynomial of degree β. Let $f(u) = \prod_{i\in I} f_i(u-i)$ be the factorization of $f(u)$ from Lemma 1.1, so $f_i(u)$ is the local factor of $f(u)$ at $u = i$. Then there is an A-linear equivalence*

$$\Psi_{A,f} \colon \mathbf{cB}(A, f) \to \mathbf{cU}^\imath(A, f_i \,|\, i \in I).$$

Proof. Let $\mathbf{cB} := \mathbf{cB}(A, f)$ and $\mathbf{cU}^\imath := \mathbf{cU}^\imath(A, f_i \,|\, i \in I)$. Let $\lambda \in X^\imath$ be the image of the weight (10.3). The local factors of $\chi_f(u)$ are

$$R_i(u) := \begin{cases} u^{\delta_{i,h}-\delta_{i,0}} \dfrac{\widetilde{f}_{-i}(u)}{f_i(u)} & \text{if } \beta \text{ is even,} \\ u^{\delta_{i,-h}-\delta_{i,0}} \dfrac{\widetilde{f}_{-i}(u)}{f_i(u)} & \text{if } \beta \text{ is odd.} \end{cases}$$

In particular, $\chi_f(u) = \prod_{i\in I} R_i(u-i)$.

To avoid ambiguity, let V and W denote the generating objects $\mathbb{1}$ and $\mathbb{1}_\lambda$ of $\mathbf{cB}$ and $\mathbf{cU}^\imath$, respectively. By Lemma 10.6 and Corollary 8.3, $\mathbf{cU}^\imath$ is a Karoubian Brauer categorification. On B_iW, the Brauer dot is $i + x$, where x is the iquantum dot. Thus f annihilates the Brauer dot on BW, since $f_i(x) = 0$ on B_iW. Moreover, (8.13) and the defining bubble relations give $O_W(u) = \prod_{i\in I} R_i(u-i) = \chi_f(u)$. Consequently, the factorization argument in Lemma 10.4 yields an A-linear morphism of $\mathbf{AB}$-module categories $F : \mathbf{cB} \to \mathbf{cU}^\imath,\ V \mapsto W$.

Conversely, equip $\mathbf{cB}$ with its even nilpotent iquantum categorification structure from Theorem 9.9. Note that $\mathrm{End}_{\mathbf{cB}}(V)$ is a non-zero quotient of the local algebra A, so V is indecomposable. It has iweight λ thanks to Remark 4.8 and Lemma 4.6. The factorization $f(x) = \prod_{i\in I} f_i(x-i)$ implies that the iquantum dot on B_iV is annihilated by f_i. Since $O_V(u) = \chi_f(u)$, uniqueness of local factorization and Lemma 9.6 give $O_{V;i}(u) = R_i(u)$. Thus the factorization argument in Lemma 10.7 yields an A-linear morphism of 2-representations $G : \mathbf{cU}^\imath \to \mathbf{cB},\ W \longmapsto V$.

Under the mutually inverse constructions of Corollary 8.3 and Theorem 9.9, a morphism of $\mathbf{AB}$ categorifications intertwines the iquantum action constructed in Theorem 9.9, and a morphism of iquantum categorifications intertwines the Brauer structures from Corollary 8.3. This follows from the uniqueness of continuous extension in the first case, and the explicit formulae in the other. Consequently, $G \circ F$ and $F \circ G$ are morphisms of Brauer and iquantum categorifications, respectively. Both preserve the generating object and its prescribed A-action. The universal properties used in Lemmas 10.4 and 10.7 determine these functors, up to natural isomorphism, from the generating object and its prescribed A-action, so they give natural isomorphisms $G \circ F \cong \mathrm{id}_{\mathbf{cB}}$ and $F \circ G \cong \mathrm{id}_{\mathbf{cU}^\imath}$. Thus F is the required A-linear equivalence $\Psi_{A,f}$. □

References

[AMR06] Susumu Ariki, Andrew Mathas, and Hebing Rui. Cyclotomic Nazarov-Wenzl algebras. *Nagoya Math. J.*, 182:47–134, 2006. doi:10.1017/S0027763000026842.

[BCNR17] Jonathan Brundan, Jonathan Comes, David Nash, and Andrew Reynolds. A basis theorem for the affine oriented Brauer category and its cyclotomic quotients. *Quantum Topol.*, 8(1):75–112, 2017. doi:10.4171/QT/87.

[BD17] Jonathan Brundan and Nicholas Davidson. Categorical actions and crystals. In *Categorification and higher representation theory*, volume 683 of *Contemp. Math.*, pages 105–147. Amer. Math. Soc., Providence, RI, 2017. doi:10.1090/conm/683.

[BK07] Arkady Berenstein and David Kazhdan. Geometric and unipotent crystals. II. From unipotent bicrystals to crystal bases. In *Quantum groups*, volume 433 of *Contemp. Math.*, pages 13–88. Amer. Math. Soc., Providence, RI, 2007. doi:10.1090/conm/433/08321.

[BK09] Jonathan Brundan and Alexander Kleshchev. Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras. *Invent. Math.*, 178(3):451–484, 2009. doi:10.1007/s00222-009-0204-8.

[BM27] Jonathan Brundan and Abhilash Mukherjee. Oriented skein categories and Khovanov's arc algebra. in preparation, 2027.

[Bru18] Jonathan Brundan. On the definition of Heisenberg category. *Algebr. Comb.*, 1(4):523–544, 2018. doi:10.5802/alco.26.

[BS12] Jonathan Brundan and Catharina Stroppel. Gradings on walled Brauer algebras and Khovanov's arc algebra. *Adv. Math.*, 231(2):709–773, 2012. doi:10.1016/j.aim.2012.05.016.

[BSW20] Jonathan Brundan, Alistair Savage, and Ben Webster. Heisenberg and Kac-Moody categorification. *Selecta Math. (N.S.)*, 26:Paper No. 74, 62, 2020. doi:10.1007/s00029-020-00602-5.

[BSW26] Jonathan Brundan, Alistair Savage, and Ben Webster. An update on Heisenberg and Kac-Moody categorification. *J. Pure Appl. Algebra*, 230(8):Paper No. 108315, 2026. doi:10.1016/j.jpaa.2026.108315.

[BSWW18] Huanchen Bao, Peng Shan, Weiqiang Wang, and Ben Webster. Categorification of quantum symmetric pairs I. *Quantum Topol.*, 9(4):643–714, 2018. doi:10.4171/QT/117.

[BV22] Jonathan Brundan and Max Vargas. A new approach to the representation theory of the partition category. *J. Algebra*, 601:198–279, 2022. doi:10.1016/j.jalgebra.2022.03.010.

[BW18a] Huanchen Bao and Weiqiang Wang. Canonical bases arising from quantum symmetric pairs. *Invent. Math.*, 213(3):1099–1177, 2018. doi:10.1007/s00222-018-0801-5.

[BW18b] Huanchen Bao and Weiqiang Wang. A new approach to Kazhdan-Lusztig theory of type B via quantum symmetric pairs. *Astérisque*, 402:vii+134, 2018. URL: https://smf.emath.fr/system/files/2017-08/smf_ast_402.pdf.

[BWW25] Jonathan Brundan, Weiqiang Wang, and Ben Webster. Categorification of quasi-split iquantum groups, 2025. arXiv:2505.22929.

[BWW26] Jonathan Brundan, Weiqiang Wang, and Ben Webster. Nil-Brauer categorifies the split iquantum group of rank one. *Forum Math. Sigma*, 14:e130, 2026. doi:10.1017/fms.2026.10278.

[CR08] Joseph Chuang and Raphaël Rouquier. Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification. *Ann. of Math. (2)*, 167(1):245–298, 2008. doi:10.4007/annals.2008.167.245.

[ES16] Michael Ehrig and Catharina Stroppel. Koszul gradings on Brauer algebras. *Int. Math. Res. Not. IMRN*, pages 3970–4011, 2016. doi:10.1093/imrn/rnv267.

[ES18] Michael Ehrig and Catharina Stroppel. Nazarov-Wenzl algebras, coideal subalgebras and categorified skew Howe duality. *Adv. Math.*, 331:58–142, 2018. doi:10.1016/j.aim.2018.01.013.

[GRS23a] Mengmeng Gao, Hebing Rui, and Linliang Song. Blocks of the Brauer category over the complex field. *Algebr. Represent. Theory*, 26(5):1319–1334, 2023. doi:10.1007/s10468-022-10133-6.

[GRS23b] Mengmeng Gao, Hebing Rui, and Linliang Song. Representations of weakly triangular categories. *J. Algebra*, 614:481–534, 2023. doi:10.1016/j.jalgebra.2022.09.025.

[GRS26] Mengmeng Gao, Hebing Rui, and Linliang Song. Affine and cyclotomic Brauer categorification via i-Kac-Moody 2-categories: the half integral type AIII case, 2026. arXiv:2609.07018.

[Kho14] Mikhail Khovanov. Heisenberg algebra and a graphical calculus. *Fund. Math.*, 225(1):169–210, 2014. doi:10.4064/fm225-1-8.

[KL09] Mikhail Khovanov and Aaron D. Lauda. A diagrammatic approach to categorification of quantum groups. I. *Represent. Theory*, 13:309–347, 2009. doi:10.1090/S1088-4165-09-00346-X.

[KL10] Mikhail Khovanov and Aaron D. Lauda. A categorification of quantum $\mathrm{sl}(n)$. *Quantum Topol.*, 1:1–92, 2010. URL: https://doi.org/10.4171/QT/1.

[Kol14] Stefan Kolb. Quantum symmetric Kac-Moody pairs. *Adv. Math.*, 267:395–469, 2014. doi:10.1016/j.aim.2014.08.010.

[Let99] Gail Letzter. Symmetric pairs for quantized enveloping algebras. *J. Algebra*, 220(2):729–767, 1999. doi:10.1006/jabr.1999.8015.

[Li14] Ge Li. A KLR grading of the Brauer algebras, 2014. arXiv:1409.1195.

[MS18] Marco Mackaay and Alistair Savage. Degenerate cyclotomic Hecke algebras and higher level Heisenberg categorification. *J. Algebra*, 505:150–193, 2018. doi:10.1016/j.jalgebra.2018.03.004.

[Rou08] Raphaël Rouquier. 2-Kac-Moody algebras, 2008. arXiv:0812.5023.

[RS19] Hebing Rui and Linliang Song. Affine Brauer category and parabolic category $\mathcal{O}$ in types B, C, D. *Math. Z.*, 293(1-2):503–550, 2019. doi:10.1007/s00209-018-2207-x.

[SS22] Steven V. Sam and Andrew Snowden. The representation theory of Brauer categories I: triangular categories. *Appl. Categ. Structures*, 30(6):1203–1256, 2022. doi:10.1007/s10485-022-09689-7.

[SW26] Alistair Savage and Ben Webster. Bubbles in the affine Brauer and Kauffman categories. *J. Aust. Math. Soc.*, 120(2):243–286, 2026. doi:10.1017/s144678872510116x.

[Web26] Ben Webster. Unfurling Khovanov-Lauda-Rouquier algebras, 2026. arXiv:1603.06311.

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