GRADED TRIANGULAR BASES

JONATHAN BRUNDAN

ABSTRACT. This article develops a practical technique for studying representations of $\Bbbk$-linear categories arising in the categorification of quantum groups. We work in terms of locally unital algebras which are $\mathbb{Z}$-graded with graded pieces that are finite-dimensional and bounded below, developing a theory of *graded triangular bases* for such algebras. The definition is a graded extension of the notion of triangular basis introduced in [BS18]. However, in the general graded setting, finitely generated projective modules often fail to be Noetherian, so that existing results from the study of highest weight categories are not directly applicable. Nevertheless, we show that there is still a good theory of *standard modules*. In motivating examples arising from Kac-Moody 2-categories, these modules categorify the PBW bases for the modified forms of quantum groups constructed by Wang.

1. Introduction

Recently, Wang [Wan21] has introduced PBW bases for the modified forms of quantum groups. Similar bases exist also for $\imath$-quantum groups. This article arose from attempts to understand the categorification of these bases. Quantum groups are categorified by the *Kac-Moody 2-categories* of Khovanov and Lauda [KL10] and Rouquier [Rou08]. From this perspective, Wang's PBW bases come from certain *standard modules* for the morphism categories of these 2-categories. This will be explained in [BWW23a]. Another example in a similar spirit is developed in [BWW23b], where we show that the split Grothendieck ring of the monoidal category of finitely generated graded projective modules for the *nil-Brauer category* from [BWW23c] is isomorphic to the split $\imath$-quantum group of rank one. Again, this $\imath$-quantum group has a PBW basis which is categorified by standard modules.

The motivating examples just mentioned are small graded $\Bbbk$-linear categories over a field $\Bbbk$. The goal of this article is to develop the algebraic tools which are applied in [BWW23a, BWW23b] in order to construct the standard modules for these categories in the first place. We find it convenient to replace the $\Bbbk$-linear category in question with its *path algebra* A. This is a locally unital graded associative algebra

$$A = \bigoplus_{i,j\in\mathbf{I}} 1_i A 1_j$$

equipped with a distinguished family of mutually orthogonal homogeneous idempotents $1_i\,(i \in \mathbf{I})$ arising from the identity endomorphisms of the objects of the underlying $\Bbbk$-linear category. The spaces 1_iA1_j are usually infinite-dimensional graded vector spaces, but they are *locally finite-dimensional*, i.e., the degree d component $1_iA_d1_j$ is finite-dimensional for all $d \in \mathbb{Z}$. Moreover, the grading is *bounded below* in the sense that for each $i, j \in \mathbf{I}$ there exists $N_{i,j} \in \mathbb{Z}$ such that $1_iA_d1_j = 0$ for all $d < N_{i,j}$.

Definition 1.1. Let $A = \bigoplus_{i,j\in\mathbf{I}} 1_iA1_j$ be a locally unital algebra that is locally finite-dimensional and bounded below. A *graded triangular basis* for A is following additional data:

- A subset $\mathbf{S} \subseteq \mathbf{I}$ indexing *special idempotents* $\{1_s \mid s \in \mathbf{S}\}$.
- A *lower finite* poset $(\Lambda, \leqslant)$, meaning that $\{\mu \in \Lambda \mid \mu \leqslant \lambda\}$ is finite for each $\lambda \in \Lambda$.
- A function $\mathbf{S} \to \Lambda$, $s \mapsto \dot{s}$ with finite fibers $\mathbf{S}_\lambda := \{s \in \mathbf{S} \mid \dot{s} = \lambda\}$.

2020 *Mathematics Subject Classification.* Primary 17B10.
Key words and phrases. Quasi-hereditary algebra, cellular basis, triangular basis.
Research supported in part by NSF grant DMS-2101783.

- Homogeneous sets $\mathrm{X}(i,s) \subset 1_i A 1_s$, $\mathrm{H}(s,t) \subset 1_s A 1_t$, $\mathrm{Y}(t,j) \subset 1_t A 1_j$ for $i,j \in \mathbf{I}$ and $s,t \in \mathbf{S}$.

For $s,t \in \mathbf{S}$, let $\mathrm{X}(s) := \bigcup_{i \in \mathbf{I}} \mathrm{X}(i,s)$ and $\mathrm{Y}(t) := \bigcup_{j \in \mathbf{I}} \mathrm{Y}(t,j)$. The axioms are as follows:

- The products xhy for $(x,h,y) \in \bigcup_{s,t \in \mathbf{S}} \mathrm{X}(s) \times \mathrm{H}(s,t) \times \mathrm{Y}(t)$ give a basis for A.
- For each $s \in \mathbf{S}$, $\mathrm{X}(s,s) = \mathrm{Y}(s,s) = \{1_s\}$.
- For $s,t \in \mathbf{S}$ with $s \neq t$, $\mathrm{X}(s,t) \neq \varnothing \Rightarrow \dot{s} > \dot{t}$, $\mathrm{H}(s,t) \neq \varnothing \Rightarrow \dot{s} = \dot{t}$, and $\mathrm{Y}(s,t) \neq \varnothing \Rightarrow \dot{s} < \dot{t}$.
- For each $i \in \mathbf{I} - \mathbf{S}$, there are only finitely many $s \in \mathbf{S}$ such that $\mathrm{X}(i,s) \cup \mathrm{Y}(s,i) \neq \varnothing$[1].

The history behind Definition 1.1 will be discussed later in the introduction. We just note for now that it is almost exactly the same as the definition of triangular basis given in [BS18, Def. 5.26], and that is equivalent to the definition of weakly triangular decomposition in [GRS23]. The main difference is that we are now in a graded setting, so that the assumption made in [BS18, GRS23] that each $1_i A 1_j$ is finite-dimensional can be weakened. We have also reversed the partial order compared to [BS18] since it seems more sensible to work in terms of lowest weight rather than highest weight modules in the sort of diagrammatical examples that we are interested in; this is the same convention as in [EL16] and [SS22].

When A has a graded triangular basis, the category A-gmod of (locally unital) graded left A-modules has properties which are reminiscent of various Abelian categories appearing in Lie theory. Here is a brief summary of the results developed in the main body of the text:

- For each $\lambda \in \Lambda$, let $e_\lambda := \sum_{s \in \mathbf{S}_\lambda} 1_s$. The *$\lambda$-weight space* of a graded left A-module V is the subspace $e_\lambda V$. Let $A_{\geqslant\lambda}$ be the quotient of A by the two-sided ideal generated by all e_μ $(\mu \not\geqslant \lambda)$. Then let $A_\lambda := \bar{e}_\lambda A_{\geqslant\lambda} \bar{e}_\lambda$, where $\bar{e}_\lambda$ is the canonical image of e_λ in $A_{\geqslant\lambda}$. These are *unital* graded algebras which are locally finite-dimensional and bounded below; in the motivating examples coming from Kac-Moody 2-categories they are some quiver Hecke algebras.
- The algebras $A_\lambda\,(\lambda \in \Lambda)$ play the role of "Cartan subalgebra" in a sort of lowest weight theory: if V is any graded left A-module and λ is a minimal weight of V, there is a naturally induced action of A_λ on the λ-weight space $e_\lambda V$. There are also exact functors $j^\lambda_! : A_\lambda\text{-gmod} \to A\text{-gmod}$ and $j^\lambda_* : A_\lambda\text{-gmod} \to A\text{-gmod}$, which are left and right adjoints of the idempotent truncation functor $j^\lambda : A_{\geqslant\lambda}\text{-gmod} \to A_\lambda\text{-gmod}$, $V \mapsto \bar{e}_\lambda V$; see Lemma 4.1. We call these the *standardization* and *costandardization functors*, respectively, following the terminology of [LW15].
- Fix also a set $\mathbf{B} = \coprod_{\lambda \in \Lambda} \mathbf{B}_\lambda$ such that $\mathbf{B}_\lambda$ indexes a complete set of irreducible graded left A_λ-modules $L_\lambda(b)\,(b \in \mathbf{B}_\lambda)$ up to isomorphism and degree shift; these modules are (globally) finite-dimensional since A_λ is unital. Also let $P_\lambda(b)$ and $I_\lambda(b)$ be a projective cover and an injective hull of $L_\lambda(b)$ in A_λ-gmod, respectively. For $b \in \mathbf{B}_\lambda$ we define *standard modules* $\Delta(b) := j^\lambda_! P_\lambda(b)$, *proper standard modules* $\bar{\Delta}(b) := j^\lambda_! L_\lambda(b)$, *costandard modules* $\nabla(b) := j^\lambda_* I_\lambda(b)$ and *proper costandard modules* $\bar{\nabla}(b) := j^\lambda_* L_\lambda(b)$. We show that $L(b) := \operatorname{cosoc} \bar{\Delta}(b) = \operatorname{soc} \bar{\nabla}(b)$ is irreducible, and the modules $L(b)$ $(b \in \mathbf{B})$ give a complete set of irreducible graded left A-modules up to isomorphism and degree shift; see Theorem 4.3.
- Let $P(b)$ be a projective cover and $I(b)$ be an injective hull of $L(b)$ in A-gmod. We show that $P(b)$ has a *Δ-flag* and $I(b)$ has a *∇-flag* with multiplicities satisfying an analog of the BGG reciprocity formula; see Corollaries 8.4 and 8.9. We also introduce $\bar{\Delta}$-flags and $\bar{\nabla}$-flags, and establish the familiar homological criteria for all of these types of "good filtrations"; see Theorems 8.3 and 8.8 (with finiteness assumptions on the flags) and Theorems 10.5 and 10.7 (with the finiteness assumptions removed).

For experts, there are probably no surprises in the above statements, but it is remarkable that it is possible to develop this theory so fully given that we have imposed very mild finiteness assumptions on A. In fact, in the motivating examples, the algebra A fails to be locally Noetherian—finitely generated projectives

[1] The final axiom is seldom needed; it is applied in Lemma 10.4.

often have submodules that are not themselves finitely generated. To deal with this, our notion of Δ-flag in this setting allows sections of such filtrations to be infinite direct sums of standard modules; see Definition 10.1. Accordingly, the Grothendieck group of the exact category of modules with Δ-flags is a certain completion with a topological basis given by the isomorphism classes of the standard modules; this is consistent with completions that can be seen in the construction of PBW bases in [Wan21]. Here is one more result proved later in the article:

- Assuming that A is unital rather than merely locally unital, graded Noetherian (both left and right), and that each of the algebras A_λ has finite graded global dimension, the algebra A has finite graded global dimension; see Theorem 11.6.

The strong finiteness assumptions in this last point are satisfied in many more classical examples. When they hold, the category of finitely generated graded left A-modules is an example of an affine properly stratified category in the sense of [Kle15a, Def. 5.1], and this result about global dimension can also be deduced from [Kle15a, Cor. 5.25].

To conclude the introduction, we make further historical remarks, with apologies to many contributions in the same spirit which we have surely missed.

- The antecedant for this genre is the notion of *cellular algebra* formulated by Graham and Lehrer [GL96]. There are many other variations in the literature, including cellular categories [Wes09], graded cellular algebras [HM10], affine cellular algebras [KX12], skew cellular algebras [HMR21], and sandwich cellular algebras [Tub22]. However, algebras with triangular bases have more in common with the quasi-hereditary algebras of [CPS88] than cellular algebras—our standard modules always have a unique irreducible quotient unlike the situation for cellular algebras where there can be strictly more cell modules than isomorphism classes of irreducible modules.
- Another influencial contribution is the definition [EL16, Def. 2.17] of *fibered object-adapted cellular basis*. Our primary motivating examples, the morphism categories of Kac-Moody 2-categories, were also one of the motivations behind [EL16]. In Definition 1.1, we have weakened some of the hypotheses compared to [EL16] but strengthened some others. However, the differences in the definitions are largely superficial—the novelty of the present article lies in the subsequent theory that we are able to develop rather than in the definition itself.
- Also providing motivation for us was the definition of *based quasi-hereditary algebra* from [KM20], and the older notion of *standardly based algebra* from [DR98]. However, [KM20] and [DR98] only consider finite-dimensional algebras, in particular, the poset Λ is finite rather than merely being lower finite. In [BS18, Def. 5.1], we simplified the definition of based quasi-hereditary algebra and upgraded it from unital to locally unital algebras. The result is equivalent to the notion of *strictly object-adapted cellular basis* from [EL16, Def. 2.4], a definition which was designed to capture the properties of Libedinsky's double leaf basis for the diagrammatic Hecke category as studied in [EW16]. In [BS18, Ch. 5], we used semi-infinite Ringel duality together with some arguments involving tilting modules adapted from [AST18] to show that all upper finite highest weight categories can be realized in terms of based quasi-hereditary algebras. Thus, there are already many important examples in the ungraded setting.
- In [BS18, Def. 5.20], the definition of based quasi-hereditary algebra was weakened to the notion of a *based stratified algebra*. This is almost the same as an algebra with a triangular basis but with one extra axiom requiring that the idempotents $\bar{1}_s (s \in S_\lambda)$ are primitive in A_λ; see also Remark 4.2 below. Upper finite fully stratified categories whose tilting modules satisfy some additional axioms can be realized in terms of based stratified algebras; see [BS18, Th. 5.24].
- Finally we would like to mention that there is a stronger notion of *triangular decomposition* formalized in [BS18, Def. 5.31], which is closely related to the notion of *triangular category*

introduced in [SS22]. The latter is particularly useful in when there is also some monoidal structure, i.e., one has what Sam and Snowden call a *triangular monoidal category*. Examples include various sorts of Brauer category (both oriented and unoriented) arising from Schur-Weyl dualities, but the notion is too restrictive to capture examples like the ones coming from Kac-Moody 2-categories.

2. Locally unital graded algebras and their modules

Throughout the article, we will work over an algebraically closed field $\Bbbk$. All algebras, categories, functors, etc. will be assumed to be $\Bbbk$-linear. We write $\mathcal{GVec}$ for the closed symmetric monoidal category of $\mathbb{Z}$-graded vector spaces with morphisms that are degree-preserving linear maps. The downward degree shift functor is denoted by q, i.e., for a graded vector space $V = \bigoplus_{d\in\mathbb{Z}} V_d$ its degree shift qV is the same underlying vector space with grading defined via $(qV)_d := V_{d+1}$ for each $d \in \mathbb{Z}$. For any sort of formal series $f = \sum_{d\in\mathbb{Z}} r_d q^d$ with each $r_d \in \mathbb{N}$, we write $V^{\oplus f}$ for $\bigoplus_{d\in\mathbb{Z}} q^d V^{\oplus r_d}$. The conjugate series $\overline{f}$ is $\sum_{d\in\mathbb{Z}} r_d q^{-d}$. For a graded vector space $V = \bigoplus_{d\in\mathbb{Z}} V_d$ with finite-dimensional graded pieces, we write

$$\dim_q V := \sum_{d\in\mathbb{Z}} (\dim V_d) q^{-d}.$$

Usually for this, V will be finite-dimensional so that $\dim_q V \in \mathbb{N}[q, q^{-1}]$, or *bounded below* in the sense that $V_d = 0$ for $d \ll 0$ so that $\dim_q V \in \mathbb{N}(\!(q^{-1})\!)$, or *bounded above* in the sense that $V_d = 0$ for $d \gg 0$ so that $\dim_q V \in \mathbb{N}(\!(q)\!)$.

By a *locally unital graded algebra* we mean a graded associative (but not necessarily unital) algebra A equipped with a distinguished system 1_i $(i \in \mathbf{I})$ of mutually orthogonal homogeneous idempotents such that

$$A = \bigoplus_{i,j\in\mathbf{I}} 1_i A 1_j. \tag{2.1}$$

By a *graded left A-module*, we mean a *locally unital* graded left A-module V, i.e., $V = \bigoplus_{i\in\mathbf{I}} 1_i V$.

For graded left A-modules V and W and $d \in \mathbb{Z}$, we write $\operatorname{Hom}_A(V, W)_d$ for the vector space of all ordinary A-module homomorphisms $f : V \to W$ such that $f(V_n) \subseteq W_{n+d}$ for each $n \in \mathbb{Z}$. Then

$$\operatorname{Hom}_A(V, W) := \bigoplus_{d\in\mathbb{Z}} \operatorname{Hom}_A(V, W)_d$$

is a morphism space in the $\mathcal{GVec}$-enriched category of graded left A-modules. We denote the underlying category consisting of the same objects with morphism spaces $\operatorname{Hom}_A(V, W)_0$ by A-gmod. This is the usual Abelian category of graded modules and degree-preserving module homomorphisms. It has enough injectives and projectives, indeed, it is a Grothendieck category, so that homological algebra makes sense in A-gmod. We define $\operatorname{Ext}^n_A(V, W)$ so that it is naturally graded just like $\operatorname{Hom}_A(V, W)$:

$$\operatorname{Ext}^n_A(V, W) = \bigoplus_{d\in\mathbb{Z}} \operatorname{Ext}^n_A(V, W)_d \qquad \text{with} \qquad \operatorname{Ext}^n_A(V, W)_d = \operatorname{Ext}^n_A(q^{-d}V, W)_0 = \operatorname{Ext}^n_A(V, q^d W)_0.$$

We use $V \cong W$ for isomorphism in A-gmod and $V \simeq W$ if $V \cong q^n W$ for some $n \in \mathbb{Z}$.

We write A-pgmod (resp., A-igmod) for the full subcategory of A-gmod consisting of finitely generated projective (resp., finitely cogenerated injective) graded modules. These are additive Karoubian categories equipped with the downward degree shift functor q. We say that a graded left A-module V is *locally finite-dimensional* if $\dim 1_i V_d < \infty$ for all $i \in \mathbf{I}$ and $d \in \mathbb{Z}$. Also it is *bounded below* (resp., *bounded above*) if for each $i \in \mathbf{I}$ there exists $N_i \in \mathbb{Z}$ such that $1_i V_d = 0$ for $d < N_i$ (resp., $d > N_i$). We denote the Abelian category of locally finite-dimensional graded left A-modules by A-lfdmod. There are also graded right A-modules, which are of course the same thing as graded left A^{op}-modules. The various categories of graded right A-modules are gmod-A, pgmod-A, igmod-A and lfdmod-A.

For any locally finite-dimensional graded A-module V and an irreducible graded A-module L, the *graded multiplicity* of L in V is the following formal series with coefficients in $\mathbb{N}$:

$$[V:L]_q := \sum_{d\in\mathbb{Z}} \max\left(\left|\{r=1,\dots,n \mid V_r/V_{r-1} \cong q^d L\}\right| \;\middle|\; \begin{array}{l}\text{for all finite graded filtrations}\\ 0=V_0\subseteq\cdots\subseteq V_n=V\end{array}\right) q^d. \tag{2.2}$$

If V is bounded below (resp., above) then this is a formal Laurent series in $\mathbb{N}(\!(q^{-1})\!)$ (resp., $\mathbb{N}(\!(q)\!)$). For example, taking A to be $\Bbbk$ itself and writing also $\Bbbk$ for the ground field viewed as a one-dimensional graded vector space concentrated in degree zero, we have that $[V:\Bbbk]_q = \dim_q V$.

There are exact contravariant functors

$$?^{\circledast} : \text{gmod-}A \to A\text{-gmod}, \qquad\qquad ?^{\circledast} : A\text{-gmod} \to \text{gmod-}A. \tag{2.3}$$

The first of these takes a graded left module V to $V^{\circledast} := \bigoplus_{i\in\mathbf{I}}\bigoplus_{d\in\mathbb{Z}}(1_i V_{-d})^*$ viewed as a graded right module with the natural action. The second functor is defined similarly. If V is locally finite-dimensional then $(V^{\circledast})^{\circledast} \cong V$ naturally. So $?^{\circledast}$ restricts to quasi-inverse contravariant equivalences

$$?^{\circledast} : \text{lfdmod-}A \to A\text{-lfdmod}, \qquad\qquad ?^{\circledast} : A\text{-lfdmod} \to \text{lfdmod-}A. \tag{2.4}$$

There is a natural isomorphism

$$\operatorname{Hom}_A(V, W^{\circledast}) \cong \operatorname{Hom}_A(W, V^{\circledast}) \tag{2.5}$$

for any graded left (resp., right) A-module V (resp., W). This implies that $?^{\circledast} : \text{gmod-}A \to (A\text{-gmod})^{\mathrm{op}}$ is left adjoint to the exact functor $?^{\circledast} : (A\text{-gmod})^{\mathrm{op}} \to \text{gmod-}A$. Hence, by properties of adjunctions, $?^{\circledast}$ takes projectives in gmod-A to projectives in $(A\text{-gmod})^{\mathrm{op}}$, i.e., injectives in A-gmod. It then follows that

$$\operatorname{Ext}^n_A(V, W^{\circledast}) \cong \operatorname{Ext}^n_A(W, V^{\circledast}) \tag{2.6}$$

for a graded left (resp., right) A-module V (resp., W) and $n \geqslant 0$. Indeed, we can compute $\operatorname{Ext}^n_A(V, W^{\circledast})$ from a projective resolution of V. Applying $?^{\circledast}$ gives an injective resolution of $V^{\circledast}$, which can be used to compute $\operatorname{Ext}^n_A(W, V^{\circledast})$. Then (2.6) follows using (2.5).

It will always be the case for us that A itself is locally finite-dimensional and bounded below, by which we mean that all of the right A-modules 1_iA ($i \in \mathbf{I}$) and all of the left A-modules $A1_j$ ($j \in \mathbf{I}$) are locally finite-dimensional and bounded below in the earlier sense. Assuming this, finitely generated (resp., finitely cogenerated) graded A-modules are locally finite-dimensional and bounded below (resp., above). In particular, if L is an irreducible graded left A-module, it is both finitely generated and finitely cogenerated, so it is locally finite-dimensional and it is bounded both below and above. This proves that

$$\dim 1_i L < \infty \tag{2.7}$$

for any $i \in \mathbf{I}$. Using also the assumption that $\Bbbk$ is algebraically closed, one deduces that

$$\operatorname{End}_A(L) = \Bbbk. \tag{2.8}$$

The functor $?^{\circledast}$ restricts to contravariant functors

$$?^{\circledast} : \text{pgmod-}A \to A\text{-igmod}, \qquad\qquad ?^{\circledast} : A\text{-pgmod} \to \text{igmod-}A. \tag{2.9}$$

For this assertion, we have used that the dual of a finitely generated projective is a finitely cogenerated injective, as follows from the discussion in the previous paragraph. It is also true that the dual of a finitely cogenerated injective is a finitely generated projective, so that restrictions of $?^{\circledast}$ also give functors

$$?^{\circledast} : A\text{-igmod} \to \text{pgmod-}A, \qquad\qquad ?^{\circledast} : \text{igmod-}A \to A\text{-pgmod}, \tag{2.10}$$

which are quasi-inverses of the ones in (2.9), i.e., these are all contravariant equivalences. The proof of this needs some further argument which will be explained in the proof of the first lemma.

Lemma 2.1. *Suppose that A is a locally unital graded algebra which is locally finite-dimensional and bounded below. Let V be any graded left A-module.*

1. *The module V is finitely cogenerated if and only if* $\operatorname{soc} V$*, the sum of its irreducible graded submodules, is an essential submodule of finite length. It always has an injective hull I_V in* A-gmod*. When V is finitely cogenerated, I_V is also finitely cogenerated and coincides with the injective hull of* $\operatorname{soc} V$ *in* A-gmod.
2. *The module V is finitely generated if and only if* $\operatorname{rad} V$*, the intersection of its maximal graded submodules, is a superfluous submodule and* $\operatorname{cosoc} V := V/\operatorname{rad} V$ *is of finite length. In that case, it has a projective cover P_V in* A-gmod*, which is itself finitely generated and coincides with the projective cover of* $\operatorname{cosoc} V$ *in* A-gmod.
3. *The module V is locally finite-dimensional if and only if* $\operatorname{Hom}_A(P_L, V) \cong \operatorname{Hom}_A(V, I_L)^{\circledast}$ *is locally finite-dimensional for all irreducible graded left A-modules L. When this holds, the graded dimension of this morphism space is equal to the graded multiplicity $[V : L]_q$ defined by (2.2).*

Proof. (1) This follows from general principles since A-gmod is a Grothendieck category.

(2) We have already noted that finitely generated (resp., finitely cogenerated) modules are locally finite-dimensional and bounded below (resp., above). Consequently, if V is finitely generated we can apply $?^{\circledast}$ then the first part of (1) (with A replaced by A^{op}) then $?^{\circledast}$ again to deduce that $\operatorname{rad} V$ is superfluous and $\operatorname{cosoc} V$ is of finite length. Conversely, if $\operatorname{rad} V$ is superfluous and $\operatorname{cosoc} V$ is of finite length then it is clear that V is finitely generated since it is generated by pre-images of generators of $\operatorname{cosoc} V$.

To complete the proof, it suffices to show that any irreducible graded left A-module L has a projective cover P_L. To see this, we pick $i \in \mathbf{I}$ such that $1_i L \neq 0$, so that L is a quotient of $q^d A 1_i$ for some $d \in \mathbb{Z}$. Since $q^d A 1_i$ is a finitely generated projective graded left A-module, its dual $(q^d A 1_i)^{\circledast}$ is finitely cogenerated and injective. So by (1), $(q^d A 1_i)^{\circledast} = I_1 \oplus \cdots \oplus I_n$ with each I_r being the injective hull of an irreducible graded right A-module. We deduce that $q^d A 1_i \cong P_1 \oplus \cdots \oplus P_n$ for $P_r := I_r^{\circledast}$. Since $q^d A 1_i$ is projective, so is each summand P_r, and duality then gives that P_r is the projective cover of its head which is an irreducible graded left A-module. One of these summands is a projective cover of the irreducible L, completing the proof. This argument shows moreover that the duals of finitely cogenerated injective graded right A-modules are projective, something which was promised just before the statement of the lemma.

(3) We just prove the assertions involving P_L; the ones involving I_L follow by the dual argument. If V is locally finite-dimensional then $\operatorname{Hom}_A(P_L, V)$ is locally finite-dimensional since P_L is finitely generated. Also its graded dimension is equal to $[V : L]_q$ by Schur's Lemma (2.8) and the definition (2.2). Conversely, suppose that $\operatorname{Hom}_A(P_L, V)$ is locally finite-dimensional for all L. We need to show that $1_i V$ is locally finite-dimensional for $i \in \mathbf{I}$. Since $A 1_i$ is finitely generated, (2) implies that there are irreducible graded left A-modules $L_1, \dots, L_n$ with $L_r \not\cong L_s$ for $r \neq s$ and $f_1, \dots, f_n \in \mathbb{N}[q, q^{-1}]$ such that

$$A 1_i \cong P_1^{\oplus f_1} \oplus \cdots \oplus P_n^{\oplus f_n},$$

where P_r is a projective cover of L_r. We deduce that $1_i V \cong \operatorname{Hom}_A(A 1_i, V)$ is locally finite-dimensional since each $\operatorname{Hom}_A(P_r, V)$ is locally finite-dimensional by assumption. $\square$

The locally unital algebra A is *unital* if and only if $|\{i \in \mathbf{I} \mid 1_i \neq 0\}| < \infty$. Then $1_A = \sum_{i \in \mathbf{I}} 1_i$. More can be said when this holds. To start with, (2.7) implies that all irreducible graded A-modules are actually finite-dimensional. Moreover, there are only finitely many of them up to isomorphism and degree shift; see [Kle15b, Lemma 2.2(i)] for the proof. The following is a graded version of the Nakayama Lemma.

Lemma 2.2. *Suppose that A is a* unital *graded algebra which is locally finite-dimensional and bounded below. Let V be a graded left A-module which is bounded below. If* $\mathrm{Hom}_A(V,L)=0$ *for all irreducible graded left A-modules L then $V=0$.*

Proof. Let $N=N(A)$ be the graded Jacobson radical of A. The quotient algebra A/N is a finite direct product of graded matrix algebras over $\Bbbk$. In particular, it is semisimple. Suppose that V is a non-zero graded module that is bounded below. Let $m\in\mathbb{Z}$ be minimal such that $V_m\neq 0$. By [Kle15a, Lem. 2.7], there exists $r\geqslant 1$ such that $N^r\subseteq\bigoplus_{d\geqslant 1}A_d$. We have that $N^rV\subseteq\bigoplus_{d\geqslant 1}A_dV\subseteq\bigoplus_{d\geqslant 1}V_{m+d}$. Hence, $N^rV\neq V$, so $NV\neq V$. As A/N is graded semisimple, V/NV is a completely reducible graded module, so there exists an irreducible graded left A-module L with $\mathrm{Hom}_A(V/NV,L)\neq 0$. This implies that $\mathrm{Hom}_A(V,L)\neq 0$ as required. $\square$

Lemma 2.3. *Suppose that A is a* unital *graded algebra that is locally finite-dimensional and bounded below. Any finitely generated (resp., finitely cogenerated) graded left A-module V has a graded filtration $V=V_0\supseteq V_1\supseteq V_2\supseteq\cdots$ (resp., $0=V_0\subseteq V_1\subseteq\cdots$) which is exhaustive in the sense that $\bigcap_{r\geqslant 0}V_r=0$ (resp., $\bigcup_{r\geqslant 0}V_r=V$) and has sections are irreducible or zero.*

Proof. We just prove the result in the finitely generated case, the other case following by duality. Let $A_{\geqslant r}:=\bigoplus_{s\geqslant r}A_s$. Let X be a finite set of homogeneous generators for V. Since $AA_{\geqslant r}/AA_{\geqslant(r+1)}$ is spanned by the image of $\sum_{s\leqslant r}A_s$, which is finite-dimensional, the sections of the exhaustive filtration

$$V=AX\supseteq AA_{\geqslant 1}X\supseteq AA_{\geqslant 2}X\supseteq\cdots$$

are all finite-dimensional. Then each section can be refined to a composition series to obtain a filtration of the desired form. $\square$

The following lemma is stronger than Lemma 2.1(1)–(2) since there is no assumption on finite generation or cogeneration here.

Lemma 2.4. *Suppose that A is a* unital *graded algebra that is locally finite-dimensional and bounded below. Let $\{L(b)\,|\,b\in\mathbf{B}\}$ be a full set of irreducible graded left A-modules up to isomorphism and degree shift. Let $P(b)$ and $I(b)$ be a projective cover and an injective hull of $L(b)$ in* A-gmod*, respectively.*

1. *Any graded left A-module V that is locally finite-dimensional and bounded below has a projective cover P_V in* A-gmod*, which is itself locally finite-dimensional and bounded below. Moreover, we have that*
$$P_V\cong\bigoplus_{b\in\mathbf{B}}P(b)^{\oplus\overline{\dim_q\mathrm{Hom}_A(V,L(b))}}\tag{2.11}$$
as a graded left A-module.
2. *Any graded left A-module V that is locally finite-dimensional and bounded above has an injective hull I_V in* A-gmod*, which is itself both locally finite-dimensional and bounded above. Moreover, we have that*
$$I_V\cong\bigoplus_{b\in\mathbf{B}}I(b)^{\oplus\dim_q\mathrm{Hom}_A(L(b),V)}\tag{2.12}$$
as a graded left A-module.

Proof. (1) Let V be a graded left A-module which is locally finite-dimensional and bounded below. The multiplication map $A\otimes_{\Bbbk}V\twoheadrightarrow V, a\otimes v\mapsto av$ is a surjective graded left A-module homomorphism. Also $A\otimes_{\Bbbk}V$ is a projective graded left A-module for the action coming from left multiplication on the first tensor factor. It is locally finite-dimensional and bounded below since both A and V are. Thus, we have constructed $f:P\twoheadrightarrow V$ for $P\in\operatorname{ob}A\text{-lfdmod}$ that is bounded below and projective in A-gmod. Next we apply the functor (2.9) to obtain $f^{\circledast}:V^{\circledast}\hookrightarrow P^{\circledast}$ with $V^{\circledast}$ and $P^{\circledast}$ being locally finite-dimensional and bounded above, and $P^{\circledast}$ being injective in gmod-A.

Let $i : V^{\circledast} \hookrightarrow I$ be an injective hull of $V^{\circledast}$ in gmod-A, which exists by general principles because gmod-A is a Grothendieck category. Using that $P^{\circledast}$ is injective, we extend $f^{\circledast} : V^{\circledast} \hookrightarrow P^{\circledast}$ to $g : I \to P^{\circledast}$ so that the following diagram commutes:

$$\begin{tikzcd} V^{\circledast} \arrow[rr, hook, "f^{\circledast}"] \arrow[dr, hook, "i"'] & & P^{\circledast} \\ & I \arrow[ur, hook, "g"'] & \end{tikzcd}$$

Thus I embeds into $P^{\circledast}$. It follows that I is locally finite-dimensional and bounded above. Also I is injective in gmod-A so it is certainly injective in the Abelian subcategory lfdmod-A, and $V^{\circledast}$ is an essential submodule of I. Finally we dualize again, making some natural identifications to get a commuting diagram

$$\begin{tikzcd} V & & P \arrow[ll, two heads, "f"] \arrow[dl, two heads, "g^{\circledast}"] \\ & I^{\circledast} \arrow[ul, two heads, "i^{\circledast}"] & \end{tikzcd}$$

By duality, $I^{\circledast}$ is projective in A-lfdmod, but we do not immediately know that it is injective in A-gmod. This follows because the surjection $g^{\circledast}$ splits to reveal that $I^{\circledast}$ is a graded summand of P, so it is projective in A-gmod as P is so. Also $\ker i^{\circledast}$ is a superfluous submodule of $I^{\circledast}$ since im i was an essential submodule of I. So $I^{\circledast}$ is a projective cover of V in A-gmod, and it is locally finite-dimensional and bounded below as required.

It remains to prove (2.11). Take $b \in \mathbf{B}$ and pick a homogeneous basis Θ for $\mathrm{Hom}_A(V, L(b))$. For each $\theta \in \Theta$, we use projectivity to construct homogeneous maps $\hat{\theta}$ making the following diagram commute:

$$\begin{tikzcd} & V \arrow[dr, two heads, "\theta"] & \\ P(b) \arrow[ur, "\hat{\theta}"] \arrow[rr, two heads] & & L(b) \end{tikzcd}$$

Note $\deg(\hat{\theta}) = -\deg(\theta)$. Let $\theta^{\vee}$ $(\theta \in \Theta)$ be the basis for $\mathrm{Hom}_A(V, L(b))^{\circledast}$ that is dual to Θ. We obtain a graded left A-module homomorphism $f_b : P(b) \otimes \mathrm{Hom}_A(V, L(b))^{\circledast} \to V, p \otimes \theta^{\vee} \mapsto \hat{\theta}(p)$. These homomorphisms for all b combine to define a graded A-module homomorphism

$$f : \bigoplus_{b \in \mathbf{B}} P(b) \otimes \mathrm{Hom}_A(V, L(b))^{\circledast} \to V.$$

This is surjective by construction. Moreover, the module P appearing on the left hand side is locally finite-dimensional, bounded below and projective in A-gmod. It follows that there is a surjection $P \twoheadrightarrow P_V$ from P to the projective cover, i.e., we have a short exact sequence $0 \to K \to P \to P_V \to 0$ for some graded submodule K of P. To complete the proof, we show that $K = 0$. Applying $\mathrm{Hom}_A(-, L(b))$ to the short exact sequence gives $0 \to \mathrm{Hom}_A(P_V, L(b)) \to \mathrm{Hom}_A(P, L(b)) \to \mathrm{Hom}_A(K, L(b)) \to 0$. As we have that $\dim_q \mathrm{Hom}_A(P, L(b)) = \dim_q \mathrm{Hom}_A(V, L(b)) = \dim_q \mathrm{Hom}_A(P_V, L(b))$ by the construction, we deduce that $\mathrm{Hom}_A(K, L(b)) = 0$ for all $b \in \mathbf{B}$. This implies that $K = 0$ by Lemma 2.2.

(2) This follows from (1) (with A replaced by A^{op}) by applying $?^{\circledast}$. □

Corollary 2.5. *Suppose once again that A is* unital*, locally finite-dimensional and bounded below. Let V be a graded left A-module which is locally finite-dimensional and bounded below. If* $\mathrm{Ext}^1_A(V, L) = 0$ *for all irreducible graded left A-modules L then V is projective in* A-gmod.

Proof. By Lemma 2.4(1), V has a projective cover P_V in A-gmod which is locally finite-dimensional and bounded below. Moreover, $\mathrm{Hom}_A(P_V, L) \cong \mathrm{Hom}_A(V, L)$ for all irreducible graded modules L. We apply $\mathrm{Hom}_A(-,L)$ to the short exact sequence $0 \to K \to P_V \to V \to 0$ using the assumption that $\mathrm{Ext}^1_A(V,L) = 0$ to get a short exact sequence $0 \to \mathrm{Hom}_A(V,L) \to \mathrm{Hom}_A(P_V,L) \to \mathrm{Hom}_A(K,L) \to 0$. We have already observed that the first map is an isomorphism. It follows that $\mathrm{Hom}_A(K,L) = 0$. By Lemma 2.2, this implies that $K = 0$, so $V \cong P_V$ as required. □

3. First properties of graded triangular bases

Throughout the section, we assume that A has a graded triangular basis in the sense of Definition 1.1. We will use obvious notations like $\mathbf{S}_{\leqslant\lambda}$ for $\{s \in \mathbf{S} \mid \dot{s} \leqslant \lambda\}$, $\mathbf{S}_{\not\geqslant\lambda}$ for $\{s \in \mathbf{S} \mid \dot{s} \not\geqslant \lambda\}$, etc. Before we do anything interesting with the axioms, we make some general remarks.

- The axioms imply that $A = \sum_{s\in\mathbf{S}} A1_sA$. It follows that A is graded Morita equivalent to the idempotent truncation $\bigoplus_{s,t\in\mathbf{S}} 1_sA1_t$. This algebra also has a graded triangular basis that is the obvious subset of the one for A. In this way, one can always reduce to the case that $\mathbf{I} = \mathbf{S}$, at the price of replacing A by a Morita equivalent algebra.
- Without changing the algebra A, merely contracting its distinguished idempotents, one can always reduce to a situation in which $\mathbf{S} = \Lambda$. To do this starting from the general setup of Definition 1.1, we first replace Λ by the image of the function $\mathbf{S} \to \Lambda, s \mapsto \dot{s}$. Assuming also that the sets Λ and $\mathbf{I} - \mathbf{S}$ are disjoint, we define $\tilde{\mathrm{X}}(i,\lambda) := \bigcup_{s\in\mathbf{S}_\lambda} \mathrm{X}(i,s)$ and $\tilde{\mathrm{Y}}(\lambda,j) := \bigcup_{t\in\mathbf{S}_\lambda} \mathrm{Y}(t,j)$ for $\lambda \in \Lambda$, $i,j \in \mathbf{I} - \mathbf{S}$. Also for $\lambda,\mu \in \Lambda$, we let $\tilde{\mathrm{H}}(\lambda,\mu) := \bigcup_{s\in\mathbf{S}_\lambda,t\in\mathbf{S}_\mu} \mathrm{H}(s,t)$, and we set $\tilde{\mathrm{X}}(\lambda,\mu) := \bigcup_{s\in\mathbf{S}_\lambda,t\in\mathbf{S}_\mu} \mathrm{X}(s,t)$ and $\tilde{\mathrm{Y}}(\lambda,\mu) := \bigcup_{s\in\mathbf{S}_\lambda,t\in\mathbf{S}_\mu} \mathrm{Y}(s,t)$ assuming that $\lambda \neq \mu$. Finally, let $\tilde{\mathbf{I}} := (\mathbf{I} - \mathbf{S}) \cup \Lambda$ and define $\tilde{1}_i$ to be 1_i for $i \in \mathbf{I} - \mathbf{S}$ or $\sum_{s\in\mathbf{S}_\lambda} 1_s$ for $i = \lambda \in \Lambda$, then set $\tilde{\mathrm{X}}(\lambda,\lambda) = \tilde{\mathrm{Y}}(\lambda,\lambda) := \{\tilde{1}_\lambda\}$. This data gives a new graded triangular basis for $A = \bigoplus_{i,j\in\tilde{\mathbf{I}}} \tilde{1}_iA\tilde{1}_j$ with special idempotents indexed by the weight poset $\Lambda \subset \tilde{\mathbf{I}}$, which is what we wanted.

Taken together, these reductions reduce to the case that $\mathbf{S} = \mathbf{I} = \Lambda$. Although harmless, we have not assumed this since it is not so convenient in the motivating examples discussed in the introduction.

Returning to the general setup, we proceed to develop some basic consequences of Definition 1.1. For $\lambda \in \Lambda$, let $e_\lambda := \sum_{s\in\mathbf{S}_\lambda} 1_s$. Note it is perfectly possible that $e_\lambda = 0$, indeed, the idempotents 1_s can already be zero, and also it could be that $\mathbf{S}_\lambda = \varnothing$ since we did not assume that the function $\mathbf{S} \to \Lambda$ is surjective. The *λ-weight space* of a graded left A-module V is the subspace $e_\lambda V$. Then the *set of weights* of V is

$$\Lambda(V) := \{\lambda \in \Lambda \mid e_\lambda V \neq 0\}. \tag{3.1}$$

Let $A_{\geqslant\lambda}$ be the quotient of A by the two-sided ideal generated by the idempotents $\{e_\mu \mid \mu \not\geqslant \lambda\}$. We often use the notation $\bar{a}$ to denote the image of $a \in A$ in $A_{\geqslant\lambda}$. The algebra $A_{\geqslant\lambda}$ is another locally unital graded algebra with distinguished idempotents $\bar{1}_i\,(i \in \mathbf{I})$, and it is locally finite-dimensional and bounded below since A is so by assumption. Let $A_\lambda := \bar{e}_\lambda A_{\geqslant\lambda}\bar{e}_\lambda$. This is a *unital* graded algebra which is locally finite-dimensional and bounded below; its identity element is $\bar{e}_\lambda$.

Lemma 3.1. *Any element f of the two-sided ideal $Ae_\lambda A$ can be written as a linear combination of elements of the form xhy for $(x,h,y) \in \bigcup_{s,t\in\mathbf{S}_{\leqslant\lambda}} \mathrm{X}(s) \times \mathrm{H}(s,t) \times \mathrm{Y}(t)$.*

Proof. We argue by induction up the poset. We may assume that $f = x_1h_1y_1x_2h_2y_2 \neq 0$ for $x_1 \in \mathrm{X}(s_1), h_1 \in \mathrm{H}(s_1,t_1), y_1 \in \mathrm{Y}(t_1,u), x_2 \in \mathrm{X}(u,t_2), h_2 \in \mathrm{H}(t_2,s_2), y_2 \in \mathrm{Y}(s_2)$, and $s_1,t_1,t_2,s_2,u \in S$ with $\dot{u} = \lambda$ and $\dot{s}_1 = \dot{t}_1 \leqslant \lambda \geqslant \dot{t}_2 = \dot{s}_2$. If $\dot{t}_1 < \lambda$ or $\lambda > \dot{t}_2$ we get done by induction, so we may assume that $\dot{t}_1 = \lambda = \dot{t}_2$. But then we must have that $t_1 = u = t_2$ and $y_1 = 1_u = x_2$. So $f = x_1h_1h_2y_2$. Then we expand h_1h_2 in terms of the basis to get a linear combination of terms $x_3h_3y_3$ for $x_3 \in \mathrm{X}(s_1,s_3), h_3 \in \mathrm{H}(s_3,t_3), y_3 \in \mathrm{Y}(t_3,s_2)$ for $s_3,t_3 \in S_\mu$ and $\mu \leqslant \lambda$. It remains to show that the resulting $x_1x_3h_3y_3y_2$ can be written in the desired form. If $\mu < \lambda$ this follows by induction, so assume that $\mu = \lambda$. Then we must have $s_1 = s_3$ and $x_3 = 1_{s_1}$, and $t_3 = s_2$ and $y_3 = 1_{s_2}$. The term simplifies to $x_1 h_3 y_2$, which is of the desired form. $\square$

Corollary 3.2. *Suppose we are given a partition $\Lambda = \hat{\Lambda} \sqcup \check{\Lambda}$ with $\hat{\Lambda}$ being an upper set, equivalently, $\check{\Lambda}$ being a lower set. The quotient $\hat{A}$ of A by the two-sided ideal I generated by $\{e_\lambda \mid \lambda \in \check{\Lambda}\}$ has basis given by the images of of all xhy for $(x,h,y) \in \bigcup_{s,t\in\hat{\mathbf{S}}} \mathrm{X}(s) \times \mathrm{H}(s,t) \times \mathrm{Y}(t)$ where $\hat{\mathbf{S}} := \{s \in \mathbf{S} \mid \dot{s} \in \hat{\Lambda}\}$.*

Proof. In view of the graded triangular basis for A, it suffices to show that I is spanned by all xhy for $(x,h,y) \in \bigcup_{s,t\in\check{\mathbf{S}}} \mathrm{X}(s) \times \mathrm{H}(s,t) \times \mathrm{Y}(t)$ where $\check{\mathbf{S}} := \{s \in \mathbf{S} \mid \dot{s} \in \check{\Lambda}\} = \mathbf{S} - \hat{\mathbf{S}}$. This follows from Lemma 3.1. $\square$

Corollary 3.3. *For $\lambda \in \Lambda$, $A_{\geqslant\lambda}$ has a basis given by all $\bar{x}\bar{h}\bar{y}$ for $(x,h,y) \in \bigcup_{s,t\in\mathbf{S}_{\geqslant\lambda}} \mathrm{X}(s) \times \mathrm{H}(s,t) \times \mathrm{Y}(t)$. Hence, $A_\lambda = \bar{e}_\lambda A_{\geqslant\lambda} \bar{e}_\lambda$ has basis consisting of all $\bar{h}$ for $h \in \bigcup_{s,t\in\mathbf{S}_\lambda} \mathrm{H}(s,t)$.*

In the setup of Corollary 3.2, we will always *identify* $\hat{A}$-gmod with the full subcategory of A-gmod consisting of the A-modules annihilated by all e_λ ($\lambda \in \check{\Lambda}$). There is an adjoint triple of functors $(i^*, i, i^!)$ with

$$i : \hat{A}\text{-gmod} \to A\text{-gmod} \tag{3.2}$$

being the (often omitted) natural inclusion functor and

$$i^* := \hat{A} \otimes_A - : A\text{-gmod} \to \hat{A}\text{-gmod}, \tag{3.3}$$

$$i^! := \bigoplus_{i\in\mathbf{I}} \mathrm{Hom}_A(\hat{A}1_i, -) : A\text{-gmod} \to \hat{A}\text{-gmod}\,. \tag{3.4}$$

We clearly have that $i^* \circ i = i^! \circ i = \mathrm{id}_{\hat{A}\text{-gmod}}$. In the special case that $\hat{A} = A_{\geqslant\lambda}$, we denote the adjoint triple $(i^*, i, i^!)$ instead by $(i^*_{\geqslant\lambda}, i_{\geqslant\lambda}, i^!_{\geqslant\lambda})$:

$$\begin{tikzcd} A_{\geqslant\lambda}\text{-gmod} \arrow[rr, "i_{\geqslant\lambda}"] && A\text{-gmod} \arrow[ll, bend right, "i^!_{\geqslant\lambda}"'] \arrow[ll, bend left, "i^*_{\geqslant\lambda}"] \end{tikzcd} \tag{3.5}$$

More explicitly, for a graded left A-module V, $i^*_{\geqslant\lambda}V$ is the largest graded quotient and $i^!_{\geqslant\lambda}V$ is the largest graded submodule of V all of whose weights are $\geqslant \lambda$.

Lemma 3.4. *Let V be a graded left A-module and λ be minimal in $\Lambda(V)$.*

(1) We have that $e_\mu A e_\lambda V = 0$ unless $\mu \geqslant \lambda$. Hence, the natural inclusion $\bar{e}_\lambda(i^!_{\geqslant\lambda}V) \hookrightarrow e_\lambda V$ is an isomorphism of graded vector spaces.

(2) We have that $e_\lambda A e_\mu V = 0$ unless $\mu \geqslant \lambda$. Hence, the natural quotient map $e_\lambda V \twoheadrightarrow \bar{e}_\lambda(i^*_{\geqslant\lambda}V)$ is an isomorphism of graded vector spaces.

Proof. (1) The subspace $e_\mu A e_\lambda V$ is spanned by vectors $xhyv$ for $x \in \mathrm{X}(s_1, s_2)$, $h \in \mathrm{H}(s_2, t_2)$, $y \in \mathrm{Y}(t_2, t_1)$ and $v \in 1_{t_1}V$ with $\dot{s}_1 = \mu$, $\dot{s}_2 = \nu = \dot{t}_2$, $\dot{t}_1 = \lambda$ and $\mu \geqslant \nu \leqslant \lambda$. The minimality of λ implies that $xhyv = 0$ unless $\nu = \lambda$, in which case $\mu \geqslant \lambda$. It follows that the submodule $Ae_\lambda V$ is contained in $i^!_{\geqslant\lambda}V$, so their λ-weight spaces coincide.

(2) The proof that $e_\lambda A e_\mu V = 0$ unless $\mu \geqslant \lambda$ is similar to the proof in (1). To deduce that $e_\lambda V \cong \bar{e}_\lambda(i^*_{\geqslant\lambda}V)$, note that $i^*_{\geqslant\lambda}V = V/\sum_{\mu\not\geqslant\lambda} Ae_\mu V$. We have shown that the λ-weight space of each $Ae_\mu V$ appearing here is zero, so the quotient map restricts to an isomorphism between the λ-weight spaces of V and $i^*_{\geqslant\lambda}V$. $\square$

4. Standard modules and the classification of irreducible modules

Suppose to start with that A is any locally unital graded algebra as in (2.1). Let e be an idempotent in A that is a finite sum of the distinguished idempotents 1_i ($i \in \mathbf{I}$). Then eAe is a unital graded algebra. Truncating a module with the idempotent e defines an exact functor

$$j : A\text{-gmod} \to eAe\text{-gmod}, V \mapsto eV. \tag{4.1}$$

It is well known that j takes irreducible graded A-modules to irreducible graded eAe-modules or to zero, and all irreducible graded left eAe-modules arise in this way. Moreover, j satisfies the universal property of quotient functor: any exact functor from A-gmod to an Abelian category which takes all of the irreducibles annihilated by j to zero factors uniquely through j. The functor j has a left adjoint $j_!$ and a right adjoint j_* defined by

$$j_! := Ae \otimes_{eAe} ? : eAe\text{-gmod} \to A_{\geqslant\lambda}\text{-gmod}, \tag{4.2}$$
$$j_* := \bigoplus_{i \in \mathbf{I}} \mathrm{Hom}_{eAe}(eA1_i, ?) : eAe\text{-gmod} \to A\text{-gmod}. \tag{4.3}$$

Neither $j_!$ nor j_* is exact in general. We obviously have that $j \circ j_! \cong j \circ j_* \cong \mathrm{id}_{eAe\text{-gmod}}$. If $P \twoheadrightarrow L$ (resp., $L \hookrightarrow I$) is a projective cover (resp., an injective hull) of an irreducible graded left A-module L such that $jL \neq 0$ then jP (resp., jI) is a projective cover (resp., an injective hull) of jL in eAe-gmod. Using properties of adjunctions, it follows that $j_! jP \cong P$ and $j_* jI \cong I$.

Now return to the setup of the previous section, so that A has a graded triangular basis. Applying the constructions just explained to the idempotent $\bar{e}_\lambda$ in the algebra $A_{\geqslant\lambda}$ produces an adjoint triple of functors which we denote by $(j^\lambda_!, j^\lambda, j^\lambda_*)$:

$$\begin{tikzcd} A_{\geqslant\lambda}\text{-gmod} \arrow[rr, "j^\lambda"] && A_\lambda\text{-gmod} \arrow[ll, bend right, "j^\lambda_*"'] \arrow[ll, bend left, "j^\lambda_!"] \end{tikzcd} \tag{4.4}$$

We call $j^\lambda_!$ and j^λ_* the *standardization* and *costandardization* functors, respectively. We are in a special situation so that these functors have additional favorable properties:

Lemma 4.1. *The functor $j^\lambda_!$ (resp., j^λ_*) is exact and it takes modules that are locally finite-dimensional and bounded below (resp., bounded above) to modules that are locally finite-dimensional and bounded below (resp., bounded above).*

Proof. The functor $j^\lambda_!$ is exact because $\bar{1}_i A_{\geqslant\lambda} \bar{e}_\lambda$ is a projective graded right A_λ-module for each $i \in \mathbf{I}$. Indeed, by Corollary 3.3, $\bar{1}_i A_{\geqslant\lambda} \bar{e}_\lambda$ has basis $\bar{x}\bar{h}$ for $(x, h) \in \bigcup_{s,t \in \mathbf{S}_\lambda} \mathrm{X}(i, s) \times \mathrm{H}(s, t)$. Hence we have that

$$\bar{1}_i A_{\geqslant\lambda} \bar{e}_\lambda = \bigoplus_{s \in \mathbf{S}_\lambda} \bigoplus_{x \in \mathrm{X}(i,s)} \bar{x} A_\lambda \tag{4.5}$$

with the summand $\bar{x}A_\lambda$ here being isomorphic to $q^{-\deg(x)} \bar{1}_s A_\lambda$ as a graded right A_λ-module, which is projective. Similarly,

$$\bar{e}_\lambda A_{\geqslant\lambda} \bar{1}_i = \bigoplus_{s \in \mathbf{S}_\lambda} \bigoplus_{y \in \mathrm{Y}(s,i)} A_\lambda \bar{y} \tag{4.6}$$

with the summand $A_\lambda \bar{y}$ being isomorphic to $q^{-\deg(y)} A_\lambda \bar{1}_s$ as a graded left A_λ-module. So $\bar{e}_\lambda A_{\geqslant\lambda} \bar{1}_i$ is a projective graded left A_λ-module, hence, j^λ_* is exact.

Now let V be a graded left A_λ-module and let $V(s)$ be a homogeneous basis for $1_s V$ for $s \in \mathbf{S}_\lambda$. The decomposition (4.5) implies that $1_i\left(j_!^\lambda V\right) = \bar{1}_i A_{\geqslant\lambda}\bar{e}_\lambda \otimes_{A_\lambda} V$ has homogeneous basis given by the vectors

$$\bar{x} \otimes v \qquad \text{for } (x, v) \in \bigcup_{s\in\mathbf{S}_\lambda} \mathrm{X}(i,s) \times V(s). \tag{4.7}$$

The vector $\bar{x} \otimes v$ is of degree $\deg(x) + \deg(v)$. Since A is locally finite-dimensional and bounded below and $\mathbf{S}_\lambda$ is finite, there are only finitely many $x \in \bigcup_{s\in\mathbf{S}_\lambda} \mathrm{X}(i,s)$ of any given degree, and these degrees are bounded below. This implies that $j_!^\lambda V$ is locally finite-dimensional and bounded below assuming V has these properties. Similarly, from (4.6), we deduce that $1_i\left(j_*^\lambda V\right) = \mathrm{Hom}_{A_\lambda}(\bar{e}_\lambda A_{\geqslant\lambda}\bar{1}_i, V)$ has basis

$$\delta_{y,v} \qquad \text{for } (y, v) \in \bigcup_{s\in\mathbf{S}_\lambda} \mathrm{Y}(s,i) \times V(s), \tag{4.8}$$

where $\delta_{y,v}$ is the unique left A_λ-module homomorphism that takes $\bar{y} \in \mathrm{Y}(s,i)$ to v and all other elements of $\bigcup_{t\in\mathbf{S}_\lambda} \mathrm{Y}(t,i)$ to zero. Since $\deg(\delta_{y,v}) = \deg(v) - \deg(y)$, it is easy to deduce that $j_*^\lambda V$ is locally finite-dimensional and bounded above assuming that V has these properties. □

Next, we fix a set $\mathbf{B} = \coprod_{\lambda\in\Lambda} \mathbf{B}_\lambda$ such that $\mathbf{B}_\lambda$ parametrizes a full set $L_\lambda(b)$ $(b \in \mathbf{B}_\lambda)$ of irreducible graded left A_λ-modules up to isomorphism and degree shift. Given $b \in \mathbf{B}$, we use the notation $\dot{b}$ to denote the unique $\lambda \in \Lambda$ such that $b \in \mathbf{B}_\lambda$. For this notation to be unamiguous, one should assume that the sets are chosen so that $\mathbf{B}_\lambda \cap \mathbf{S} = \mathbf{B} \cap \mathbf{S}_\lambda$. As we did with $\mathbf{S}$, we also use notations like $\mathbf{B}_{\leqslant\lambda}, \mathbf{B}_{\geqslant\lambda}$, etc. Since A_λ is a unital graded algebra which is locally finite-dimensional and bounded below, the set $\mathbf{B}_\lambda$ is finite and each $L_\lambda(b)$ is finite-dimensional. Also let $P_\lambda(b)$ (resp., $I_\lambda(b)$) be a projective cover (resp., injective hull) of $L_\lambda(b)$ in A_λ-gmod; these modules may be infinite-dimensional. For any $b \in \mathbf{B}$, we let

$$\Delta(b) := j_!^\lambda P_\lambda(b), \qquad \bar{\Delta}(b) := j_!^\lambda L_\lambda(b), \qquad \bar{\nabla}(b) := j_*^\lambda L_\lambda(b), \qquad \nabla(b) := j_*^\lambda I_\lambda(b), \tag{4.9}$$

where $\lambda := \dot{b}$. We view all of these as graded left A-modules via the natural inclusion $i_{\geqslant\lambda}$. We call them the *standard, proper standard, proper costandard* and *costandard* modules, respectively. If one knows bases for $P_\lambda(b), L_\lambda(b)$ and $I_\lambda(b)$, one obtains bases for $\Delta(b)$ and $\bar{\Delta}(b)$ from (4.7), and bases for $\nabla(b)$ and $\bar{\nabla}(b)$ from (4.8).

In general, there is no reason for any of the modules (4.9) to have finite length. However, by Lemma 2.3, each $P_\lambda(b)$ $(b \in \mathbf{B}_\lambda)$ admits an exhaustive descending filtration with irreducible sections. By exactness of $j_!^\lambda$, it follows that $\Delta(b)$ has an exhaustive descending filtration with top section $\bar{\Delta}(b)$ and other sections that are degree shifts of $\bar{\Delta}(c)$ for $c \in \mathbf{B}_\lambda$. Similarly, $\nabla(b)$ has an exhaustive ascending filtration with bottom section $\bar{\nabla}(b)$ and other sections that are degree shifts of $\bar{\nabla}(c)$ for $c \in \mathbf{B}_\lambda$.

Remark 4.2. It is especially convenient when the sets $\mathbf{B}$ and $\mathbf{S}$ are naturally identified. We record here two special cases of Definition 1.1 where this can be achieved.

- We call A a *based affine quasi-hereditary algebra* if $\mathbf{S} = \Lambda$ with the map $\mathbf{S} \to \Lambda, s \mapsto \dot{s}$ being the identity, and each A_λ $(\lambda \in \Lambda)$ is graded local, i.e., the quotient of A_λ by its graded Jacobson radical $N(A_\lambda)$ is $\Bbbk$. In this situation, $\mathbf{B}_\lambda$ is a singleton. Then one can choose notation so that $\mathbf{S} = \Lambda = \mathbf{B}$ and $P_\lambda(\lambda) = A_\lambda$ for each $\lambda \in \Lambda$. When the grading is concentrated in degree zero, this setup recovers the based quasi-hereditary algebras of [KM20] if Λ is finite, or their semi-infinite analog from [BS18, Def. 5.1] when Λ is infinite.
- We call A a *based affine stratified algebra* if $A_\lambda/N(A_\lambda) \cong \prod_{s\in\mathbf{S}_\lambda} \Bbbk$ for each $\lambda \in \Lambda$. In this situation, we can choose notation so that $\mathbf{S}_\lambda = \mathbf{B}_\lambda$ for each $\lambda \in \Lambda$ and $P_\lambda(b) = A_\lambda\bar{1}_b$ for each $b \in \mathbf{B}_\lambda$. When the grading is concentrated in degree zero, this setup recovers the based stratified algebras of [BS18, Def. 5.20].

If V is any graded left A-module and λ is minimal in $\Lambda(V)$, the weight space $e_\lambda V$ is naturally an A_λ-module, with the basis vector $\bar{h}$ of A_λ acting simply by multiplication by $h \in \bigcup_{s,t\in\mathbf{S}_\lambda} \mathrm{H}(s,t)$. This follows from Lemma 3.4(1). Clearly, both of the isomorphisms $e_\lambda V \cong j^\lambda i^!_{\geqslant\lambda} V$ and $e_\lambda V \cong j^\lambda i^*_{\geqslant\lambda} V$ from Lemma 3.4 are A_λ-module homomorphisms. If we take V here to be one of the modules $\Delta(b), \bar{\Delta}(b), \bar{\nabla}(b)$ or $\nabla(b)$ for $b \in \mathbf{B}_\lambda$ then λ is the *lowest weight* of V, i.e., it is the unique minimal weight in $\Lambda(V)$. Moreover, in view of the bases (4.7) and (4.8), the lowest weight space $e_\lambda V$ simply recovers the A_λ-module from which V was constructed in the first place in (4.9). This is a familiar situation since it is entirely analogous to the construction of Verma and dual Verma modules for semisimple Lie algebras. In view of this, the following theorem (and its proof) should come as no surprise.

Theorem 4.3 (Classification of irreducible modules)**.** *For $b \in \mathbf{B}$, the module $\Delta(b)$ has a unique irreducible quotient denoted $L(b)$. This is also the unique irreducible submodule of $\nabla(b)$. Moreover, the modules $L(b)$ $(b \in \mathbf{B})$ give a full set of irreducible graded left A-modules up to isomorphism and degree shift.*

Proof. Take $b \in \mathbf{B}$ and let $\lambda := \dot{b}$. The indecomposable projective A_λ-module $P_\lambda(b)$ has a unique maximal graded submodule $\operatorname{rad} P_\lambda(b)$. Since $e_\lambda\Delta(b) = \bar{e}_\lambda \otimes P_\lambda(b) \cong P_\lambda(b)$ as an A_λ-module, and $\Delta(b)$ is generated as an A-module by its lowest weight space $e_\lambda\Delta(b)$, it follows that $\Delta(b)$ has a unique maximal graded submodule, namely, $(\bar{e}_\lambda \otimes \operatorname{rad} P_\lambda(b)) \oplus \bigoplus_{\mu>\lambda} e_\mu\Delta(b)$. Hence, $\Delta(b)$ has a unique irreducible quotient $L(b)$. Moreover, λ is lowest weight of $L(b)$, and $e_\lambda L(b) \cong L_\lambda(b)$ as a graded A_λ-module. This implies that $L(a) \not\cong L(b)$ for $a \neq b$.

Now we show that any irreducible graded A-module L is isomorphic to $L(b)$ for some $b \in B$. Pick λ minimal in $\Lambda(L)$, so that $e_\lambda L$ is naturally an A_λ-module. There is a non-zero homogeneous A_λ-module homomorphism $f : P_\lambda(b) \to e_\lambda L$ for some $b \in \mathbf{B}_\lambda$. Since $e_\lambda L \cong j^\lambda i^!_{\geqslant\lambda} L$ as an A_λ-module and $\Delta(b) = i_{\geqslant\lambda} j^\lambda_! P_\lambda(b)$, the adjunctions produce a non-zero homogeneous A-module homomorphism $\Delta(b) \to L$, which is necessarily surjective. We deduce that $L \simeq L(b)$. The classification of irreducible modules is now proved.

It remains to show that $\nabla(b)$ has irreducible socle $L(b)$. For this, we take $a \in \mathbf{B}$ and compute:

$$\mathrm{Hom}_A(L(a), \nabla(b)) = \mathrm{Hom}_A(L(a), i_{\geqslant\lambda} j^\lambda_* I_\lambda(b)) \cong \mathrm{Hom}_{A_\lambda}(j^\lambda i^*_{\geqslant\lambda} L(a), I_\lambda(b)).$$

Since $j^\lambda i^*_{\geqslant\lambda} L(a) = 0$ unless $a \in \mathbf{B}_\lambda$, in which case it is $L_\lambda(a)$, we deduce that $\mathrm{Hom}_A(L(a), \nabla(b))$ is zero unless $a = b$, when it is $\mathbb{k}$. This proves that $\operatorname{soc} \nabla(b) = L(b)$. □

For $b \in \mathbf{B}$, we let $P(b)$ be a projective cover and $I(b)$ be an injective hull of the irreducible module $L(b)$ in A-gmod. For $b \in \mathbf{B}_\lambda$, $\Delta(b)$ (resp., $\nabla(b)$) can also be described as the projective cover (resp., injective hull) of $L(b)$ in $A_{\geqslant\lambda}$-gmod, and we have that $P(b) \twoheadrightarrow \Delta(b)$ and $\nabla(b) \hookrightarrow I(b)$. The following lemma gives characterizations of $\bar{\Delta}(b)$ and $\bar{\nabla}(b)$ in a similar vein.

Lemma 4.4. *Suppose that $b \in \mathbf{B}$ and let $\lambda := \dot{b}$.*

(1) The proper standard module $\bar{\Delta}(b)$ is the largest graded quotient of $\Delta(b)$ with the properties $[\bar{\Delta}(b) : L(b)]_q = 1$ and $[\bar{\Delta}(b) : L(c)]_q = 0$ for $b \neq c \in \mathbf{B}_{\not>\lambda}$.

(2) The proper costandard module $\bar{\nabla}(b)$ is the largest graded submodule of $\nabla(b)$ with the properties $[\bar{\nabla}(b) : L(b)]_q = 1$ and $[\bar{\nabla}(b) : L(c)]_q = 0$ for $b \neq c \in \mathbf{B}_{\not>\lambda}$.

Proof. (1) Let $\lambda := \dot{b}$. As noted earlier, $\Delta(b)$ has an exhaustive descending filtration $V = V_0 \supset V_1 \supseteq \cdots$ with top section $V_0/V_1 = \bar{\Delta}(b)$ and other sections $\simeq \bar{\Delta}(c)$ for $c \in \mathbf{B}_\lambda$. It follows that any strictly larger quotient Q of $\Delta(b)$ than $\bar{\Delta}(b)$ has an irreducible quotient of the form $q^d L(c)$ for some $c \in \mathbf{B}_\lambda$. Hence, either $[Q : L(b)]_q \neq 1$ or $[Q : L(c)]_q \neq 0$ for $b \neq c \in \mathbf{B}_{\not>\lambda}$, violating the properties we wanted. It remains to see that the quotient $\bar{\Delta}(b)$ does have these properties. We certainly have that $[\bar{\Delta}(b) : L(c)]_q = 0$ if

$\dot{c} \not\geqslant \lambda$ since λ is the lowest weight of $\bar{\Delta}(b)$. If $\dot{c} = \lambda$ then $L(c)$ can be viewed an irreducible $A_{\geqslant\lambda}$-module with $j^\lambda L(c) \cong L_\lambda(c)$, and we have by exactness of j^λ that

$$[\bar{\Delta}(b) : L(c)]_q = [j^\lambda_! L_\lambda(b) : L(c)]_q = [j^\lambda j^\lambda_! L_\lambda(b) : j^\lambda L(c)]_q = [L_\lambda(b) : L_\lambda(c)]_q = \delta_{b,c}.$$

(2) Similar. ◻

Corollary 4.5. *For $b, c \in \mathbf{B}$, we have that* $\dim_q \mathrm{Hom}_A(\Delta(b), \bar{\nabla}(c)) = \dim_q \mathrm{Hom}_A(\bar{\Delta}(b), \nabla(c)) = \delta_{b,c}$.

Proof. We just explain for $\mathrm{Hom}_A(\Delta(b), \bar{\nabla}(c))$. If $b = c$ there is by Schur's Lemma a unique (up to scalars) non-zero homomorphism taking the irreducible head of $\Delta(b)$ to the irreducible socle of $\bar{\nabla}(c)$. Any non-zero homogeneous homomorphism $\Delta(b) \to \bar{\nabla}(c)$ that is not of this form takes the head $L(b)$ of $\Delta(b)$ to an irreducible subquotient of $\bar{\nabla}(c)$ different from $L(c)$, so we get that $\dot{b} > \dot{c}$ thanks to the lemma. Also we have that $\dot{c} \geqslant \dot{b}$ since there must be an irreducible subquotient of $\Delta(b)$ isomorphic to the socle $L(c)$ of $\bar{\nabla}(c)$. This contradiction shows that there are no such homomorphisms. ◻

Corollary 4.6. *For $b, c \in \mathbf{B}$, $f \in \mathbb{N}(\!(q^{-1})\!)$ and $g \in \mathbb{N}(\!(q)\!)$, we have that*

$$\dim_q \mathrm{Hom}_A\left(\Delta(b)^{\oplus f}, \bar{\nabla}(c)^{\oplus g}\right) = \dim_q \mathrm{Hom}_A\left(\bar{\Delta}(b)^{\oplus f}, \nabla(c)^{\oplus g}\right) = \delta_{b,c}\,\overline{f}\,g \in \mathbb{N}(\!(q)\!).$$

Proof. Again we just treat $\mathrm{Hom}_A\left(\Delta(b)^{\oplus f}, \bar{\nabla}(c)^{\oplus g}\right)$. Say $f = \sum_{m\in\mathbb{Z}} r_m q^{-m}$ and $g = \sum_{n\in\mathbb{Z}} s_n q^n$. We need to show that

$$\dim \mathrm{Hom}_A\left(\bigoplus_{m\in\mathbb{Z}} q^{-m}\Delta(b)^{\oplus r_m}, \bigoplus_{n\in\mathbb{Z}} q^n\bar{\nabla}(c)^{\oplus s_n}\right)_{-d} = \delta_{b,c}\sum_{m+n=d} r_m s_n,$$

which makes sense because $r_m = s_n = 0$ for $m, n \ll 0$. Using that $\Delta(b)$ is finitely generated, we have that

$$\begin{aligned}\mathrm{Hom}_A\left(\bigoplus_{m\in\mathbb{Z}} q^{-m}\Delta(b)^{\oplus r_m}, \bigoplus_{n\in\mathbb{Z}} q^n\bar{\nabla}(c)^{\oplus s_n}\right)_{-d} &\cong \prod_{m\in\mathbb{Z}} \mathrm{Hom}_A\left(\Delta(b), \bigoplus_{n\in\mathbb{Z}} q^n\bar{\nabla}(c)^{\oplus s_n}\right)^{\oplus r_m}_{m-d}\\ &\cong \prod_{m\in\mathbb{Z}}\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_A(\Delta(b), \bar{\nabla}(c))^{\oplus r_m s_n}_{m+n-d}.\end{aligned}$$

By Corollary 4.5, the Hom space here is zero unless $b = c$ and $m + n = d$, when it is 1-dimensional. So the dimension is $\delta_{b,c}\sum_{m+n=d} r_m s_n$ as required. ◻

We record also a useful consequence of the Nakayama Lemma for the algebras A_λ.

Lemma 4.7. *Let V be a graded left A-module that is bounded below. If* $\mathrm{Hom}_A(V, \bar{\nabla}(b)) = 0$ *for all $b \in \mathbf{B}$ then $V = 0$.*

Proof. Suppose that $V \neq 0$. Let λ be minimal in $\Lambda(V)$. By Lemma 2.2, there is a non-zero A_λ-module homomorphism $e_\lambda V \to L_\lambda(b)$ for some $b \in \mathbf{B}_\lambda$. Since $e_\lambda V \cong j^\lambda i^*_{\geqslant\lambda} V$ and $\bar{\nabla}(b) = i_{\geqslant\lambda} j^\lambda_* L_\lambda(b)$, we get induced a non-zero homomorphism $V \to \bar{\nabla}(b)$. So $\mathrm{Hom}_A(V, \bar{\nabla}(b)) \neq 0$. ◻

5. Duality

The definition of graded triangular basis is symmetric in the sense that if we are given a graded triangular basis of A, then it also gives one for A^{op}. One just has to swap the sets $\mathrm{X}(s)$ and $\mathrm{Y}(s)$. Clearly the algebras $(A^{\mathrm{op}})_\lambda$ arising from this new triangular basis for A^{op} are the duals $(A_\lambda)^{\mathrm{op}}$ of the algebras A_λ from before. Letting $L^{\mathrm{op}}_\lambda(b) := L_\lambda(b)^\circledast$ for each $b \in \mathbf{B}_\lambda$, we obtain a full set of irreducible graded right A_λ-modules up to isomorphism and degree shift. Then one can apply the general theory to this basis of A^{op} to obtain graded right A-modules $P^{\mathrm{op}}(b)$, $\Delta^{\mathrm{op}}(b)$, $\bar{\Delta}^{\mathrm{op}}(b)$, $L^{\mathrm{op}}(b)$, $\bar{\nabla}^{\mathrm{op}}(b)$, $\nabla^{\mathrm{op}}(b)$ and $I^{\mathrm{op}}(b)$ indexed by $b \in \mathbf{B}$. By properties of adjunctions, we have that

$$j^\lambda_! \circ ?^\circledast \cong ?^\circledast \circ j^\lambda_*, \qquad\qquad j^\lambda_* \circ ?^\circledast \cong ?^\circledast \circ j^\lambda_!. \tag{5.1}$$

Also duality obviously commutes with the inclusion functor $i_{\geqslant\lambda}$. It follows that $?^{\circledast}$ takes $\Delta^{\mathrm{op}}(b)$, $\bar\Delta^{\mathrm{op}}(b)$, $\bar\nabla^{\mathrm{op}}(b)$ and $\nabla^{\mathrm{op}}(b)$ to $\nabla(b)$, $\bar\nabla(b)$, $\bar\Delta(b)$ and $\Delta(b)$, respectively. By Theorem 4.3, we deduce that $L^{\mathrm{op}}(b)^{\circledast} \cong L(b)$, so $I^{\mathrm{op}}(b)^{\circledast} \cong P(b)$ and $P^{\mathrm{op}}(b)^{\circledast} \cong I(b)$.

In examples, it is often the case that A admits a graded algebra anti-automorphism $\tau : A \to A$ fixing each 1_i ($i \in \mathbf{I}$). Then we can compose the functor $?^{\circledast} : A\text{-gmod} \to \text{gmod-}A$ from (2.3) with restriction along τ to obtain a contravariant functor

$$?^{\circledτ} : A\text{-gmod} \to A\text{-gmod}\,. \tag{5.2}$$

This restricts to a contravariant graded auto-equivalence on A-lfdmod. It is easy to see that τ descends to anti-automorphisms $\tau : A_{\geqslant\lambda} \to A_{\geqslant\lambda}$ and $\tau : A_\lambda \to A_\lambda$ for each $\lambda \in \Lambda$. Using this, we define dualities $?^{\circledτ}$ on $A_{\geqslant\lambda}$-gmod and A_λ-gmod too. If it happens that $L_\lambda(b)^{\circledτ} \cong L_\lambda(b)$ for each $b \in \mathbf{B}_\lambda$, we get that

$$\Delta(b)^{\circledτ} \cong \nabla(b), \qquad \bar\Delta(b)^{\circledτ} \cong \bar\nabla(b), \qquad L(b)^{\circledτ} \cong L(b), \qquad P(b)^{\circledτ} \cong I(b), \qquad I(b)^{\circledτ} \cong P(b). \tag{5.3}$$

It follows that

$$[\bar\Delta(a) : L(b)]_q = \overline{[\bar\nabla(a) : L(b)]_q} \tag{5.4}$$

for any $a, b \in \mathbf{B}$. We say simply that the graded triangular basis admits a *duality* τ when all of this holds.

6. Good filtrations

Continue with A being an algebra with a graded triangular basis. The next important theorem is similar to [BS18, Th. 5.28], which treats the ungraded setting. A key difference in the graded case is that the direct sums in the sections of the filtration may be infinite.

Theorem 6.1. *Take any $b \in \mathbf{B}$ and let $\lambda := \dot b$. Let $\lambda = \lambda_1, \dots, \lambda_n$ be $\{\mu \in \Lambda \mid \mu \leqslant \lambda\}$ ordered so that $\lambda_p < \lambda_q \Rightarrow p > q$.*

(1) There exists a (non-unique) module $Q(b) \in \mathrm{ob}\,A\text{-pgmod}$ with a graded filtration

$$Q(b) = Q_0(b) \supset Q_1(b) \supseteq \cdots \supseteq Q_n(b) = 0$$

such that each $Q_{r-1}(b)/Q_r(b)$ is a (possibly infinite) direct sum of degree-shifted copies of standard modules $\Delta(a)$ for $a \in \mathbf{B}_{\lambda_r}$. Moreover, the top section $Q_0(b)/Q_1(b)$ is actually a finite direct sum of these standard modules, with one of them being $\Delta(b)$.

(2) There exists a (non-unique) module $J(b) \in \mathrm{ob}\,A\text{-igmod}$ with a graded filtration

$$0 = J_0(b) \subset J_1(b) \subseteq \cdots \subseteq J_n(b) = J(b)$$

such that each $J_r(b)/J_{r-1}(b)$ is a (possibly infinite) direct sum of degree-shifted copies of co-standard modules $\nabla(a)$ for $a \in \mathbf{B}_{\lambda_r}$. Moreover, the bottom section $J_1(b)/J_0(b)$ is actually a finite direct sum of these costandard modules, with one of them being $\nabla(b)$.

Proof. We just explain the proof of (1). Then (2) follows by applying $?^{\circledast}$ to the conclusion of (1) for A^{op} . Pick $u \in \mathbf{S}_\lambda$ and $d \in \mathbb{Z}$ such that $\bar 1_u L_\lambda(b)_{-d} \neq 0$. Equivalently, $P_\lambda(b)$ is a summand of $q^d A_\lambda \bar 1_u$. We define $Q(b)$ to be the finitely generated projective graded left A-module $q^d A 1_u$. It has basis xhy for $(x, h, y) \in \bigcup_{s,t \in \mathbf{S}_{\leqslant\lambda}} \mathrm{X}(s) \times \mathrm{H}(s,t) \times \mathrm{Y}(t,u)$. Let $Q_r(b)$ be the subspace of $Q(b)$ spanned by all xhy for $(x, h, y) \in \bigcup_{r<f\leqslant n} \bigcup_{s,t\in\mathbf{S}_{\lambda_f}} \mathrm{X}(s) \times \mathrm{H}(s,t) \times \mathrm{Y}(t,u)$.

We show in this paragraph that $Q_r(b)$ is an A-submodule of $Q(b)$. It suffices to see that $axhy \in Q_r(b)$ for any $i \in \mathbf{I}$, $r < f \leqslant n$, $s, t \in \mathbf{S}_{\lambda_f}$, $a \in A1_i$, and $(x, h, y) \in \mathrm{X}(i, s) \times \mathrm{H}(s, t) \times \mathrm{Y}(t, u)$. This follows by applying Lemma 3.1 to get that $axhy$ is a linear combination of elements of the form $x'h'y'$ for $x' \in \mathrm{X}(s') \times \mathrm{H}(s', t') \times \mathrm{Y}(t', u)$ and $s', t' \in \mathbf{S}_{\lambda_g}$ ($g \geqslant f$). Hence, we have constructed a filtration of $Q(b)$.

Consider some $1 \leqslant r \leqslant n$. The module $Q_{r-1}(b)/Q_r(b)$ has basis given by the canonical images of the vectors xhy for $(x, h, y) \in \bigcup_{s,t\in\mathbf{S}_{\lambda_r}} \mathrm{X}(s) \times \mathrm{H}(s,t) \times \mathrm{Y}(t,u)$. By (4.7), the vectors $\bar x \otimes \bar h \bar y$ for

$(x,h,y) \in \bigcup_{s,t\in\mathbf{S}_{\lambda_r}} \mathrm{X}(s)\times\mathrm{H}(s,t)\times\mathrm{Y}(t,u)$ give a basis for $A_{\geqslant\lambda_j}\bar{e}_{\lambda_r}\otimes_{A_{\lambda_r}}\bar{e}_{\lambda_r}A_{\geqslant\lambda_r}\bar{1}_u$. It follows that there is a degree-preserving isomorphism of graded vector spaces

$$f: q^d A_{\geqslant\lambda_r}\bar{e}_{\lambda_r}\otimes_{A_{\lambda_r}}\bar{e}_{\lambda_r}A_{\geqslant\lambda_r}\bar{1}_u \xrightarrow{\sim} Q_{r-1}(b)/Q_r(b), \qquad \bar{x}\otimes\bar{h}\bar{y}\mapsto xhy+Q_r(b).$$

This is actually an isomorphism of A-modules. To see this, we take $s,t\in\mathbf{S}_{\lambda_r}$, $a\in A1_i$ and $(x,h,y)\in\mathrm{X}(i,s)\times\mathrm{H}(s,t)\times\mathrm{Y}(t)$ then apply Lemma 3.1 again to write $axhy$ as a linear combination of basis elements $x'h'y'$ for $(x',h',y')\in\bigcup_{s',t'\in\mathbf{S}_{\geqslant\lambda_r}}\mathrm{X}(i,s')\times\mathrm{H}(s',t')\times\mathrm{Y}(t',s)$. It just remains to observe that when $s',t'\in\mathbf{S}_{>\lambda_r}$ both $\bar{x}'\otimes\bar{h}'\bar{y}'$ and $xhy+Q_r(b)$ are zero.

We have now proved that $Q_{r-1}(b)/Q_r(b)\cong q^d A_{\geqslant\lambda_r}\bar{e}_{\lambda_r}\otimes_{A_{\lambda_r}}\bar{e}_{\lambda_r}A_{\geqslant\lambda_r}\bar{1}_u$ as graded left A-modules. The basis implies that $\bar{e}_{\lambda_r}A_{\geqslant\lambda_r}\bar{1}_u = \bigoplus_{s\in\mathbf{S}_{\lambda_r}}\bigoplus_{y\in\mathrm{Y}(s,u)}A_{\lambda_r}\bar{y}$ as a graded left A_{λ_r}-module, and $A_{\lambda_r}\bar{y}\cong q^{-\deg(y)}A_{\lambda_r}\bar{1}_s$ for $s\in\mathbf{S}_{\lambda_r}$ and $y\in\mathrm{Y}(s,u)$. We deduce that

$$q^d A_{\geqslant\lambda_r}\bar{e}_{\lambda_r}\otimes_{A_{\lambda_r}}\bar{e}_{\lambda_r}A_{\geqslant\lambda_r}\bar{1}_u\cong\bigoplus_{s\in\mathbf{S}_{\lambda_r}}\bigoplus_{y\in\mathrm{Y}(s,u)}q^{d-\deg(y)}j_!^{\lambda_r}A_{\lambda_r}\bar{1}_s$$

as graded left A-modules. Since $A_{\lambda_r}\bar{1}_s$ is a finitely generated projective, it is a finite direct sum of degree-shifted copies of $P_{\lambda_r}(a)$ for $a\in\mathbf{B}_{\lambda_r}$. Now our decomposition implies that $Q_{r-1}(b)/Q_r(b)$ is a (possibly infinite) direct sum of degree-shifted copies of $\Delta_{\lambda_r}(a)$ for $a\in\mathbf{B}_{\lambda_r}$. In the case $r=1$, the argument shows further that $Q_0(b)/Q_1(b)\cong q^d j_!^\lambda A_\lambda\bar{1}_u$, which is a *finite* direct sum of degree-shifted standard modules since $A_\lambda\bar{1}_u$ is a finitely generated projective, and it contains $j_!^\lambda P_\lambda(b)\cong\Delta(b)$ as a summand by the choice of u. □

Corollary 6.2. *Suppose that λ is minimal in Λ. Then $\Delta(b)=P(b)$ and $\nabla(b)=I(b)$ for any $b\in\mathbf{B}_\lambda$.*

Proof. We just prove the first statement. It suffices to show that $\Delta(b)$ is projective. This follows from Theorem 6.1(1): the filtration of $Q(b)$ constructed there has just one layer by the minimality of λ so it shows that $\Delta(b)$ is a summand of the projective module $Q(b)$. □

Theorem 6.1 reveals that we are in a situation which is similar in some respects to the semi-infinite fully stratified categories of [BS18], and in other respects to the affine highest weight categories of [Kle15a]. However, in [BS18], there is no grading and the algebras A_λ are assumed to be finite-dimensional, while in [Kle15a] the graded algebra A is assumed to be both unital and Noetherian. In the examples of interest to us, the sections of the filtration constructed in Theorem 6.1 usually involve infinite direct sums, so that our indecomposable projectives $P(b)$ $(b\in\mathbf{B})$ are seldom Noetherian. So we need to develop some new theory to proceed.

Definition 6.3. By a Δ-*layer* (resp., a $\bar{\Delta}$-*layer*) of type λ, we mean a graded A-module that is isomorphic to $j_!^\lambda\bar{V}$ for a projective (resp., an arbitrary) graded left A_λ-module $\bar{V}$ that is locally finite-dimensional and bounded below. We say that $V\in\mathrm{ob}\,A\text{-gmod}$ has a Δ-*flag* (resp., a $\bar{\Delta}$-*flag*) if for some $n\geqslant 0$ there is a graded filtration

$$0=V_0\subset V_1\subset\cdots\subset V_n=V$$

and distinct weights $\lambda_1,\dots,\lambda_n\in\Lambda$ such that V_r/V_{r-1} is a Δ-layer (resp., a $\bar{\Delta}$-layer) of type λ_r for each $r=1,\dots,n$.

Definition 6.4. By a ∇-*layer* (resp., a $\bar{\nabla}$-*layer*) of type λ, we mean a graded A-module that is isomorphic to $j_*^\lambda\bar{V}$ for an injective (resp., an arbitrary) graded left A_λ-module $\bar{V}$ that is locally finite-dimensional and bounded above. We say that $V\in\mathrm{ob}\,A\text{-gmod}$ has a ∇-*flag* (resp., a $\bar{\nabla}$-*flag*) if for some $n\geqslant 0$ there is a graded filtration

$$V=V_0\supset V_1\supset\cdots\supset V_n=0$$

and distinct weights $\lambda_1,\dots,\lambda_n\in\Lambda$ such that V_{r-1}/V_r is a ∇-layer (resp., a $\bar{\nabla}$-layer) of type λ_r for each $r=1,\dots,n$.

Remark 6.5. Our Δ-layers of type λ can be defined equivalently as modules of the form $\bigoplus_{b\in\mathbf{B}_\lambda}\Delta(b)^{\oplus f_b}$ for $f_b\in\mathbb{N}(\!(q^{-1})\!)$. Similarly, ∇-layers of type λ are modules of the form $\bigoplus_{b\in\mathbf{B}_\lambda}\nabla(b)^{\oplus f_b}$ for $f_b\in\mathbb{N}(\!(q)\!)$. Using this, it follows that the module $Q(b)$ in Theorem 6.1(1) has a Δ-flag, and the module $J(b)$ in Theorem 6.1(2) has a ∇-flag.

The full subcategory of A-gmod consisting of modules with Δ-flags (resp., $\bar\Delta$-flags, ∇-flags, $\bar\nabla$-flags) will be denoted $A\text{-gmod}_\Delta$ (resp., $A\text{-gmod}_{\bar\Delta}$, $A\text{-gmod}_\nabla$, $A\text{-gmod}_{\bar\nabla}$). Evidently, Definitions 6.3 and 6.4 are dual to each other. Note also by Lemma 4.1 that modules in $A\text{-gmod}_\Delta$ or $A\text{-gmod}_{\bar\Delta}$ are locally finite-dimensional and bounded below, and modules in $A\text{-gmod}_\nabla$ or $A\text{-gmod}_{\bar\nabla}$ are locally finite-dimensional and bounded above. Consequently, all subsequent results about Δ- or $\bar\Delta$-flags have dual formulations involving ∇- or $\bar\nabla$-flags.

Noting that Δ-layers are $\bar\Delta$-layers, $A\text{-gmod}_\Delta$ is a subcategory of $A\text{-gmod}_{\bar\Delta}$. The next result allows sections in $\bar\Delta$-flags, hence, in Δ-flags, to be reordered so that the biggest weights are at the top.

Lemma 6.6. *If V is a $\bar\Delta$-layer of type λ and W is a $\bar\Delta$-layer of type μ for $\lambda\not\geq\mu$ then* $\operatorname{Ext}^1_A(V,W)=0$.

Proof. We have that $V=j_!^\lambda\bar V$ for some $\bar V\in\operatorname{ob}A_\lambda\text{-gmod}$ that is locally finite-dimensional and bounded below. Take the start of a projective resolution of $\bar V\in\operatorname{ob}A_\lambda\text{-gmod}$: $\bar P_1\to\bar P_0\to\bar V\to 0$. Then apply the exact functor $j_!^\lambda$ to deduce that there is an exact sequence $P_1\xrightarrow{f}P_0\to V\to 0$ in A-gmod such that P_0,P_1 both (possibly infinite) are direct sums of degree-shifted modules of the form $\Delta(b)$ for $b\in\mathbf{B}_\lambda$. Next, we apply Theorem 6.1(1) to construct projective resolutions $\cdots\to P_{0,1}\to P_{0,0}\to P_0\to 0$ and $\cdots\to P_{1,0}\to P_1\to 0$ such that $P_{0,0}$ and $P_{1,0}$ are (possibly infinite) direct sum of degree-shifted copies of $Q(b)$ for $b\in\mathbf{B}_\lambda$ and $P_{0,1}$ is a (possibly infinite) direct sum of degree-shifted copies of $Q(a)$ for $a\in\mathbf{B}_{\leqslant\lambda}$. By the nature of the filtration from Theorem 6.1(1), it follows that all irreducible quotients of $P_{0,0},P_{0,1}$ and $P_{1,0}$ are degree-shifted copies of $L(a)$ for $a\in\mathbf{B}_{\leqslant\lambda}$. We lift $f:P_1\to P_0$ to these resolutions then take the total complex to obtain the beginning of a projective resolution

$$P_{1,0}\oplus P_{0,1}\to P_{0,0}\to V\to 0$$

of V. Then apply $\operatorname{Hom}_A(-,W)$ and take homology to deduce that $\operatorname{Ext}^1_A(V,W)$ is a subquotient of $\operatorname{Hom}_A(P_{1,0}\oplus P_{0,1},W)$. But the module W has lowest weight μ, while all non-zero quotients of $P_{1,0}\oplus P_{0,1}$ have a weight that is $\leqslant\lambda$. Since $\lambda\not\geq\mu$ this means that $\operatorname{Hom}_A(P_{1,0}\oplus P_{0,1},W)=0$, so $\operatorname{Ext}^1_A(V,W)=0$ too. □

Corollary 6.7. *If V is a Δ-layer of type λ and W is a Δ-layer of type μ for $\lambda\not\geq\mu$ then* $\operatorname{Ext}^1_A(V,W)=0$.

Proof. This follows immediately from the lemma since Δ-layers are $\bar\Delta$-layers. □

Remark 6.8. In fact, the following slightly stronger statement than Corollary 6.7 is true: if V is a Δ-layer of type λ and W is a Δ-layer of type μ for $\lambda\not>\mu$ then $\operatorname{Ext}^1_A(V,W)=0$. We are not in a position to be able to prove this yet, but it follows from Corollary 8.4 and Lemma 4.4 since they imply that $P(b)$ has a Δ-flag with top section $\Delta(b)$ and other sections that are Δ-layers of type μ for $\mu<\dot b$. In view of this, if V is a Δ-layer of type λ, we can construct a projective resolution $\cdots\to P_1\to P_0\to V$ such that P_0 is a direct sum of degree-shifted copies of $P(b)$ ($b\in\mathbf{B}_\lambda$) and P_1 is a direct sum of degree-shifted copies of $P(c)$ ($c\in\mathbf{B}_{<\lambda}$). We deduce that $\operatorname{Ext}^1_A(V,W)=0$ for a Δ-layer W of type $\mu\not<\lambda$ since we have that $\operatorname{Hom}_A(P_1,W)=0$ like at the end of the proof of Lemma 6.6.

Later on, the next lemma (which is really two lemmas since there are two cases in the statement) will be used at a crucial point in an inductive argument; see the proof of Theorem 8.3.

Lemma 6.9. *Suppose that $\lambda\in\Lambda$ is minimal and $V\in A$-gmod has the following properties:*

(1) V is locally finite-dimensional and bounded below;

(2) $V = Ae_\lambda V$;
(3) $\mathrm{Ext}^1_A(V, \bar\nabla(b)) = 0$ *(resp.,* $\mathrm{Ext}^1_A(V, \nabla(b)) = 0$*) for all* $b \in \mathbf{B}$.

Then V *is a* Δ*-layer (resp., a* $\bar\Delta$*-layer) of type* λ.

Proof. The assumption (2) plus Lemma 3.4(1) implies that all weights of V are $\geqslant \lambda$, hence, V is an $A_{\geqslant\lambda}$-module. The counit of adjunction gives a homomorphism $\varepsilon^\lambda_V : j^\lambda_! j^\lambda V \to V$. This becomes an isomorphism when we apply j^λ, so the λ-weight space of coker f is zero. But by (2) we know that every quotient of V is generated by its λ-weight space, so this implies that coker $\varepsilon^\lambda_V = 0$. Thus, we have proved that ε^λ_V is surjective.

Let $K := \ker \varepsilon^\lambda_V$ so that there is a short exact sequence $0 \to K \to j^\lambda_! j^\lambda V \to V \to 0$. Let $Y := \bar\nabla(b)$ (resp., $\nabla(b)$) for some $b \in \mathbf{B}$. We claim that $\mathrm{Hom}_A(K, Y) = 0$. To see this, we apply $\mathrm{Hom}_A(-, Y)$ to the short exact sequence and use (3) to get another short exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(V, Y) \longrightarrow \mathrm{Hom}_A(j^\lambda_! j^\lambda V, Y) \longrightarrow \mathrm{Hom}_A(K, Y) \to 0$$

If $\dot b \not\geqslant \lambda$ then $\mathrm{Hom}_A(K, Y) = 0$ because $\dot b$ is a weight of soc Y but it is not a weight of K. If $\dot b > \lambda$ then $\mathrm{Hom}_A(j^\lambda_! j^\lambda V, Y) \cong \mathrm{Hom}_{A_\lambda}(j^\lambda V, j^\lambda Y)$, which is zero since $j^\lambda Y = 0$. Hence, $\mathrm{Hom}_A(K, Y) = 0$ in this case. If $\dot b = \lambda$, both $\mathrm{Hom}_A(j^\lambda_! j^\lambda V, Y)$ and $\mathrm{Hom}_A(V, Y)$ are isomorphic to $\mathrm{Hom}_{A_\lambda}(j^\lambda V, j^\lambda Y)$; in the second case this follows because $Y = j^\lambda_* j^\lambda Y$. Hence, they have the same graded dimensions. It follows that the first map in the displayed short exact sequence is an isomorphism. Again, this gives that $\mathrm{Hom}_A(K, Y) = 0$, and the claim is proved.

By the claim, we deduce in either case that $\mathrm{Hom}_A(K, \bar\nabla(b)) = 0$ for all $b \in \mathbf{B}$. So $K = 0$ thanks to Lemma 4.7. Now we have proved that $V \cong j^\lambda_! j^\lambda V$. This already shows that V is a $\bar\Delta$-layer. To complete the proof, we need to show that $j^\lambda V$ is projective in A_λ-gmod in the case that $\mathrm{Ext}^1_A(V, \bar\nabla(b)) = 0$ for all $b \in \mathbf{B}$. The functor j^λ_* is right adjoint to an exact functor, so it takes injective graded A_λ-modules to injective graded $A_{\geqslant\lambda}$-modules. It is also exact by Lemma 4.1. Since $\mathrm{Hom}_{A_\lambda}(j^\lambda V, -) \cong \mathrm{Hom}_{A_{\geqslant\lambda}}(V, -) \circ j^\lambda_*$, a standard degenerate Grothendieck spectral sequence argument gives that

$$\mathrm{Ext}^n_{A_\lambda}(j^\lambda V, -) \cong \mathrm{Ext}^n_{A_{\geqslant\lambda}}(V, j^\lambda_* -) \tag{6.1}$$

for any $V \in A_{\geqslant\lambda}$-gmod and $n \geqslant 0$. Using this, we deduce that

$$\mathrm{Ext}^1_{A_\lambda}(j^\lambda V, L_\lambda(b)) \cong \mathrm{Ext}^1_{A_{\geqslant\lambda}}(V, j^\lambda_* L_\lambda(b)) = \mathrm{Ext}^1_A(V, \bar\nabla(b)) = 0$$

for all $b \in \mathbf{B}_\lambda$. This implies that $j^\lambda V$ is projective according to Corollary 2.5. □

7. Truncation to upper sets

In this section, we assume that $\hat\Lambda$ is an upper set in Λ. Let $\hat{\mathbf{S}} := \{s \in \mathbf{S} \mid \dot s \in \hat\Lambda\}$, $\hat{\mathbf{B}} := \{b \in \mathbf{B} \mid \dot b \in \hat\Lambda\}$ and $\check\Lambda := \Lambda - \hat\Lambda$, $\check{\mathbf{S}} := \mathbf{S} - \hat{\mathbf{S}}$, $\check{\mathbf{B}} := \mathbf{B} - \hat{\mathbf{B}}$. Let $\hat A$ be the quotient of A by the two-sided ideal generated by the idempotents e_λ $(\lambda \in \check\Lambda)$. Let

$$i : \hat A\text{-gmod} \to A\text{-gmod}$$

be the canonical inclusion functor, which is part of the adjoint triple $(i^*, i, i^!)$ discussed in (3.3) and (3.4). So $i^* = \hat A \otimes_A -$ and $i^! = \bigoplus_{i \in \mathbf{I}} \mathrm{Hom}_A(\hat A \bar 1_i, -)$.

By Corollary 3.2, $\hat A = \bigoplus_{i,j \in \mathbf{I}} 1_i \hat A 1_j$ has a graded triangular basis with special idempotents indexed by $\hat{\mathbf{S}} \subseteq \mathbf{I}$, weight poset $\hat\Lambda$, and bases arising from the sets $\hat{\mathrm{X}}(i,s), \hat{\mathrm{H}}(s,t), \hat{\mathrm{Y}}(s,j)$ that are the canonical images of $\mathrm{X}(i,s), \mathrm{H}(s,t), \mathrm{Y}(t,j)$ for $i,j \in \mathbf{I}$, $s,t \in \hat{\mathbf{S}}$. Using the decoration "$\wedge$" in other notation related to $\hat A$ in the obvious way, the algebras $\hat A_{\geqslant\lambda}$ $(\lambda \in \hat\Lambda)$ are naturally identified with the algebras $A_{\geqslant\lambda}$. So we also have that $\hat A_\lambda = A_\lambda$, and the adjoint triple $(\hat j^\lambda_!, \hat j, \hat j^\lambda_*)$ defined for $\hat A$ is just the same triple of functors $(j^\lambda_!, j^\lambda, j^\lambda_*)$ as for A, still assuming that $\lambda \in \hat\Lambda$.

The various modules for $\hat{A}$ arising from the triangular basis are

$$\hat{\Delta}(b) := j_!^\lambda P_\lambda(b), \qquad \hat{\bar{\Delta}}(b) := j_!^\lambda L_\lambda(b), \qquad \hat{\bar{\nabla}}(b) := j_*^\lambda L_\lambda(b), \qquad \hat{\nabla}(b) := j_*^\lambda I_\lambda(b) \tag{7.1}$$

for $b \in \hat{\mathbf{B}}$ and $\lambda := \dot{b}$. Then the modules $\hat{L}(b) := \operatorname{cosoc} \hat{\Delta}(b) = \operatorname{soc} \hat{\nabla}(b)$ for $b \in \hat{\mathbf{B}}$ give a complete set of irreducible graded left $\hat{A}$-modules up to isomorphism and degree shift. We denote a projective cover and an injective hull of $\hat{L}(b)$ by $\hat{P}(b)$ and $\hat{I}(b)$, respectively.

Lemma 7.1. *For $b \in \hat{\mathbf{B}}$, we have that $\hat{\Delta}(b) = \Delta(b)$, $\hat{\bar{\Delta}}(b) = \bar{\Delta}(b)$, $\hat{L}(b) = L(b)$, $\hat{\bar{\nabla}}(b) = \bar{\nabla}(b)$ and $\hat{\nabla}(b) = \nabla(b)$. Also $i^*P(b) \cong \hat{P}(b)$, $i^!I(b) \cong \hat{I}(b)$ if $b \in \hat{\mathbf{B}}$, and $i^*P(b) = i^*\Delta(b) = i^*\bar{\Delta}(b) = i^*L(b) = i^!L(b) = i^!\bar{\nabla}(b) = i^!\nabla(b) = i^!I(b) = 0$ if $b \in \check{\mathbf{B}}$.*

Proof. We have that $i_{\geqslant\lambda} = i \circ \hat{\imath}_{\geqslant\lambda}$, which implies the assertions about $\Delta(b)$, $\bar{\Delta}(b)$, $\bar{\nabla}(b)$ and $\nabla(b)$ for $b \in \hat{\mathbf{B}}$. Clearly we also have that $\hat{L}(b) = L(b)$ since it is the irreducible head of $\hat{\Delta}(b) = \Delta(b)$. To see that $i^*P(b) \cong \hat{P}(b)$, note that i^* is left adjoint to an exact functor, so $i^*P(b)$ is a finitely generated projective for any $b \in \mathbf{B}$. It remains to observe for $c \in \hat{\mathbf{B}}$ that $\operatorname{Hom}_{\hat{A}}(i^*P(b), L(c)) \cong \operatorname{Hom}_A(P(b), iL(c))$, which is zero unless $c = b$. This gives that $i^*P(b) \cong \hat{P}(b)$ for $b \in \hat{\mathbf{B}}$ and it is zero otherwise. A similar argument proves the assertion about $i^!I(b)$. Everything else follows by right exactness of i^* and left exactness of $i^!$. □

Lemma 7.2. *For $V \in \operatorname{ob} A\text{-gmod}_\Delta$ and $i \in \mathbf{I}$, we have that $\operatorname{Tor}_m^A(1_i\hat{A}, V) = 0$ for all $m \geqslant 1$.*

Proof. In the next paragraph, we show that $\operatorname{Tor}_m^A(1_i\hat{A}, \Delta(b)) = 0$ for $b \in \mathbf{B}$ and $m \geqslant 1$. To deduce the lemma from this, Δ-layers are (possibly infinite) direct sums of standard modules as noted in Remark 6.5, so we get that $\operatorname{Tor}_m^A(1_i\hat{A}, V) = 0$ for all Δ-layers V and $m \geqslant 1$. Then one deduces the result for all V with a Δ-flag by induction on the length of the filtration.

Take $b \in \mathbf{B}$ and let Q be the module $Q(b)$ from Theorem 6.1(1). There is a short exact sequence $0 \to K \to Q \to \Delta(b) \to 0$ with K and Q having a Δ-flags. Applying $1_i\hat{A} \otimes_A -$ gives the long exact sequence

$$0 \to \operatorname{Tor}_1^A(1_i\hat{A}, \Delta(b)) \longrightarrow 1_i\hat{A} \otimes_A K \longrightarrow 1_i\hat{A} \otimes_A Q \longrightarrow 1_i\hat{A} \otimes_A \Delta(b) \to 0$$

and isomorphisms $\operatorname{Tor}_{m+1}^A(1_i\hat{A}, \Delta(b)) \cong \operatorname{Tor}_m^A(1_i\hat{A}, K)$ for $m \geqslant 1$. Now we use Corollary 6.7 to see that the Δ-flags of K and Q can be ordered to obtain short exact sequences $0 \to K^- \to K \to K^+ \to 0$ and $0 \to Q^- \to Q \to Q^+ \to 0$ so that K^- and Q^- (resp., K^+ and Q^+) have a Δ-flags with all sections being Δ-layers of types in $\check{\Lambda}$ (resp., $\hat{\Lambda}$). It is then clear that $1_i\hat{A} \otimes_A K = 1_iK^+$ and $1_i\hat{A} \otimes_A Q = 1_iQ^+$, since K^+ and Q^+ are the largest quotients of K and Q with all weights in $\hat{\Lambda}$. If $b \notin \hat{\mathbf{B}}$ then $K^+ = 0$, so we have that $\operatorname{Tor}_1^A(1_i\hat{A}, \Delta(b)) = 0$ at once. If $b \in \hat{\mathbf{B}}$ then there is a short exact sequence $0 \to 1_iK^+ \to 1_iQ^+ \to 1_i\Delta(b) \to 0$. This is just the same as the rightmost terms $1_i\hat{A} \otimes_A K \to 1_i\hat{A} \otimes_A Q \to 1_i\hat{A} \otimes_A \Delta(b) \to 0$ of the long exact sequence displayed above. So again we deduce that $\operatorname{Tor}_1^A(1_i\hat{A}, \Delta(b)) = 0$. So now we have shown that $\operatorname{Tor}_1^A(1_i\hat{A}, \Delta(b)) = 0$ for all $b \in \mathbf{B}$. For K as before, it follows that $\operatorname{Tor}_1^A(1_i\hat{A}, K) = 0$, hence, we get that $\operatorname{Tor}_2^A(1_i\hat{A}, \Delta(b)) = 0$ for all $b \in \mathbf{B}$. Further degree shifting like this gives the conclusion in general. □

Corollary 7.3. *The functor $i^* = \hat{A} \otimes_A -$ takes short exact sequences of modules with Δ-flags to short exact sequences of modules with Δ-flags. Similarly, the functor $i^! = \bigoplus_{i \in \mathbf{I}} \operatorname{Hom}_A(\hat{A}1_i, -)$ takes short exact sequences of modules with ∇-flags to short exact sequences of modules with ∇-flags.*

Proof. The lemma shows that i^* takes short exact sequences of modules with Δ-flags to short exact sequences. Hence, to prove that i^* takes modules with Δ-flags to Δ-flags, it suffices to check that i^* takes Δ-layers to Δ-layers. This follows from Lemma 7.1 since i^* commutes with direct sums. This proves the first statement. Then the second statement follows by duality, i.e., we apply $?^{\circledast}$ then the analog of the first statement for the opposite algebras, then apply $?^{\circledast}$ again. □

Lemma 7.4. *For $V \in \mathrm{ob}\, A\text{-gmod}_\Delta$ and $W \in \mathrm{ob}\, \hat{A}\text{-gmod}$, we have that $\mathrm{Ext}^n_A(V, iW) \cong \mathrm{Ext}^n_{\hat{A}}(i^*V, W)$ for all $n \geqslant 0$. Similarly, for $V \in \mathrm{ob}\, \hat{A}\text{-gmod}$ and $W \in \mathrm{ob}\, A\text{-gmod}_\nabla$, we have that $\mathrm{Ext}^n_A(iV, W) \cong \mathrm{Ext}^n_{\hat{A}}(V, i^! W)$ for all $n \geqslant 0$.*

Proof. To prove the first statement, take $W \in \mathrm{ob}\, \hat{A}\text{-gmod}$. The adjunction gives an isomorphism of functors $\mathrm{Hom}_{\hat{A}}(-, W) \circ i^* \cong \mathrm{Hom}_A(-, iW)$. Also the functor $i^* = \hat{A} \otimes_A -$ takes projectives to projectives as it is left adjoint to an exact functor. By a Grothendieck spectral sequence argument, it follows that $\mathrm{Ext}^n_{\hat{A}}(i^*V, W) \cong \mathrm{Ext}^n_A(V, iW)$ for all $n \geqslant 0$ and V such that $(\mathbb{L}_m i^*)V = \mathrm{Tor}^A_m(\hat{A}, V) = 0$ for all $m \geqslant 1$. It remains to apply Lemma 7.2.

The second statement follows from the first statement by duality. This is a bit more complicated than it sounds, so we go through the details. We show equivalently that $\mathrm{Ext}^n_A(iV, W^\circledast) \cong \mathrm{Ext}^n_{\hat{A}}\left(V, i^!(W^\circledast)\right)$ for $V \in \mathrm{ob}\, \hat{A}\text{-gmod}$ and $W \in \mathrm{ob}\ \text{gmod-}A$ such that $W^\circledast$ has a ∇-flag (equivalently, W has a Δ^{op}-flag). We have that $i \circ ?^\circledast \cong ?^\circledast \circ i$ viewed as covariant functors from $(\text{gmod-}\hat{A})^{\mathrm{op}}$ to A-gmod. Taking left adjoints gives that $?^\circledast \circ i^* \cong i^! \circ ?^\circledast$ viewed as functors from A-gmod to $(\text{gmod-}\hat{A})^{\mathrm{op}}$. So

$$\mathrm{Ext}^n_{\hat{A}}\left(V, i^!(W^\circledast)\right) \cong \mathrm{Ext}^n_{\hat{A}}\left(V, (i^*W)^\circledast\right) \overset{(2.6)}{\cong} \mathrm{Ext}^n_{\hat{A}}\left(i^*W, V^\circledast\right).$$

Then we apply the analog of the first statement for the opposite algebras to see that

$$\mathrm{Ext}^n_{\hat{A}}\left(i^*W, V^\circledast\right) \cong \mathrm{Ext}^n_A\left(W, i(V^\circledast)\right) \cong \mathrm{Ext}^n_A\left(W, (iV)^\circledast\right) \overset{(2.6)}{\cong} \mathrm{Ext}^n_A\left(iV, W^\circledast\right),$$

as required. □

Now we can prove the hallmark property of highest weight categories and their generalizations:

Theorem 7.5. *If $V \in \mathrm{ob}\, A\text{-gmod}_\Delta$ and $W \in \mathrm{ob}\, A\text{-gmod}_{\bar{\nabla}}$, or if $V \in \mathrm{ob}\, A\text{-gmod}_{\bar{\Delta}}$ and $W \in \mathrm{ob}\, A\text{-gmod}_\nabla$, we have that $\mathrm{Ext}^n_A(V, W) = 0$ for all $n \geqslant 1$.*

Proof. We prove this assuming $V \in \mathrm{ob}\, A\text{-gmod}_\Delta$ and $W \in \mathrm{ob}\, A\text{-gmod}_{\bar{\nabla}}$; the result in the other case then follows by duality. The proof reduces easily to the case that V is a single Δ-layer and $W = j^\lambda_! \bar{W}$ is a single $\bar{\nabla}$-layer of type λ. By Remark 6.5, V is a (possibly infinite) direct sum of degree-shifted standard modules, and the proof reduces further to checking that $\mathrm{Ext}^n_A(\Delta(b), j^\lambda_! \bar{W}) = 0$ for all $b \in \mathbf{B}$ and $n \geqslant 1$. By Lemma 7.4, we have that

$$\mathrm{Ext}^n_A(\Delta(b), j^\lambda_! \bar{W}) \cong \mathrm{Ext}^n_{A_{\geqslant\lambda}}(i^*_{\geqslant\lambda}\Delta(b), j^\lambda_! \bar{W}).$$

If $\dot{b} \not\geqslant \lambda$ then $i^*_{\geqslant\lambda}\Delta(b) = 0$ and the conclusion follows at once. If $\dot{b} \geqslant \lambda$ then we are in the same situation as (6.1), and applying that isomorphism gives that $\mathrm{Ext}^n_{A_{\geqslant\lambda}}(\Delta(b), j^\lambda_* \bar{W}) \cong \mathrm{Ext}^n_{A_\lambda}(j^\lambda \Delta(b), \bar{W})$. This is zero for $n \geqslant 1$ as required since $j^\lambda\Delta(b) \cong P_\lambda(b)$ is projective in A_λ-gmod if $\dot{b} = \lambda$, and $j^\lambda \Delta(b) = 0$ otherwise. □

8. BGG reciprocity

Using Theorem 7.5, we can make sense of multiplicities in Δ- and $\bar{\Delta}$-flags. First, for any $V \in A$-gmod, we define the Δ- and $\bar{\Delta}$-*supports* of V:

$$\mathrm{supp}_\Delta(V) := \left\{\dot{b} \;\middle|\; b \in \mathbf{B} \text{ such that } \mathrm{Hom}_A(V, \bar{\nabla}(b)) \neq 0\right\}, \tag{8.1}$$

$$\mathrm{supp}_{\bar{\Delta}}(V) := \left\{\dot{b} \;\middle|\; b \in \mathbf{B} \text{ such that } \mathrm{Hom}_A(V, \nabla(b)) \neq 0\right\}. \tag{8.2}$$

Since $\bar{\nabla}(b) \hookrightarrow \nabla(b)$, we have that $\mathrm{supp}_\Delta(V) \subseteq \mathrm{supp}_{\bar{\Delta}}(V)$. When A is not unital, i.e., infinitely many of the e_λ $(\lambda \in \Lambda)$ are non-zero, these sets could be infinite, but they are always finite if V is finitely generated:

Lemma 8.1. *If V is a finitely generated graded left A-module then* $\mathrm{supp}_\Delta(V)$ *and* $\mathrm{supp}_{\bar\Delta}(V)$ *are finite.*

Proof. Suppose that V is generated by finitely many weight vectors. Let $\lambda_1,\dots,\lambda_n$ be their weights. Then $\mathrm{Hom}_A(V,\nabla(b)) = 0$ unless one of $\lambda_1,\dots,\lambda_n$ is a weight of $\nabla(b)$. But this implies that $b \in \mathbf{B}_{\leqslant\lambda_1} \cup \cdots \cup \mathbf{B}_{\leqslant\lambda_r}$, which is finite. □

For V with a Δ-flag, we define

$$(V : \Delta(b))_q := \overline{\dim_q \mathrm{Hom}_A(V, \bar\nabla(b))} \in \mathbb{N}(\!(q^{-1})\!). \tag{8.3}$$

This is non-zero if and only if $b \in \mathrm{supp}_\Delta(V)$. If $0 = V_0 \subseteq \cdots \subseteq V_n = V$ is a Δ-flag, the section V_r/V_{r-1} being a Δ-layer of type λ_r, we have that

$$(V : \Delta(b))_q = \sum_{r=1}^n (V_r/V_{r-1} : \Delta(b))_q. \tag{8.4}$$

This follows from Theorem 7.5. Moreover, Corollary 4.6 implies that

$$V_r/V_{r-1} \cong \bigoplus_{b\in\mathbf{B}_{\lambda_r}} \Delta(b)^{\oplus (V_r/V_{r-1}:\Delta(b))_q}. \tag{8.5}$$

Thus, $(V : \Delta(b))_q$ counts the graded multiplicity of $\Delta(b)$ as a summand of the layers of the Δ-flag as one would expect. Instead, if V has a $\bar\Delta$-flag, we set

$$(V : \bar\Delta(b))_q := \overline{\dim_q \mathrm{Hom}_A(V, \nabla(b))_q} \in \mathbb{N}(\!(q^{-1})\!), \tag{8.6}$$

which is non-zero if and only if $b \in \mathrm{supp}_{\bar\Delta}(V)$. Again, we have that

$$(V : \bar\Delta(b))_q = \sum_{r=1}^n (V_r/V_{r-1} : \bar\Delta(b))_q \tag{8.7}$$

if $0 = V_0 \subseteq \cdots \subseteq V_n = V$ is a $\bar\Delta$-flag; now this follows by Theorem 7.5. So $(V : \bar\Delta(b))_q$ computes the sum of the graded multiplicities of $\bar\Delta(b)$ in each of the $\bar\Delta$-layers, with the understanding that for a single $\bar\Delta$-layer $W \cong j_!^\lambda \bar W$ of type λ and $b \in \mathbf{B}_\lambda$ we have that

$$(W : \bar\Delta(b))_q = \overline{\dim_q \mathrm{Hom}_A(W, \nabla(b))} = \overline{\dim_q \mathrm{Hom}_{A_\lambda}(\bar W, I_\lambda(b))} = [\bar W : L_\lambda(b)]_q. \tag{8.8}$$

For example, every $\Delta(a)$ has a $\bar\Delta$-flag, and we have that

$$(\Delta(a) : \bar\Delta(b))_q = \begin{cases} [P_\lambda(a) : L_\lambda(b)]_q & \text{if } a, b \in \mathbf{B}_\lambda \text{ for some } \lambda \in \Lambda \\ 0 & \text{if } \dot a \neq \dot b. \end{cases} \tag{8.9}$$

Lemma 8.2. *If V has a Δ-flag then* $(V : \bar\Delta(b))_q = \sum_{a\in\mathbf{B}} (V : \Delta(a))_q (\Delta(a) : \bar\Delta(b))_q$.

Proof. It suffices to prove this when V is a Δ-layer of type λ, so $V \cong \bigoplus_{a\in\mathbf{B}_\lambda} \Delta(a)^{\oplus(V:\Delta(a))_q}$. Then

$$(V : \bar\Delta(b))_q = [j^\lambda V : L_\lambda(b)]_q = \sum_{a\in\mathbf{B}_\lambda} (V : \Delta(a))_q [P_\lambda(a) : L_\lambda(b)]_q = \sum_{a\in\mathbf{B}_\lambda} (V : \Delta(a))_q (\Delta(a) : \bar\Delta(b))_q.$$

Here, we used (8.8) and (8.9). □

Theorem 8.3 (Homological criteria for Δ- and $\bar\Delta$-flags)**.** *Assume that $V \in$* ob A-gmod *is locally finite-dimensional and bounded below.*

(1) *The following are equivalent:*
   (a) *V has a Δ-flag;*
   (b) *$|\mathrm{supp}_\Delta(V)| < \infty$ and $\mathrm{Ext}^1_A(V, \bar\nabla(b)) = 0$ for all $b \in \mathbf{B}$;*
   (c) *$|\mathrm{supp}_\Delta(V)| < \infty$ and $\mathrm{Ext}^n_A(V, \bar\nabla(b)) = 0$ for all $b \in \mathbf{B}$ and $n \geqslant 1$;*

(2) *The following are equivalent:*

(a) V *has a* $\bar{\Delta}$*-flag;*
(b) $|\operatorname{supp}_{\bar{\Delta}}(V)| < \infty$ *and* $\operatorname{Ext}^1_A(V, \nabla(b)) = 0$ *for all* $b \in \mathbf{B}$*;*
(c) $|\operatorname{supp}_{\bar{\Delta}}(V)| < \infty$ *and* $\operatorname{Ext}^n_A(V, \nabla(b)) = 0$ *for all* $b \in \mathbf{B}$ *and* $n \geqslant 1$.

Proof. (1) Clearly (c)⇒(b). Also (a)⇒(c) by Theorem 7.5. It remains to prove that (b)⇒(a). Suppose that (b) holds. We show that V has a Δ-flag by induction on the size of the support $\operatorname{supp}_\Delta(V)$. If $\operatorname{supp}_\Delta(V) = \varnothing$ then we have that $V = 0$ by Lemma 4.7, and the conclusion is clear. Now assume that $\operatorname{supp}_\Delta(V)$ is non-empty and pick a maximal element λ. Let $W := i^*_{\geqslant\lambda} V$. We are going to apply Lemma 6.9 (with A replaced by $A_{\geqslant\lambda}$, Λ replaced by the upper set generated by λ and $\mathbf{B}$ replaced by $\mathbf{B}_{\geqslant\lambda}$) to show that W is a Δ-layer of type λ; it is important to note here that "Δ-layer of type λ" means the same thing for $A_{\geqslant\lambda}$-gmod as it does for A-gmod because $\hat{j}^\lambda_! = j^\lambda_!$, notation as in section 7. Since W is a quotient of V, the choice of λ implies that $\operatorname{Hom}_A(W, \bar{\nabla}(b)) = 0$ unless $\dot{b} = \lambda$. Since $W/Ae_\lambda W$ does not have λ as a weight, we deduce that $\operatorname{Hom}_A(W/Ae_\lambda W, \bar{\nabla}(b)) = 0$ for all $b \in \mathbf{B}$. Applying Lemma 4.7 again, it follows that $W = Ae_\lambda W = A_{\geqslant\lambda} e_\lambda W$. Thus W satisfies property (2) from Lemma 6.9. Also, W is finitely generated, so it satisfies property (1). To show that it satisfies property (3) too, let K be the kernel of the quotient map $V \twoheadrightarrow W$ and take any $b \in \mathbf{B}$. Applying $\operatorname{Hom}_A(-, \bar{\nabla}(b))$ to the short exact sequence $0 \to K \to V \to W \to 0$ gives the long exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(W, \bar{\nabla}(b)) \longrightarrow \operatorname{Hom}_A(V, \bar{\nabla}(b)) \longrightarrow \operatorname{Hom}_A(K, \bar{\nabla}(b)) \longrightarrow \operatorname{Ext}^1_A(W, \bar{\nabla}(b)) \longrightarrow 0,$$

plus an isomorphism $\operatorname{Ext}^1_A(K, \bar{\nabla}(b)) \cong \operatorname{Ext}^2_A(W, \bar{\nabla}(b))$. Now suppose that $b \in \mathbf{B}_{\geqslant\lambda}$, so that all weights of $\bar{\nabla}(b)$ are $\geqslant \lambda$ too. By the definition of W, K does not have a proper quotient whose weights are all $\geqslant \lambda$, so $\operatorname{Hom}_A(K, \bar{\nabla}(b)) = 0$. We deduce that $\operatorname{Ext}^1_A(W, \bar{\nabla}(b)) = \operatorname{Ext}^1_{A_{\geqslant\lambda}}(W, \bar{\nabla}(b)) = 0$ for all $b \in \mathbf{B}_{\geqslant\lambda}$. Now we have checked all of the properties, so we can now apply Lemma 6.9 to deduce that W is indeed a Δ-layer of type λ.

From Theorem 7.5, it follows that $\operatorname{Ext}^2_A(W, \bar{\nabla}(b)) = 0$, hence, we get also that $\operatorname{Ext}^1_A(K, \bar{\nabla}(b)) = 0$ for all $b \in \mathbf{B}$. Also $|\operatorname{supp}_\Delta(K)| < |\operatorname{supp}_\Delta(V)|$ since $\operatorname{Hom}_A(K, \bar{\nabla}(b))$ is a quotient of $\operatorname{Hom}_A(V, \bar{\nabla}(b))$ for all $b \in \mathbf{B}$, and $\operatorname{Hom}_A(K, \bar{\nabla}(b)) = 0$ for $b \in \mathbf{B}_\lambda$ so $\lambda \notin \operatorname{supp}_\Delta(K)$. This means that we can apply the induction hypothesis to the module K to deduce that it has a Δ-flag. Also none of the layers in such a flag are of type λ, again because $\operatorname{Hom}_A(K, \bar{\nabla}(b)) = 0$ for $b \in \mathbf{B}_\lambda$. Now we have in our hands a Δ-flag of V coming from the Δ-flag of K plus the top section that is the Δ-layer W of type λ. Thus, (a) is proved.

(2) This is a very similar argument. For the hardest implication (b)⇒(a), one proceeds by induction on the size of the set $\operatorname{supp}_{\bar{\Delta}}(V)$. Noting that $\operatorname{supp}_\Delta(V) \subseteq \operatorname{supp}_{\bar{\Delta}}(V)$, we are done trivially in case $\operatorname{supp}_{\bar{\Delta}}(V) = \varnothing$ as before. Then we repeat the arguments in (a) replacing $\operatorname{supp}_\Delta(V)$ and $\bar{\nabla}(b)$ with $\operatorname{supp}_{\bar{\Delta}}(V)$ and $\nabla(V)$. ◻

Corollary 8.4 (BGG reciprocity for projectives)**.** *For* $b \in \mathbf{B}$*, the indecomposable projective* $P(b)$ *has a* Δ*-flag with* $(P(b) : \Delta(a))_q = \overline{[\bar{\nabla}(a) : L(b)]_q}$ *for all* $a \in \mathbf{B}$*. If the graded triangular basis admits a duality then* $(P(b) : \Delta(a))_q = [\bar{\Delta}(a) : L(b)]_q$*.*

Proof. The fact that $P(b)$ has a Δ-flag follows from Lemma 8.1 and the homological criterion of Theorem 8.3. For the multiplicities, we compute from the definition (8.3):

$$(P(b) : \Delta(a))_q = \overline{\dim_q \operatorname{Hom}_A(P(b), \bar{\nabla}(a))} = \overline{[\bar{\nabla}(a) : L(b)]_q}.$$

◻

Corollary 8.5. *Suppose that* $0 \to U \to V \to W \to 0$ *is a short exact sequence of graded left* A*-modules. Assuming that* W *has a* $\bar{\Delta}$*-flag,* U *has a* $\bar{\Delta}$*-flag if and only if* V *has a* $\bar{\Delta}$*-flag. Similarly for* Δ*-flags.*

Proof. We explain for $\bar\Delta$-flags, the case of Δ-flags being similar. Since W is locally finite-dimensional bounded below as it has a $\bar\Delta$-flag, it is clear that U is locally finite-dimensional and bounded below if and only if V has these properties. Also, this is the case if either U or V has a $\bar\Delta$-flag. Applying $\mathrm{Hom}_A(-,\nabla(b))$ to the short exact sequence using the vanishing of $\mathrm{Ext}^n_A(W,\nabla(b))$ for $n \geqslant 1$ gives short exact sequences

$$0 \longrightarrow \mathrm{Hom}_A(W,\nabla(b)) \longrightarrow \mathrm{Hom}_A(V,\nabla(b)) \longrightarrow \mathrm{Hom}_A(U,\nabla(b)) \longrightarrow 0$$

and isomorphisms $\mathrm{Ext}^1_A(V,\nabla(b)) \cong \mathrm{Ext}^1_A(U,\nabla(b))$ for all $b \in \mathbf{B}$. The short exact sequences imply that

$$\mathrm{supp}_{\bar\Delta}(V) = \mathrm{supp}_{\bar\Delta}(U) \cup \mathrm{supp}_{\bar\Delta}(W). \tag{8.10}$$

Hence, $\mathrm{supp}_{\bar\Delta}(U)$ is finite if and only if $\mathrm{supp}_{\bar\Delta}(V)$ is finite. Now we can apply the homological criterion for $\bar\Delta$-flags from Theorem 8.3 to deduce the result. □

Corollary 8.6. *The categories A-gmod_Δ and A-$\mathrm{gmod}_{\bar\Delta}$ are closed under degree shift, finite direct sum and passing to graded direct summands.*

Since they are often useful, we take the time to formulate the dual results too. The ∇- and $\bar\nabla$-*supports* of $V \in \mathrm{ob}\,A$-gmod are

$$\mathrm{supp}_\nabla(V) := \left\{\dot b \,\middle|\, b \in \mathbf{B} \text{ such that } \mathrm{Hom}_A(\bar\Delta(b),V) \neq 0\right\}, \tag{8.11}$$

$$\mathrm{supp}_{\bar\nabla}(V) := \left\{\dot b \,\middle|\, b \in \mathbf{B} \text{ such that } \mathrm{Hom}_A(\Delta(b),V) \neq 0\right\}. \tag{8.12}$$

We have that $\mathrm{supp}_\nabla(V) \subseteq \mathrm{supp}_{\bar\nabla}(V)$. These sets are necessarily finite if A is unital, or if V is finitely cogenerated (this statement is dual to Lemma 8.1). Multiplicities in ∇- and $\bar\nabla$-flags are defined by

$$(V : \nabla(b))_q := \dim_q \mathrm{Hom}_A(\bar\Delta(b),V) \in \mathbb{N}(\!(q)\!), \tag{8.13}$$

$$(V : \bar\nabla(b))_q := \dim_q \mathrm{Hom}_A(\Delta(b),V) \in \mathbb{N}(\!(q)\!), \tag{8.14}$$

with interpretations similar to the ones explained for Δ- and $\bar\Delta$-flags. For example, every $\nabla(a)$ has a $\bar\nabla$-flag with

$$(\nabla(a) : \bar\nabla(b))_q = \begin{cases} [I_\lambda(a) : L_\lambda(b)]_q & \text{if } a,b \in \mathbf{B}_\lambda \text{ for some } \lambda \in \Lambda \\ 0 & \text{if } \dot a \neq \dot b. \end{cases} \tag{8.15}$$

The dual results to Lemma 8.2, Theorem 8.3 and its corollaries are as follows:

Lemma 8.7. *If V has a ∇-flag then $(V : \bar\nabla(b))_q = \sum_{a\in\mathbf{B}} (V : \nabla(a))_q(\nabla(a) : \bar\nabla(b))_q$.*

Theorem 8.8 (Homological criteria for ∇- and $\bar\nabla$-flags)**.** *Assume that $V \in \mathrm{ob}\,A$-gmod is locally finite-dimensional and bounded above.*

(1) The following are equivalent:
  - *(a) V has a ∇-flag;*
  - *(b) $|\mathrm{supp}_\nabla(V)| < \infty$ and $\mathrm{Ext}^1_A(\bar\Delta(b),V) = 0$ for all $b \in \mathbf{B}$;*
  - *(c) $|\mathrm{supp}_\nabla(V)| < \infty$ and $\mathrm{Ext}^n_A(\bar\Delta(b),V) = 0$ for all $b \in \mathbf{B}$ and $n \geqslant 1$.*

(2) The following are equivalent:
  - *(a) V has a $\bar\nabla$-flag;*
  - *(b) $|\mathrm{supp}_{\bar\nabla}(V)| < \infty$ and $\mathrm{Ext}^1_A(\Delta(b),V) = 0$ for all $b \in \mathbf{B}$;*
  - *(c) $|\mathrm{supp}_{\bar\nabla}(V)| < \infty$ and $\mathrm{Ext}^n_A(\Delta(b),V) = 0$ for all $b \in \mathbf{B}$ and $n \geqslant 1$.*

Corollary 8.9 (BGG reciprocity for injectives)**.** *For $b \in \mathbf{B}$, the indecomposable injective $I(b)$ has a ∇-flag with $(I(b) : \nabla(a))_q = \overline{[\bar\Delta(a) : L(b)]_q}$ for all $a \in \mathbf{B}$. If the graded triangular basis admits a duality then $(I(b) : \nabla(a))_q = [\bar\nabla(a) : L(b)]_q$.*

Corollary 8.10. *Suppose that* $0 \to U \to V \to W \to 0$ *is a short exact sequence of graded left* A*-modules. Assuming that* U *has a* $\bar\nabla$*-flag,* V *has a* $\bar\nabla$*-flag if and only if* W *has a* $\bar\nabla$*-flag. Similarly for* ∇*-flags.*

Corollary 8.11. *The categories* $A\text{-gmod}_{\nabla}$ *and* $A\text{-gmod}_{\bar\nabla}$ *are closed under degree shift, finite direct sum and passing to graded direct summands.*

We record one more lemma which will be needed in the next section.

Lemma 8.12. *If* V *has a* $\bar\Delta$*-flag (resp., a* $\bar\nabla$*-flag) then* $[V : L(b)]_q = \sum_{a\in\mathbf{B}}(V : \bar\Delta(a))_q[\bar\Delta(a) : L(b)]_q$ *(resp.,* $\sum_{a\in\mathbf{B}}(V : \bar\nabla(a))_q[\bar\nabla(a) : L(b)]_q$*).*

Proof. We just prove the result when V has a $\bar\nabla$-flag, the other case being the dual statement. We may assume that V is a single $\bar\nabla$-layer, so $V \cong j^\lambda_* \bar V$ for a graded left A_λ-module $\bar V$ that is locally finite-dimensional and bounded above. By Corollary 8.4, $P(b)$ has a Δ-flag with sections given by the Δ-layers $\bigoplus_{a\in\mathbf{B}_\mu} \Delta(a)^{\oplus\overline{[\bar\nabla(a):L(b)]_q}}$ of type μ for all $\mu \in \Lambda$ (this being zero unless $\mu \leqslant \dot b$). Using Theorem 7.5, we deduce that

$$[V : L(b)]_q = \dim_q \mathrm{Hom}_A(P(b), V) = \sum_{\substack{\mu\in\Lambda\\ a\in\mathbf{B}_\mu}} \dim_q \mathrm{Hom}_A\left(\Delta(a)^{\oplus\overline{[\bar\nabla(a):L(b)]_q}}, j^\lambda_*\bar V\right)$$

$$= \sum_{\substack{\mu\in\mathbf{B}_{\geqslant\lambda}\\ a\in\mathbf{B}_\mu}} \dim_q \mathrm{Hom}_{A_{\geqslant\lambda}}\left(\Delta(a)^{\oplus\overline{[\bar\nabla(a):L(b)]_q}}, j^\lambda_*\bar V\right) = \sum_{a\in\mathbf{B}_\lambda} \dim_q \mathrm{Hom}_{A_\lambda}\left(P_\lambda(a)^{\oplus\overline{[\bar\nabla(a):L(b)]_q}}, \bar V\right).$$

To complete the proof, we show that the q^d-coefficient of $\dim_q \mathrm{Hom}_{A_\lambda}\left(P_\lambda(a)^{\oplus\overline{[\bar\nabla(a):L(b)]_q}}, \bar V\right)$ is equal to the q^d-coefficient of $(V : \bar\nabla(a))_q[\bar\nabla(a) : L(b)]_q$ for each $a \in \mathbf{B}_\lambda$ and $d \in \mathbb{Z}$. Like in (8.8), we have that

$$(V : \bar\nabla(a))_q = \dim_q \mathrm{Hom}_A(\Delta(a), V) = \dim_q \mathrm{Hom}_{A_\lambda}\left(P_\lambda(a), \bar V\right) = [\bar V : L_\lambda(a)]_q. \tag{8.16}$$

Assuming that $(V : \bar\nabla(a))_q = \sum_{m\in\mathbb{Z}} r_m q^m$ and $[\bar\nabla(a) : L(b)]_q = \sum_{n\in\mathbb{Z}} s_n q^n$, we deduce that

$$\dim \mathrm{Hom}_{A_\lambda}\left(P_\lambda(a)^{\oplus\overline{[\bar\nabla(a):L(b)]_q}}, \bar V\right)_{-d} = \dim \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{A_\lambda}\left(P_\lambda(a), \bar V\right)^{\oplus s_n}_{n-d} = \sum_{n\in\mathbb{Z}} r_{d-n}s_n,$$

which is the q^d-coefficient of $(V : \bar\nabla(a))_q[\bar\nabla(a) : L(b)]_q$ as we wanted. $\square$

9. Truncation to finite lower sets

Now let Γ be a *finite* lower set in Λ and set $\mathbf{S}_\Gamma := \{s \in \mathbf{S} \mid \dot s \in \Gamma\}$, $\mathbf{B}_\Gamma := \{b \in \mathbf{B} \mid \dot b \in \Gamma\}$. Let $e_\Gamma := \sum_{\lambda\in\Gamma} e_\lambda$. Then $A_\Gamma := e_\Gamma A e_\Gamma = \bigoplus_{s,t\in\mathbf{S}_\Gamma} 1_s A 1_t$ is a unital graded algebra which is locally finite-dimensional and bounded below. We let

$$j^\Gamma : A\text{-gmod} \to A_\Gamma\text{-gmod} \tag{9.1}$$

be the quotient functor defined by truncating with the idempotent e_Γ. As explained at the start of section 4, j^Γ fits into an adjoint triple $(j^\Gamma_!, j^\Gamma, j^\Gamma_*)$.

The algebra A_Γ has a graded triangular basis with distinguished=special idempotents 1_s ($s \in \mathbf{S}_\Gamma$), the finite weight poset $(\Gamma, \leqslant)$, and basis elements arising from the sets $\mathrm{X}(s,t), \mathrm{H}(s,t)$ and $\mathrm{Y}(s,t)$ for all $s,t \in \mathbf{S}_\Gamma$. For $\lambda \in \Gamma$, it is clear by considering the bases that the quotient algebra $(A_\Gamma)_{\geqslant\lambda}$ of A_Γ may be identified with the idempotent truncation $(A_{\geqslant\lambda})_\Gamma = \bar e_\Gamma A_{\geqslant\lambda}\bar e_\Gamma$ of $A_{\geqslant\lambda}$. Hence, $(A_\Gamma)_\lambda$ is identified with exactly the same algebra $A_\lambda = \bar e_\lambda A_{\geqslant\lambda}\bar e_\lambda$ as before. The analog of the adjoint triple $(j^\lambda_!, j^\lambda, j^\lambda_*)$ for A_Γ will be denoted $(j^{\Gamma,\lambda}_!, j^{\Gamma,\lambda}, j^{\Gamma,\lambda}_*)$. So

$$j^{\Gamma,\lambda} : (A_\Gamma)_{\geqslant\lambda}\text{-gmod} \to A_\gamma\text{-gmod} \tag{9.2}$$

is the idempotent truncation functor defined by $\bar{e}_\lambda$, and $j_!^{\Gamma,\lambda}$ and $j_*^{\Gamma,\lambda}$ are its left and right adjoints.

The standard, proper standard, costandard and proper costandard modules for A_Γ arising from the graded triangular basis are

$$\Delta_\Gamma(b) := j_!^{\Gamma,\lambda} P_\lambda(b), \qquad \bar{\Delta}_\Gamma(b) := j_!^{\Gamma,\lambda} L_\lambda(b), \qquad \bar{\nabla}_\Gamma(b) := j_*^{\Gamma,\lambda} L_\lambda(b), \qquad \nabla_\Gamma(b) := j_*^{\Gamma,\lambda} I_\lambda(b) \tag{9.3}$$

for $b \in \mathbf{B}_\Gamma$ and $\lambda := \dot{b}$. Then, by Theorem 4.3, the modules $L_\Gamma(b) := \operatorname{cosoc} \Delta_\Gamma(b) = \operatorname{soc} \nabla_\Gamma(b)$ for $b \in \mathbf{B}_\Gamma$ give a complete set of irreducible graded left A_Γ-modules up to isomorphism and degree shift. We denote a projective cover and an injective hull of $L_\Gamma(b)$ by $P_\Gamma(b)$ and $I_\Gamma(b)$, respectively.

Lemma 9.1. *For $b \in \mathbf{B}_\Gamma$, we have that $j_!^\Gamma P_\Gamma(b) \cong P(b)$, $j_!^\Gamma \Delta_\Gamma(b) \cong \Delta(b)$, $j_!^\Gamma \bar{\Delta}_\Gamma(b) \cong \bar{\Delta}(b)$, $j_*^\Gamma \bar{\nabla}_\Gamma(b) \cong \bar{\nabla}(b)$. $j_*^\Gamma \nabla_\Gamma(b) \cong \nabla(b)$, $j_*^\Gamma I_\Gamma(b) \cong I(b)$. Also $j^\Gamma L(b) \cong L_\Gamma(b)$ for $b \in \mathbf{B}_\Gamma$, and $j^\Gamma \Delta(b) = j^\Gamma \bar{\Delta}(b) = j^\Gamma L(b) = j^\Gamma \bar{\nabla}(b) = j^\Gamma \nabla(b) = 0$ for $b \in \mathbf{B} - \mathbf{B}_\Gamma$.*

Proof. The functor $j^\lambda : A_{\geqslant\lambda}\text{-gmod} \to A_\lambda\text{-gmod}$ is the composition of $j^\Gamma : A_{\geqslant\lambda}\text{-gmod} \to \bar{e}_\Gamma A_{\geqslant\lambda} \bar{e}_\Gamma\text{-gmod}$ followed by $j^{\Gamma,\lambda} : \bar{e}_\Gamma A_{\geqslant\lambda} \bar{e}_\Gamma\text{-gmod} \to A_\lambda\text{-gmod}$. Hence, $j_!^\lambda \cong j_!^\Gamma \circ j_!^{\Gamma,\lambda}$, giving that $j_!^\Gamma \Delta_\Gamma(b) \cong \Delta(b)$ and $j_!^\Gamma \bar{\Delta}_\Gamma(b) \cong \bar{\Delta}(b)$. Similarly, $j_*^\lambda \cong j_*^\Gamma \circ j_*^{\Gamma,\lambda}$, giving that $j_*^\Gamma \bar{\nabla}_\Gamma(b) \cong \bar{\nabla}(b)$. $j_*^\Gamma \nabla_\Gamma(b) \cong \nabla(b)$.

Next we show that $jL(b) \cong L_\Gamma(b)$ for $b \in \mathbf{B}_\lambda$ and $\lambda \in \Gamma$. This follows because $j^{\Gamma,\lambda}\left(j^\Gamma L(b)\right) = j^\lambda L(b) \cong L_\lambda(b)$ as A_λ-modules. Then we deduce that $j_!^\Gamma P_\Gamma(b) \cong P(b)$ and $j_*^\Gamma I_\Gamma(b) \cong I(b)$ for $b \in \mathbf{B}_\Gamma$ using adjunction properties.

Finally, it is clear that $j\Delta(b) = j\bar{\Delta}(b) = jL(b) = j\bar{\nabla}(b) = j\nabla(b) = 0$ for $b \in \mathbf{B} - \mathbf{B}_\Gamma$, since all these have lowest weight $\dot{b}$, hence, they have no weights that are in Γ. □

Corollary 9.2. *For $b \in \mathbf{B}_\Gamma$, we have that $j^\Gamma P(b) \cong P_\Gamma(b)$, $j^\Gamma \Delta(b) \cong \Delta_\Gamma(b)$, $j^\Gamma \bar{\Delta}(b) \cong \bar{\Delta}_\Gamma(b)$, $j^\Gamma \bar{\nabla}(b) \cong \bar{\nabla}_\Gamma(b)$, $j^\Gamma \nabla(b) \cong \nabla_\Gamma(b)$ and $j^\Gamma I(b) \cong I_\Gamma(b)$.*

Proof. This follows from the lemma since $j^\Gamma \circ j_!^\Gamma \cong \mathrm{id}_{A_\Gamma\text{-gmod}} \cong j^\Gamma \circ j_*^\Gamma$. □

Lemma 9.3. *For $V \in \operatorname{ob} A\text{-gmod}$ and $W \in \operatorname{ob} A_\Gamma\text{-gmod}_\nabla$, we have that $\operatorname{Ext}^n_{A_\Gamma}\left(j^\Gamma V, W\right) \cong \operatorname{Ext}^n_A\left(V, j_*^\Gamma W\right)$ for all $n \geqslant 0$.*

Proof. This is another Grothendieck spectral sequence argument. We have that $\operatorname{Hom}_{A_\Gamma}(-, W) \circ j^\Gamma \cong \operatorname{Hom}_A(-, j_*^\Gamma W)$. Also j^Γ is exact. To deduce that $\operatorname{Ext}^n_{A_\Gamma}(j^\Gamma -, W) \cong \operatorname{Hom}_A(-, j_*^\Gamma W)$, it remains to show that j^Γ sends projectives in A-gmod to modules that are acyclic for $\operatorname{Hom}_{A_\Gamma}(-, W)$. Since any projective in A-gmod is a summand of a direct sum of degree-shifts of the projective modules $Q(b)$ from Theorem 6.1(1), and j^Γ commutes with direct sum and with Q, the proof of this reduces to checking that $\operatorname{Ext}^n_{A_\Gamma}(j^\Gamma Q(b), W) = 0$ for all $b \in \mathbf{B}$ and $n \geqslant 1$. Since j^Γ is exact and $j^\Gamma \Delta(b)$ is either zero or a standard module for A_Γ by Lemma 9.1 and Corollary 9.2, we deduce that $j^\Gamma Q(b)$ has a Δ-flag. Hence, $\operatorname{Ext}^n_{A_\Gamma}(j^\Gamma Q(b), W) = 0$ for $n \geqslant 1$ thanks to Theorem 7.5. □

The dual result to Lemma 9.3 will be formulated and proved in Lemma 9.8 below. It does not follow immediately at this point since we have not included any assumption of locally finite-dimensionality on W in the statement.

Lemma 9.4. *If $V \in \operatorname{ob} A_\Gamma\text{-gmod}_\Delta$ then $j_!^\Gamma V \in \operatorname{ob} A\text{-gmod}_\Delta$ with $\operatorname{supp}_\Delta\left(j_!^\Gamma V\right) = \operatorname{supp}_\Delta(V)$, indeed, we have $\left(j_!^\Gamma V : \Delta(b)\right) = (V : \Delta_\Gamma(b))$ for $b \in \mathbf{B}_\Gamma$. The same statement with Δ replaced by $\bar{\Delta}$ everywhere also holds. Similarly, if $V \in \operatorname{ob} A_\Gamma\text{-gmod}_\nabla$ then $j_*^\Gamma V \in \operatorname{ob} A\text{-gmod}_\nabla$ with $\operatorname{supp}_\nabla\left(j_*^\Gamma V\right) = \operatorname{supp}_\nabla(V)$, indeed, we have $\left(j_*^\Gamma V : \nabla(b)\right) = (V : \nabla_\Gamma(b))$ for $b \in \mathbf{B}_\Gamma$. The same statement with ∇ replaced by $\bar{\nabla}$ everywhere also holds.*

Proof. The statements for Δ and $\bar{\Delta}$ follow from the ones for ∇ and $\bar{\nabla}$ by the usual duality argument. Now we proceed to prove the statement for ∇, with a similar argument proving the one for $\bar{\nabla}$. For $i \in \mathbf{I}$, we have that $1_i(j^\Gamma_* V) = \mathrm{Hom}_{A_\Gamma}(e_\Gamma A 1_i, V) \subseteq \mathrm{Hom}_\Bbbk(e_\Gamma A 1_i, V)$. Since $e_\Gamma A 1_i$ is locally finite-dimensional and bounded below and V is locally finite-dimensional and bounded above, we deduce that $j^\Gamma_* V$ is locally finite-dimensional and bounded above.

We have that $\mathrm{Hom}_A\left(\bar\Delta(b), j^\Gamma_* V\right) \cong \mathrm{Hom}_{A_\Gamma}\left(j^\Gamma \bar\Delta(b), V\right)$, and $j^\Gamma \bar\Delta(b) = \bar\Delta_\Gamma(b)$ if $b \in \mathbf{B}_\Gamma$ or 0 otherwise thanks to Lemma 9.1 and Corollary 9.2. Once we have proved that $j^\Gamma_* V$ has a ∇-flag, this will imply the statement about the multiplicities $\left(j^\Gamma_* V : \nabla(b)\right)$. It shows already that $\left|\mathrm{supp}_\nabla(j^\Gamma_* V)\right| = |\,\mathrm{supp}_\nabla(V)| < \infty$. Also $\mathrm{Ext}^1_A\left(\bar\Delta(b), j^\Gamma_* V\right) \cong \mathrm{Ext}^1_{A_\Gamma}\left(j^\Gamma \bar\Delta(b), V\right)$ by Lemma 9.3, which is zero for all $b \in \mathbf{B}$ as V has a $\bar\Delta$-flag. It remains to apply Theorem 8.8. ◻

Lemma 9.5. *For $V \in \mathrm{ob}\, A_\Gamma\text{-gmod}_{\bar\Delta}$ and $i \in \mathbf{I}$, we have that $\mathrm{Tor}^{A_\Gamma}_m(1_i A e_\Gamma, V) = 0$ for all $m \geqslant 1$.*

Proof. Consider the short exact sequence $0 \to K \to P \to V$ where $P := P_V$ is the projective cover of V in A_Γ-gmod from Lemma 2.4(1). We note that P has a $\bar\Delta$-flag. This follows from Theorem 8.3 (the condition $|\,\mathrm{sup}_{\bar\Delta}(P)| < \infty$ holds automatically since Γ is finite). Also V has a $\bar\Delta$-flag by assumption. Hence, K has a $\bar\Delta$-flag by Corollary 8.5. Now Lemma 9.4 implies that $j^\Gamma_! V$, $j^\Gamma_! P$ and $j^\Gamma_! L$ all have $\bar\Delta$-flags, and moreover $\left(j^\Gamma_! P : \bar\Delta(b)\right) = \left(j^\Gamma_! K : \bar\Delta(b)\right) + \left(j^\Gamma_! V : \bar\Delta(b)\right)$ for all $b \in \mathbf{B}$. Applying Lemma 8.12, we deduce that

$$\left[j^\Gamma_! P : L(b)\right]_q = \left[j^\Gamma_! K : L(b)\right]_q + \left[j^\Gamma_! V : L(b)\right]_q$$

for each $b \in \mathbf{B}$. It follows that $\dim_q 1_i A e_\Gamma \otimes_{A_\Gamma} P = \dim_q 1_i A e_\Gamma \otimes_{A_\Gamma} K + \dim_q 1_i A e_\Gamma \otimes_{A_\Gamma} V$. Applying $1_i A e_\Gamma \otimes_{A_\Gamma} -$ to the short exact sequence gives the long exact sequence

$$0 \longrightarrow \mathrm{Tor}^{A_\Gamma}_1(1_i A e, V) \longrightarrow 1_i A e_\Gamma \otimes_{A_\Gamma} K \longrightarrow 1_i A e_\Gamma \otimes_{A_\Gamma} P \longrightarrow 1_i A e_\Gamma \otimes_{A_\Gamma} V \longrightarrow 0$$

and isomorphisms $\mathrm{Tor}^{A_\Gamma}_m(1_i A e, K) \cong \mathrm{Tor}^{A_\Gamma}_{m+1}(1_i A e, V)$ for all $m \geqslant 1$. From this long exact sequence and the equality of dimensions already established, we deduce that $\mathrm{Tor}^{A_\Gamma}_1(1_i A e, V) = 0$. This applies equally well to K, so we get that $\mathrm{Tor}^{A_\Gamma}_1(1_i A e, K) = 0$, hence, $\mathrm{Tor}^{A_\Gamma}_2(1_i A e, V) = 0$. Further degree shifting like this completes the proof. ◻

Corollary 9.6. *The functor $j^\Gamma_! = A e_\Gamma \otimes_{\check A_\Gamma} -$ takes short exact sequences of modules with Δ-flags (resp., $\bar\Delta$-flags) to short exact sequences of modules with Δ-flags (resp., $\bar\Delta$-flags). Similarly, The functor $j^\Gamma_* = \bigoplus_{i \in \mathbf{I}} \mathrm{Hom}_{A_\Gamma}(e_\Gamma A 1_i,)$ takes short exact sequences of modules with ∇-flags (resp., $\bar\nabla$-flags) to short exact sequences of modules with ∇-flags (resp., $\bar\nabla$-flags).*

Proof. The results for ∇-flags and $\bar\nabla$-flags follow for the ones for Δ-flags and $\bar\Delta$-flags by duality. The proofs of Δ-flags and $\bar\Delta$-flags are similar. In the case of Δ-flags, the functor $j_!$ takes modules with Δ-flags to modules with Δ-flags by Lemma 9.4. It is exact on $A_\Gamma\text{-gmod}_{\bar\Delta}$ by Lemma 9.5, hence, it is exact on $A_\Gamma\text{-gmod}_\Delta$ too since this is a subcategory. ◻

The next theorem will be useful in the next section. For $V \in A$-gmod, we let

$$V_\Gamma := A e_\Gamma V, \qquad\qquad V^\Gamma := \{v \in V \mid e_\Gamma A v = 0\}. \tag{9.4}$$

The counit of adjunction for the adjoint pair $(j^\Gamma_!, j^\Gamma)$ defines a homomorphism $\varepsilon^\Gamma_V : j^\Gamma_! j^\Gamma V \to V$. This is just the natural multiplication map $A e_\Gamma \otimes_{A_\Gamma} e_\Gamma V \to V$, so its image is the submodule V_Γ just defined. Also the unit of adjunction for the adjoint pair (j^Γ, j^Γ_*) defines a homomorphism $\eta^\Gamma_V : V \to j^\Gamma_* j^\Gamma V$. This takes $v \in V$ to the element of $j^\Gamma_* j^\Gamma V = \bigoplus_{i \in \mathbf{I}} \mathrm{Hom}_{A_\Gamma}(e_\Gamma A 1_i, V)$ that maps $e_\Gamma a 1_i \in e_\Gamma A 1_i$ to $e_\Gamma a 1_i v$. From this, we see that $\ker \eta^\Gamma_V = V^\Gamma$.

Theorem 9.7. *Suppose that $V \in A$-gmod.*

(1) *If V_Γ has a $\bar\Delta$-flag then the counit of adjunction defines an isomorphism $\varepsilon^\Gamma_V : j^\Gamma_! j^\Gamma V \xrightarrow{\sim} V_\Gamma$.*
(2) *If V/V^Γ has a $\bar\nabla$-flag then the unit of adjunction defines an isomorphism $\eta^\Gamma_V : V/V^\Gamma \xrightarrow{\sim} j^\Gamma_* j^\Gamma V$.*

Proof. (1) Suppose that $V_\Gamma = Ae_\Gamma V$ has a $\bar{\Delta}$-flag. Let $K := \ker \varepsilon_V^\Gamma$ so that there is a short exact sequence $0 \to K \to j_!^\Gamma j^\Gamma V \to V_\Gamma \to 0$. We need to show that $K = 0$. Note that the second map in this short exact sequence becomes an isomorphism when we apply j^Γ, so we have that $j^\Gamma K = 0$. Since j^Γ is exact, it is clear from Lemma 9.1 that $j^\Gamma V_\Gamma$ has a $\bar{\Delta}$-flag. Since $j^\Gamma V_\Gamma = e_\Gamma A e_\Gamma V = j^\Gamma V$, we deduce that $j^\Gamma V$ has a $\bar{\Delta}$-flag. Now Lemma 9.4 gives that $j_!^\Gamma j^\Gamma V$ has a $\bar{\Delta}$-flag with $\operatorname{supp}_{\bar{\Delta}} \left(j_!^\Gamma j^\Gamma V\right) \subseteq \Gamma$. By Corollary 8.5 and (8.10), we deduce that K has a $\bar{\Delta}$-flag with $\operatorname{supp}_{\bar{\Delta}}(K) \subseteq \Gamma$ too. Since $j^\Gamma K = 0$ and j^Γ is non-zero on any $\bar{\Delta}(b)$, we must have that $K = 0$.
(2) This is quite similar. Start from the short exact sequence $0 \to V/V^\Gamma \to j_*^\Gamma j^\Gamma V \to Q \to 0$. We must show that $Q = 0$. The first map becomes an isomorphism when we apply j^γ, so $j^\gamma Q = 0$. It remains to show that Q has a $\bar{\nabla}$-flag with $\operatorname{supp}_{\bar{\nabla}}(Q) \subseteq \Gamma$. This follows from Corollary 8.11 and the obvious analog of (8.10) because V/V^Γ has a $\bar{\nabla}$-flag by assumption and $j_*^\Gamma j^\Gamma V \cong j_*^\Gamma j^\Gamma (V/V^\Gamma)$ has a $\bar{\nabla}$-flag with the appropriate support by Lemma 9.4. □

The final lemma is the dual version of Lemma 9.3 promised earlier.

Lemma 9.8. *For $V \in \operatorname{ob} A_\Gamma\text{-gmod}_{\bar{\Delta}}$ and $W \in \operatorname{ob} A\text{-gmod}$, we have that $\operatorname{Ext}^n_{A_\Gamma}\left(V, j^\Gamma W\right) \cong \operatorname{Ext}^n_A\left(j_!^\Gamma V, W\right)$ for all $n \geqslant 0$.*

Proof. We have that $\operatorname{Hom}_A(-, W) \circ j_!^\Gamma \cong \operatorname{Hom}_{A_\Gamma}(-, j^\Gamma W)$. Also $j_!^\Gamma$ takes projectives to projectives since it is left adjoint to an exact functor. Therefore, by the usual argument, we have that $\operatorname{Ext}^n_A(j_!^\Gamma V, W) \cong \operatorname{Ext}^n_{A_\Gamma}(V, j^\Gamma W)$ for all $n \geqslant 0$ and $V \in A_\Gamma\text{-gmod}$ such that $(\mathbb{L}_m j_!^\Gamma)V = \operatorname{Tor}_m^{A_\Gamma}(Ae_\Gamma, V) = 0$ for $m \geqslant 1$. This holds for $V \in A_\Gamma\text{-gmod}_{\bar{\Delta}}$ by Lemma 9.5. □

10. Semi-infinite flags

When the algebra A (still possessing a graded triangular basis) is not unital, it also makes sense to consider certain semi-infinite Δ-flags, $\bar{\Delta}$-flags, ∇-flags and $\bar{\nabla}$-flags. These were introduced in [BS18, Def. 3.35] in the ungraded setting, and then they were there used to introduce tilting modules. In this section, we make some first steps in this direction in the graded setting by setting up the basic facts about semi-infinite flags. Throughout the section, we will make use of the notation from the previous section for a finite lower set $\Gamma \subseteq \Lambda$, especially (9.4).

Definition 10.1. We say that a graded left A-module V has an *ascending Δ-flag* (resp., an *ascending $\bar{\Delta}$-flag*) if the A-submodule V_Γ has a Δ-flag (resp., a $\bar{\Delta}$-flag) for all finite lower sets $\Gamma \subseteq \Lambda$.

Definition 10.2. We say that a graded left A-module V has a *descending ∇-flag* (resp., a *descending $\bar{\nabla}$-flag*) if the quotient module V/V^Γ has a ∇-flag (resp., a $\bar{\nabla}$-flag) for all finite lower sets $\Gamma \subseteq \Lambda$.

Our first lemma shows that in order to check the conditions in Definitions 10.1 and 10.2, it suffices just to consider finite lower sets $\Gamma \subseteq \Lambda$ that are sufficiently large. In particular, if A is unital (i.e., $\{\lambda \in \Lambda \mid e_\lambda \neq 0\}$ is finite), we deduce that V has an ascending Δ-flag if and only if V has a Δ-flag in the earlier sense, and similarly for $\bar{\Delta}$-flags, ∇-flags and $\bar{\nabla}$-flags. So these new notions are only interesting in the non-unital case.

Lemma 10.3. *Let $\Gamma \subseteq \Pi$ be two finite lower sets in Λ and $V \in \operatorname{ob} A\text{-gmod}$.*

(1) If V_Π has a Δ-flag (resp., a $\bar{\Delta}$-flag) then so do V_Γ and V_Π/V_Γ.
(2) If V/V^Π has a ∇-flag (resp., a $\bar{\nabla}$-flag) then so do V/V^Γ and V^Γ/V^Π.

Proof. We just go through the details for Δ-flags, the other cases are similar. Since $e_\Gamma = e_\Pi e_\Gamma = e_\Gamma e_\Pi$, we have that $V_\Gamma \subseteq V_\Pi$. We are given that V_Π has a Δ-flag. Clearly its sections are Δ-layers of types from Π. Using Corollary 6.7, we can arrange the layers to obtain a short exact sequence $0 \to K \to V_\Pi \to Q \to 0$ so that K has a Δ-flag with Δ-layers of types from Γ and Q has a Δ-flag with layers from $\Pi - \Gamma$.

But e_Γ is zero on Δ-layers of types from $\Pi - \Gamma$, and any Δ-layer W of type from Γ is generated by $e_\Gamma W$. It follows that $K = V_\Gamma$, $Q = V_\Pi / V_\Gamma$, so both have Δ-flags. □

For V with an ascending Δ-flag or an ascending $\bar{\Delta}$-flag, we define the multiplicities $(V : \Delta(b))_q$ and $(V : \bar{\Delta}(b))_q$ by the same formulae (8.3) and (8.6) as before. They both belong to $\mathbb{N}(\!(q^{-1})\!)$ thanks to the next lemma. Similarly, we define $(V : \nabla(b))_q$ and $(V : \bar{\nabla}(b))_q$ for V with a descending ∇-flag or a descending $\bar{\nabla}$-flag by (8.13) and (8.14); these necessarily belong to $\mathbb{N}(\!(q)\!)$.

Lemma 10.4. *Let V be a graded left A-module.*

(1) If V has an ascending Δ-flag (resp., an ascending $\bar{\Delta}$-flag) then V is locally finite-dimensional and bounded below.
(2) If V has a descending ∇-flag (resp., a descending $\bar{\nabla}$-flag) then V is locally finite-dimensional and bounded above.

Proof. (1) It suffices to prove that V is locally finite-dimensional and bounded below assuming if it has an ascending $\bar{\Delta}$-flag. Fix a choice of $i \in \mathbf{I}$. If $1_i(j_!^\lambda \bar{V}) \neq 0$ for some $\lambda \in \Lambda$ and a graded left A_λ-module that is locally finite-dimensional and bounded below, then by (4.7) $\bar{x} \otimes v \neq 0$ for some $x \in \mathrm{X}(i, s)$, $v \in 1_s \bar{v}$ and $s \in \mathbf{S}_\lambda$. By the final axiom in Definition 1.1, there are only finitely many possibilities for λ. Let Γ be the finite lower set in Λ generated by all of them. Then we have proved that $1_i V = 1_i V_\Gamma$. Since V_Γ has a $\bar{\Delta}$-flag, it is locally finite-dimensional and bounded below by Lemma 4.1. Hence, so is V.

(2) This follows by the dual argument. □

Now we are ready for the main results of the section. These are almost the same as Theorems 8.3 and 8.8, it is just that the conditions on finite support have been removed.

Theorem 10.5 (Homological criteria for ascending Δ- and $\bar{\Delta}$-flags)**.** *Assume that $V \in \mathrm{ob}\, A\text{-gmod}$ is locally finite-dimensional and bounded below.*

(1) The following are equivalent:
  (a) V has an ascending Δ-flag;
  (b) $j^\Gamma V$ has a Δ-flag for all finite lower sets $\Gamma \subseteq \Lambda$;
  (c) $\mathrm{Ext}^1_A(V, \bar{\nabla}(b)) = 0$ for all $b \in \mathbf{B}$;
  (d) $\mathrm{Ext}^n_A(V, \bar{\nabla}(b)) = 0$ for all $b \in \mathbf{B}$ and $n \geqslant 1$.
  When this holds, for any finite lower set $\Gamma \subseteq \Lambda$, both V_Γ and V/V_Γ have ascending Δ-flags with

$$(V_\Gamma : \Delta(b))_q = \begin{cases} (V : \Delta(b))_q & \text{if } b \in \mathbf{B}_\Gamma \\ 0 & \text{otherwise;} \end{cases} \qquad (V/V_\Gamma : \Delta(b))_q = \begin{cases} 0 & \text{if } b \in \mathbf{B}_\Gamma \\ (V : \Delta(b))_q & \text{otherwise.} \end{cases} \tag{10.1}$$

(2) The following are equivalent:
  (a) V has an ascending $\bar{\Delta}$-flag;
  (b) $j^\Gamma V$ has a $\bar{\Delta}$-flag for all finite lower sets $\Gamma \subseteq \Lambda$;
  (c) $\mathrm{Ext}^1_A(V, \nabla(b)) = 0$ for all $b \in \mathbf{B}$;
  (d) $\mathrm{Ext}^n_A(V, \nabla(b)) = 0$ for all $b \in \mathbf{B}$ and $n \geqslant 1$.
  When this holds, for any finite lower set $\Gamma \subseteq \Lambda$, both V_Γ and V/V_Γ have ascending $\bar{\Delta}$-flags with

$$(V_\Gamma : \bar{\Delta}(b))_q = \begin{cases} (V : \bar{\Delta}(b))_q & \text{if } b \in \mathbf{B}_\Gamma \\ 0 & \text{otherwise;} \end{cases} \qquad (V/V_\Gamma : \bar{\Delta}(b))_q = \begin{cases} 0 & \text{if } b \in \mathbf{B}_\Gamma \\ (V : \bar{\Delta}(b))_q & \text{otherwise.} \end{cases} \tag{10.2}$$

Proof. (1) It is clear that (d)⇒(c).

To prove that (a)⇒(d), the canonical map $\varinjlim_\Gamma V_\Gamma \to V$ is an isomorphism, where the direct limit is over all finite lower sets $\Gamma \subset \Lambda$ with maps given by the natural inclusions. This follows because V is generated by all of its weight spaces $e_\lambda V$ ($\lambda \in \Lambda$), and the poset is lower finite so every weight space is a subset of V_Γ for some finite lower set Γ. So

$$\mathrm{Ext}^n_A(V, \bar\nabla(b)) \cong \mathrm{Ext}^n_A\left(\varinjlim_{\Gamma} V_\Gamma, \bar\nabla(b)\right) \cong \varprojlim_{\Gamma} \mathrm{Ext}^n_A\left(V_\Gamma, \bar\nabla(b)\right).$$

This is 0 for $n \geqslant 1$ thanks to Theorem 7.5 as each V_Γ has a Δ-flag by the definition of ascending Δ-flag.

In this paragraph, we prove that (c)⇒(b). Take a finite lower set Γ. Note that $j^\Gamma V$ is locally finite-dimensional and bounded below since V has these properties. Also for $b \in \mathbf{B}_\Gamma$, we have that $\mathrm{Ext}^1_{A_\Gamma}(j^\Gamma V, \bar\Delta_\Gamma(b)) \cong \mathrm{Ext}^1_A(V, j^\Gamma_* \bar\Delta_\Gamma(b))$ by Lemma 9.3. Since $j^\Gamma_*\bar\Delta_\Gamma(b) \cong \bar\Delta(b)$ by Lemma 9.1, the assumed property (3) gives that $\mathrm{Ext}^1_{A_\Gamma}(j^\Gamma V, \bar\Delta_\Gamma(b)) = 0$. Now we can apply Theorem 8.3 (using that Γ is finite so the support condition is automatic) to establish (2).

For (b)⇒(a), assume that (b) holds. Lemma 9.4 implies that $j^\Gamma_! j^\Gamma V$ has a Δ-flag. We claim that the counit of adjunction $\varepsilon^\Gamma_V : j^\Gamma_! j^\Gamma V \to V$ is injective. Given this, the image of ε^Γ_V is V_Γ, so we deduce that V_Γ has a Δ-flag, as needed to prove (a). Suppose for a contradiction that ε^Γ_V is not injective. Then we can find $\lambda \in \Lambda$ such that the restriction of ε^Γ_V to the λ-weight space is not injective. Let Π be the finite lower set generated by Γ and λ. Consider the following diagram:

$$\begin{tikzcd} j^\Pi_! j^\Pi V \arrow[r, "\varepsilon^\Pi_V"] & V \\ j^\Gamma_! j^\Gamma\left(j^\Pi_! j^\Pi V\right) \arrow[u, hook, "\varepsilon^\Gamma_{j^\Pi_! j^\Pi V}"] \arrow[r, "\sim"] & j^\Gamma_! j^\Gamma V \arrow[u, "\varepsilon^\Gamma_V"'] \end{tikzcd}$$

The bottom map is

$$Ae_\Gamma \otimes_{A_\Gamma} e_\Gamma A e_\Pi \otimes_{A_\Pi} e_\Pi V \to Ae_\Gamma \otimes_{A_\Gamma} e_\Gamma V, \qquad ae_\Gamma \otimes e_\Gamma b e_\Pi \otimes e_\Pi v \mapsto ae_\Gamma \otimes e_\Gamma b e_\Pi v,$$

which is clearly an isomorphism. The left hand map is

$$Ae_\Gamma \otimes_{A_\Gamma} e_\Gamma A e_\Pi \otimes_{A_\Pi} e_\Pi V \to Ae_\Pi \otimes_{A_\Pi} e_\Pi V, \qquad ae_\Gamma \otimes e_\Gamma b e_\Pi \otimes e_\Pi v \mapsto ae_\Gamma b e_\Pi \otimes e_\Pi v.$$

This map is injective. To see this, let $W := j^\Pi_! j^\Pi V$. We know it has a $\bar\Delta$-flag, so by Lemma 10.3(1) we deduce that W_Γ has a $\bar\Delta$-flag too. Hence, by Theorem 9.7(1), $\varepsilon^\Gamma_W : j^\Gamma_! j^\Gamma W \to W_\Gamma$ is an isomorphism. The top and right hand maps in the diagram are the natural multiplication maps, and it is easily checked that the diagram commutes. Finally, the top map becomes an isomorphism when we apply j^Π, hence, it is a bijection on λ-weight spaces. It follows that $\varepsilon^\Pi_V \circ \varepsilon^\Gamma_W$ is injective on the λ-weight space. Hence, ε^Γ_V is injective on the λ-weight space, contradicting the earlier assumption.

Finally, we assume (a) and deduce (10.1). Let $\Gamma \subseteq \Lambda$ be a finite lower set. We have that V_Γ has a Δ-flag by the definition. Also V/V_Γ has an ascending Δ-flag, as follows directly from the definition using Lemma 10.3(1). Now to establish (10.1), one just has to apply $\mathrm{Hom}_A(-, \bar\nabla(b))$ to the short exact sequence $0 \to V_\Gamma \to V \to V/V_\Gamma \to 0$. Since $\mathrm{Ext}^1_A(V/V_\Gamma, \bar\nabla(b)) = 0$, one obtains a short exact sequence showing that

$$(V : \Delta(b))_q = (V_\Gamma : \Delta(b))_q + (V/V_\Gamma : \Delta(b))_q.$$

Defining supports as in (8.1) and (8.2), we also have that $\mathrm{supp}_\Delta(V_\Gamma) \subseteq \Gamma$, e.g., this follows from Lemma 9.4 because $V_\Gamma \cong j^\Gamma_! j^\Gamma V$ by Theorem 9.7(1) and $j^\Gamma V \in A_\Gamma\text{-gmod}_\Delta$. Also $\mathrm{supp}_\Delta(V/V_\Gamma) \subseteq \Lambda - \Gamma$ since it has no weights belonging to Γ. Now (10.1) is clear.

(2) Similar. ∎

Corollary 10.6. *Suppose that $0 \to U \to V \to W \to 0$ is a short exact sequence of graded left A-modules. Assuming that W has an ascending Δ-flag, U has an ascending Δ-flag if and only if V has an ascending Δ-flag. Similarly for $\bar\Delta$-flags.*

Finally, we state the dual results which, as usual, follow from by dualizing the above.

Theorem 10.7 (Homological criteria for descending ∇- and $\bar{\nabla}$-flags)**.** *Assume that $V \in \operatorname{ob} A\text{-gmod}$ is locally finite-dimensional and bounded above.*

(1) The following are equivalent:
  (a) V has an descending ∇-flag;
  (b) $j^\Gamma V$ has a ∇-flag for all finite lower sets $\Gamma \subseteq \Lambda$;
  (c) $\operatorname{Ext}^1_A(\bar{\Delta}(b), V) = 0$ for all $b \in \mathbf{B}$;
  (d) $\operatorname{Ext}^n_A(\bar{\Delta}(b), V) = 0$ for all $b \in \mathbf{B}$ and $n \geqslant 1$.

When this holds, for any finite lower set $\Gamma \subseteq \Lambda$, both V/V^Γ and V^Γ have ascending ∇-flags with

$$(V/V^\Gamma : \nabla(b))_q = \begin{cases} (V : \nabla(b))_q & \text{if } b \in \mathbf{B}_\Gamma \\ 0 & \text{otherwise;} \end{cases} \qquad (V^\Gamma : \nabla(b))_q = \begin{cases} 0 & \text{if } b \in \mathbf{B}_\Gamma \\ (V : \nabla(b))_q & \text{otherwise.} \end{cases} \tag{10.3}$$

(2) The following are equivalent:
  (a) V has an descending $\bar{\nabla}$-flag;
  (b) $j^\Gamma V$ has a $\bar{\nabla}$-flag for all finite lower sets $\Gamma \subseteq \Lambda$;
  (c) $\operatorname{Ext}^1_A(\Delta(b), V) = 0$ for all $b \in \mathbf{B}$;
  (d) $\operatorname{Ext}^n_A(\Delta(b), V) = 0$ for all $b \in \mathbf{B}$ and $n \geqslant 1$.

When this holds, for any finite lower set $\Gamma \subseteq \Lambda$, both V/V^Γ and V^Γ have ascending $\bar{\nabla}$-flags with

$$(V/V^\Gamma : \bar{\nabla}(b))_q = \begin{cases} (V : \bar{\nabla}(b))_q & \text{if } b \in \mathbf{B}_\Gamma \\ 0 & \text{otherwise;} \end{cases} \qquad (V^\Gamma : \bar{\nabla}(b))_q = \begin{cases} 0 & \text{if } b \in \mathbf{B}_\Gamma \\ (V : \bar{\nabla}(b))_q & \text{otherwise.} \end{cases} \tag{10.4}$$

Corollary 10.8. *Suppose that $0 \to U \to V \to W \to 0$ is a short exact sequence of graded left A-modules. Assuming that U has an ascending ∇-flag, V has an ascending ∇-flag if and only if W has an ascending ∇-flag. Similarly for $\bar{\nabla}$-flags.*

11. Homological dimensions

In this section, with give some applications to homological dimensions. Often these require some Noetherian assumptions (something we have sought to avoid up until now). Continue with A having a graded triangular basis. We say that A is *locally left* (resp., *right*) *graded Notherian* if each finitely generated projective graded left (resp., right) A-module has the descending chain condition (DCC) on graded submodules. Since A is locally finite-dimensional, this is obviously equivalent by duality to each finitely cogenerated injective graded right (resp., left) A-module having ACC. If A is both locally left and locally right graded Noetherian, then its (possibly infinite) left and right graded global dimensions coincide, and we refer to them both just as the graded global dimension of A. Without this assumption, one must talk about the left and right graded global dimensions of A separately. This is the same as for ordinary (graded) algebras, e.g., see [Wei94, Ch. 4].

Lemma 11.1. *For $\lambda \in \Lambda$, let $\ell(\lambda)$ be the maximal length of a descending chain $\lambda = \lambda_0 > \lambda_1 > \cdots > \lambda_\ell$ in the poset Λ. For any $b \in \mathbf{B}_\lambda$, the graded projective (resp., injective) dimension of a Δ-layer (resp., a ∇-layer) of type λ is $\leqslant \ell(\lambda)$.*

Proof. We just explain for Δ-layers; the argument for ∇-layers is similar. By [Wei94, Ex. 4.1.3(1)], it suffices to show that the graded projective dimension of $\Delta(b)$ is $\leqslant \ell(\lambda)$ for $b \in \mathbf{B}_\lambda$. We prove this by induction on $\ell(\lambda)$. If $\ell(\lambda) = 0$ then $\Delta(b)$ is projective by Corollary 6.2, giving the induction base. Now suppose that $\ell(\lambda) > 0$. By Corollary 8.4, there is a short exact sequence $0 \to K \to P(b) \to \Delta(b) \to 0$ such that K has a Δ-flag with sections that are Δ-layers of types μ with $\ell(\mu) < \ell(\lambda)$. By [Wei94, Ex. 4.1.2(1)] and the induction hypothesis, it follows that the graded projective dimension of K is $< \ell(\lambda)$. Another application of [Wei94, Ex. 4.1.2(1)] shows that the graded projective dimension of $\Delta(b)$ is at most one more than that of K. Hence, the graded projective dimension of $\Delta(b)$ is $\leqslant \ell(\lambda)$. $\square$

Lemma 11.2. *Suppose that we are given $\lambda \in \Lambda$ such that A_λ has finite left (resp., right) graded global dimension $d(\lambda)$. Then any $\bar{\Delta}$-layer (resp., $\bar{\nabla}$-layer) of type λ has finite graded projective (resp., injective) dimension that is $\leqslant \ell(\lambda) + d(\lambda)$.*

Proof. We go through the argument for a $\bar{\Delta}$-layer $V = j_!^\lambda \bar{V}$ of type λ. Since $\bar{V}$ is locally finite-dimensional and bounded below, Lemma 2.4(1) implies that it has a projective cover P_0 in A_λ-gmod which is again locally finite-dimensional and bounded below. It follows that the kernel of $P_0 \twoheadrightarrow \bar{V}$ is locally finite-dimensional and bounded below. Repeating the argument, we end up with a minimal graded projective resolution of $\bar{V}$ of the form $0 \to P_n \to \cdots \to P_0 \to \bar{V} \to 0$ for $n \leqslant d(\lambda)$ and each P_r being a projective graded module that is locally finite-dimensional and bounded below. Then we apply $j_!^\lambda$ to obtain an exact sequence $0 \to j_!^\lambda P_n \to \cdots \to j_!^\lambda P_0 \to V \to 0$ with $n \leqslant d(\lambda)$ and each $j_!^\lambda P_r$ being Δ-layer of type λ. We deduce that V is of finite graded projective dimension $\leqslant \ell(\lambda) + d(\lambda)$ using Lemma 11.1. □

Lemma 11.3. *If A is locally left (resp., right) graded Noetherian then all A_λ $(\lambda \in \Lambda)$ are left (resp., right) graded Noetherian.*

Proof. Assume that A is locally left graded Noetherian. Take $b \in \mathbf{B}_\lambda$ and a descending chain $P_\lambda(b) = P_0 \supseteq P_1 \supseteq \cdots$ of graded submodules. Apply the exact functor $j_!^\lambda$ to get a descending chain of graded submodules of the standard module $\Delta(b)$. Since A is locally left graded Noetherian and this module is finitely generated, it follows that the chain stabilizes. Then apply j using $j \circ j_!^\lambda \cong \mathrm{id}_{A_\lambda\text{-gmod}}$ to deduce that the original chain stabilizes too. This proves that A_λ is left graded Noetherian. A similar argument starting with an ascending chain of graded submodules of $I_\lambda(b)$ and using $j \circ j_*^\lambda \cong \mathrm{id}_{A_\lambda\text{-gmod}}$ proves that A_λ is right graded Notherian when A has this property. □

Lemma 11.4. *Assume that A is locally left (resp., right) graded Noetherian. Then $(P(b) : \Delta(a))_q$ and $[\bar{\nabla}(a) : L(b)]_q$ (resp., $(I(b) : \nabla(a))_q$ and $[\bar{\Delta}(a) : L(b)]_q$) are Laurent polynomials in $\mathbb{N}[q, q^{-1}]$ for all $a, b \in \mathbf{B}$.*

Proof. We just explain for the case of left Noetherian. If $(P(b) : \Delta(a))_q = \overline{[\bar{\nabla}(a) : L(b)]_q}$ is *not* in $\mathbb{N}[q, q^{-1}]$ for some $a, b \in \mathbf{B}$ then Corollary 8.4 implies that there is a Δ-flag $P(b) = P_0 \supseteq P_1 \supseteq \cdots \supseteq P_n = 0$ such that for some r the section P_{r-1}/P_r is an infinite direct sum of degree-shifted standard modules. This implies that P_{r-1} is not finitely generated, hence, $P(b)$ is not graded Noetherian, contradicting the assumption that A is locally left graded Noetherian. □

Corollary 11.5. *If A is unital and locally left (resp., right) graded Noetherian, then all of the proper standard modules $\bar{\Delta}(b)$ (resp., the proper costandard modules $\bar{\nabla}(b)$) are of finite length.*

Theorem 11.6. *Assume A is unital, both left and right graded Noetherian, and that each A_λ $(\lambda \in \Lambda)$ has finite graded global dimension. Then A has finite graded global dimension.*

Proof. It suffices to show that A has finite left graded global dimension. By [Wei94, Th. 4.1.2(3)], we need to show that there is $N \in \mathbb{N}$ such that $\mathrm{Ext}^n_A(V, W) = 0$ for $n > N$, all finitely generated graded left A-modules V and arbitrary graded left A-modules W. In fact, we may also assume that W is finitely generated. To prove this, we use that A is graded left Noetherian to construct a graded projective resolution $\cdots \to P_{n+1} \xrightarrow{\partial_n} P_n \xrightarrow{\partial_{n-1}} P_{n-1} \to \cdots \to P_0 \to V \to 0$ all of whose terms are finitely generated. Any element of $\mathrm{Ext}^n_A(V, W)$ is represented by a homomorphism $f : P_n \to W$ such that $f \circ \partial_n = 0$. The image of f is a finitely generated submodule W' of W. If we know $\mathrm{Ext}^n_A(V, W') = 0$, then $f = g \circ \partial_{n-1}$ for some $g : P_{n-1} \to W'$, and we deduce that the image of f in $\mathrm{Ext}^n_A(V, W)$ is zero, hence, $\mathrm{Ext}^n_A(V, W) = 0$. So now we have reduced the problem to showing that there exists $N \in \mathbb{N}$ such that $\mathrm{Ext}^n_A(V, W) = 0$ for $n > N$ and all finitely generated graded left A-modules V and W. By [BKM14, Lem. 1.1], the proof reduces further to checking this statement just for all *irreducible* W.

Thus, the proof has been reduced to showing that all of the irreducible modules $L(b)$ $(b \in \mathbf{B})$ have finite graded injective dimension. Replacing Λ by $\{\dot{b} \mid b \in \mathbf{B}\}$, we may assume that the poset Λ is finite, and proceed by downward induction on this poset. Take any $b \in \mathbf{B}$ and consider the short exact sequence $0 \to L(b) \to \bar{\nabla}(b) \to Q \to 0$. By Corollary 11.5, Q is of finite length. Moreover, all of its composition factors are degree shifts of $L(c)$ for $c \in \mathbf{B}$ with $\dot{c} > \dot{b}$. By induction, they are all of finite injective dimension, hence, Q is of finite injective dimension. Also A_λ is graded right Noetherian by Lemma 11.3, so $\bar{\nabla}(b)$ is of finite injective dimension by Lemma 11.2. It follows that $L(b)$ has finite graded injective dimension. □

12. Refinement

What happens if the algebras A_λ have additional structure? The results in this section address this question in the situation that each A_λ is itself a based affine quasi-hereditary algebra in the sense of Remark 4.2. This means that we are given partial orders $\leqslant_\lambda$ on $\mathbf{B}_\lambda$ and "local" graded triangular bases for each $\lambda \in \Lambda$ making the unital graded algebras A_λ into based affine quasi-hereditary algebras with respect to the posets $(\mathbf{B}_\lambda, \leqslant_\lambda)$. Then we can define a refined partial order $\leqslant$ on $\mathbf{B}$ by

$$b \leqslant c \qquad \Leftrightarrow \qquad \dot{b} < \dot{c} \text{ or } (\lambda := \dot{b} = \dot{c} \text{ and } b \leqslant_\lambda c). \tag{12.1}$$

One might hope to be able to assemble the various local triangular bases into a new global triangular basis making A into a based affine quasi-hereditary algebra with weight poset $(\mathbf{B}, \leqslant)$. Unfortunately this seems to be difficult to do directly. However, it is still possible to prove the representation theoretical consequences of the existence of such a basis.

So assume from now on that we are given a graded triangular basis for A as usual, and additional partial orders $\leqslant_\lambda$ on each of the finite sets $\mathbf{B}_\lambda$ $(\lambda \in \Lambda)$. Define $\leqslant$ on $\mathbf{B}$ as in (12.1). For each $\lambda \in \Lambda$, we assume that A_λ has some extra structure making it into a based affine quasi-hereditary algebra with respect to the poset $(\mathbf{B}_\lambda, \leqslant_\lambda)$. We will never refer explicitly to these bases, rather, we will work with them implicitly in terms of the consequences of the existence of these bases for the categories A_λ-gmod. We denote the various families of graded modules for A_λ arising from the extra structure by $P_\lambda(b), \Delta_\lambda(b), \bar{\Delta}_\lambda(b), L_\lambda(b), \bar{\nabla}_\lambda(b), \nabla_\lambda(b)$ and $I_\lambda(b)$, all for $b \in \mathbf{B}_\lambda$. There are corresponding notions of Δ-layers, $\bar{\Delta}$-layers, Δ-flags, $\bar{\Delta}$-flags, etc. for A_λ-modules, which we will call Δ_λ-layers, $\bar{\Delta}_\lambda$-layers, Δ_λ-flags, $\bar{\Delta}_\lambda$-flags, etc. for extra clarity. As well the usual A-modules $\Delta(\lambda), \bar{\Delta}(\lambda), \bar{\nabla}(\lambda)$ and $\nabla(\lambda)$ defined as in (4.9), we also have

$$\blacktriangle(b) := j_!^\lambda \Delta_\lambda(b), \qquad \bar{\blacktriangle}(b) := j_!^\lambda \bar{\Delta}_\lambda(b), \qquad \bar{\blacktriangledown}(b) := j_*^\lambda \bar{\nabla}_\lambda(b), \qquad \blacktriangledown(b) := j_*^\lambda \nabla_\lambda(b) \tag{12.2}$$

for $b \in \mathbf{B}_\lambda$. We call these the *pure standard*, *pure proper standard*, *pure costandard* and *pure proper costandard modules*, respectively. Note by the definitions that $P(b) \twoheadrightarrow \Delta(b) \twoheadrightarrow \blacktriangle(b) \twoheadrightarrow \bar{\blacktriangle}(b) \twoheadrightarrow \bar{\Delta}(b) \twoheadrightarrow L(b) \hookrightarrow \bar{\nabla}(b) \hookrightarrow \bar{\blacktriangledown}(b) \hookrightarrow \blacktriangledown(b) \hookrightarrow \nabla(b) \hookrightarrow I(b)$.

Lemma 12.1. *For $b, c \in \mathbf{B}$, $f \in \mathbb{N}(\!(q^{-1})\!)$ and $g \in \mathbb{N}(\!(q)\!)$, we have that*

$$\dim_q \operatorname{Hom}_A\left(\blacktriangle(b)^{\oplus f}, \bar{\blacktriangledown}(c)^{\oplus g}\right) = \dim_q \operatorname{Hom}_A\left(\bar{\blacktriangle}(b)^{\oplus f}, \blacktriangledown(c)^{\oplus g}\right) = \delta_{b,c}\, \overline{f}\, g \in \mathbb{N}(\!(q)\!).$$

Proof. We just explain for the first space. Since $\blacktriangle(b)$ has lowest weight $\dot{b}$ and $\bar{\blacktriangledown}(c)$ has lowest weight $\dot{c}$, the space is zero unless $\lambda := \dot{b} = \dot{c}$. Assuming this, we have that

$$\begin{aligned} \operatorname{Hom}_A(\blacktriangle(b)^{\oplus f}, \bar{\blacktriangledown}(c)^{\oplus g}) &= \operatorname{Hom}_{A_{\geqslant\lambda}}(\blacktriangle(b)^{\oplus f}, \bar{\blacktriangledown}(c)^{\oplus g}) = \operatorname{Hom}_{A_{\geqslant\lambda}}(j_!^\lambda \Delta_\lambda(b)^{\oplus f}, j_*^\lambda \bar{\nabla}_\lambda(c)^{\oplus g}) \\ &\cong \operatorname{Hom}_{A_\lambda}(\Delta_\lambda(b)^{\oplus f}, j^\lambda j_*^\lambda \bar{\nabla}_\lambda(b)^{\oplus g}) = \operatorname{Hom}_{A_\lambda}(\Delta_\lambda(b)^{\oplus f}, \bar{\nabla}_\lambda(b)^{\oplus g}), \end{aligned}$$

which is of graded dimension $\delta_{b,c}\overline{f}g$ by Corollary 4.6. □

Definition 12.2. By a $\blacktriangle$-*layer* (resp., a $\bar{\blacktriangle}$-*layer*) of type $b \in \mathbf{B}_\lambda$, we mean a graded A-module that is isomorphic to $j_!^\lambda \bar{V}$ for a graded left A_λ-module $\bar{V}$ which is a Δ_λ-layer (resp., a $\bar{\Delta}_\lambda$-layer) of type b. We say that $V \in \operatorname{ob} A\text{-gmod}$ has a $\blacktriangle$-*flag* (resp., a $\bar{\blacktriangle}$-*flag*) if for some $n \geqslant 0$ there is a graded filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

and distinct $b_1, \dots, b_n \in \mathbf{B}$ such that V_r/V_{r-1} is a $\blacktriangle$-layer (resp., a $\bar{\blacktriangle}$-layer) of type b_r for each $r = 1, \dots, n$. We say that V has an *ascending* $\blacktriangle$-*flag* (resp., an *ascending* $\bar{\blacktriangle}$-*flag*) if the A-submodule V_Γ defined in (9.4) has a $\blacktriangle$-flag (resp., a $\bar{\blacktriangle}$-flag) for all finite lower sets $\Gamma \subseteq \Lambda$.

Definition 12.3. By a $\blacktriangledown$-*layer* (resp., a $\bar{\blacktriangledown}$-*layer*) of type $b \in \mathbf{B}_\lambda$, we mean a graded A-module that is isomorphic to $j_*^\lambda \bar{V}$ for a graded left A_λ-module $\bar{V}$ which is a ∇_λ-layer (resp., a $\bar{\nabla}_\lambda$-layer) of type b. We say that $V \in \operatorname{ob} A\text{-gmod}$ has a $\blacktriangledown$-*flag* (resp., a $\bar{\blacktriangledown}$-*flag*) if for some $n \geqslant 0$ there is a graded filtration

$$V = V_0 \supset V_1 \supset \cdots \supset V_n = 0$$

and distinct $b_1, \dots, b_n \in \mathbf{B}$ such that V_{r-1}/V_r is a $\blacktriangledown$-layer (resp., a $\bar{\blacktriangledown}$-layer) of type b_r for each $r = 1, \dots, n$. We say that V has an *ascending* $\blacktriangledown$-*flag* (resp., an *ascending* $\bar{\blacktriangledown}$-*flag*) if the quotient module V/V^Γ defined in (9.4) has a $\blacktriangledown$-flag (resp., a $\bar{\blacktriangledown}$-flag) for all finite lower sets $\Gamma \subseteq \Lambda$.

Remark 12.4. A $\blacktriangle$-layer of type b means just the same thing as a direct sum $\blacktriangle(b)^{\oplus f}$ for $f \in \mathbb{N}(\!(q^{-1})\!)$. Similarly, a $\blacktriangledown$-layer of type b is a direct sum $\blacktriangledown(b)^{\oplus g}$ for $g \in \mathbb{N}(\!(q)\!)$.

The full subcategory of A-gmod consisting of modules with $\blacktriangle$-flags (resp., $\bar{\blacktriangle}$-flags, $\blacktriangledown$-flags, $\bar{\blacktriangledown}$-flags) will be denoted $A\text{-gmod}_\blacktriangle$ (resp. $A\text{-gmod}_{\bar{\blacktriangle}}$, $A\text{-gmod}_\blacktriangledown$, $A\text{-gmod}_{\bar{\blacktriangledown}}$). Since Definitions 12.2 and 12.3 are dual to each other, from now on, we will explain results just in the case of $\blacktriangle$- and $\bar{\blacktriangle}$-flags, leaving the dual statements for $\blacktriangledown$- and $\bar{\blacktriangledown}$-flags to the reader.

Noting that $\blacktriangle$-layers are $\bar{\blacktriangle}$-layers, $A\text{-gmod}_\blacktriangle$ is a subcategory of $A\text{-gmod}_{\bar{\blacktriangle}}$. It is also the case that $A\text{-gmod}_\Delta$ is a subcategory of $A\text{-gmod}_\blacktriangle$ and $A\text{-gmod}_{\bar{\blacktriangle}}$ is a subcategory of $A\text{-gmod}_{\bar{\Delta}}$. These statements are not quite obvious; they are justified by the corollary appearing after the next lemma.

Lemma 12.5. *In either of the following situations, we have that* $\operatorname{Ext}^1_A(V, W) = 0$*:*

(1) V is a $\bar{\blacktriangle}$-layer of type b and W is a $\bar{\blacktriangle}$-layer of type c for $b \not\geqslant c$;
(2) V is a $\blacktriangle$-layer of type b and W is a $\blacktriangle$-layer of type c for $b \not> c$.

Proof. (1) Suppose that $V = j_!^\lambda \bar{V}$ for a $\bar{\Delta}_\lambda$-layer $\bar{V}$ of type $b \in \mathbf{B}_\lambda$ and $W = j_!^\mu \bar{V}$ for a $\bar{\Delta}_\mu$-layer $\bar{V}$ of type $c \in \mathbf{B}_\mu$. The hypothesis that $b \not\geqslant c$ means either that $\lambda \not\geqslant \mu$, or $\lambda = \mu$ and $b \not\geqslant_\lambda c$. Since $\bar{\blacktriangle}$-layers are $\bar{\Delta}$-layers, Lemma 6.6 gives $\operatorname{Ext}^1_A(V, W) = 0$ if $\lambda \neq \mu$. Now suppose that $\lambda = \mu$. We have that $\operatorname{Ext}^1_A(V, W) \cong \operatorname{Ext}^1_{A_{\geqslant\lambda}}(V, W)$ which, is isomorphic to $\operatorname{Ext}^1_{A_\lambda}(\bar{V}, \bar{W})$ by Lemma 9.8. As $b \not\geqslant_\lambda c$, this is zero thanks to Lemma 6.6 in A_λ-gmod.

(2) This is similar to (1) using Remark 6.8 in place of Lemma 6.6. □

Corollary 12.6. *If V has a Δ-flag (resp., a $\bar{\blacktriangle}$-flag) then it has a $\blacktriangle$-flag (resp., a $\bar{\Delta}$-flag).*

Proof. First suppose that V has a Δ-flag. Take $b \in \mathbf{B}_\lambda$. Applying $j_!^\lambda$ to a Δ_λ-flag for $P_\lambda(b)$ arising from Corollary 8.4, we deduce that $\Delta(b)$ has a filtration of finite length with top section $\blacktriangle(b)$ and other sections that are $\blacktriangle$-layers of types $c \in \mathbf{B}_\lambda$ with $c <_\lambda b$. Using Remark 6.5, it follows easily that any Δ-layer of type λ has a filtration of finite length with sections that are $\blacktriangle$-layers of types $c \in \mathbf{B}_\lambda$. Hence, V itself has a filtration of finite length with sections that are $\blacktriangle$-layers of types $c \in \mathbf{B}$. However, this is not yet a $\blacktriangle$-flag of V due to the requirement that $b_1, \dots, b_n$ are distinct in Definition 12.3. To fix the problem, we first use Lemma 12.5(2) to order the $\blacktriangle$-layers in some order refining the order $\leqslant$ on $\mathbf{B}$ (biggest at the top). It could still be that there are several neighboring layers of the same type, but these can be combined into a single $\blacktriangle$-layer by taking their direct sum. This uses the fact that $\mathrm{Ext}^1_A(V, W) = 0$ if V and W are $\blacktriangle$-layers of the same type, which is Lemma 12.5(2) again.

Now assume that V has a $\bar{\blacktriangle}$-flag. Since $\bar{\blacktriangle}$-layers are $\bar{\Delta}$-layers, this means that V has a finite filtration with sections that are $\bar{\Delta}$-layers of types $\lambda \in \mathbf{B}$. However this is not a $\bar{\Delta}$-flag due to the requirement that $\lambda_1, \dots, \lambda_n$ are distinct in Definition 6.3. To fix the problem, we first use Lemma 6.6 to reorder the sections if necessary. Then we have to merge neighboring $\bar{\Delta}$-layers of the same type into a single $\bar{\Delta}$-layer. This follows because if $0 \to U \to V \to W \to 0$ is an extension of two $\bar{\Delta}$-layers of type λ then $V \cong j^\lambda_! j^\lambda V$, so it is itself a $\bar{\Delta}$-layer of type λ. Indeed, the counit of adjunction gives a homomorphism $j^\lambda_! j^\lambda V \to V$. This homomorphism is an isomorphism because $j^\lambda_! \circ j^\lambda$ is exact and the counit of adjunction is an isomorphism on $U = j^\lambda_! \bar{U}$ and $W = j^\lambda_! \bar{W}$. □

Corollary 12.7. *For $b \in \mathbf{B}$, the indecomposable projective module $P(b)$ has a $\blacktriangle$-flag with top section $\blacktriangle(b)$ and other sections that are $\blacktriangle$-layers of types $a < b$.*

Proof. By Corollary 8.4, we know that $P(b)$ has a Δ-flag with top section $\Delta(b)$ and other sections that are Δ-layers of types $\mu < \dot{b}$. Also $\Delta(b)$ has a $\blacktriangle$-flag with top section $\blacktriangle(b)$ and other sections that are $\blacktriangle$-layers of types $a <_\lambda b$. This filtation can be converted into the desired $\blacktriangle$-flag by using Lemma 12.5(2) as in the proof of the previous corollary. □

Corollary 12.7 is the key property needed to upgrade other results about Δ- and $\bar{\Delta}$-flags to $\blacktriangle$- and $\bar{\blacktriangle}$-flags. To start with, the results about truncation to upper sets from section 7 carry over to the refined setting. In particular, letting $\hat{\Lambda}$ be an upper set in Λ and $\hat{A}$ be as in section 7, we have the following:

- For $V \in \mathrm{ob}\, A\text{-gmod}_{\blacktriangle}$ and $i \in \mathbf{I}$, we have that $\mathrm{Tor}^A_m(1_i\hat{A}, V) = 0$ for all $m \geqslant 1$.
- The functor $i^* = A \otimes_{\hat{A}} -$ takes short exact sequences of modules with $\blacktriangle$-flags to short exact sequences of modules with $\blacktriangle$-flags.
- For $V \in \mathrm{ob}\, A\text{-gmod}_{\blacktriangle}$ and $W \in \mathrm{ob}\, \hat{A}\text{-gmod}$, we have that $\mathrm{Ext}^n_A(V, iW) \cong \mathrm{Ext}^n_{\hat{A}}(i^*V, W)$ for all $n \geqslant 0$.

These follow by mimicking the proofs of Lemma 7.2, Corollary 7.3 and Lemma 7.4, respectively, using the $\blacktriangle$-flag of $P(b)$ from Corollary 12.7 in place of the arguments with the Δ-flag of $Q(b)$ given before.

Theorem 12.8. *In either of the following situations, we have that $\mathrm{Ext}^n_A(V, W) = 0$ for all $n \geqslant 1$:*

(1) $V \in \mathrm{ob}\, A\text{-gmod}_{\blacktriangle}$ and $W \in \mathrm{ob}\, A\text{-gmod}_{\bar{\blacktriangledown}}$;
(2) $V \in \mathrm{ob}\, A\text{-gmod}_{\bar{\blacktriangle}}$ and $W \in \mathrm{ob}\, A\text{-gmod}_{\blacktriangledown}$.

Proof. (1) The strategy is similar to the proof of Theorem 7.5. Using Remark 12.4, we reduce to checking that $\mathrm{Ext}^n_A(\blacktriangle(b), W) = 0$ for $b \in \mathbf{B}$, $n \geqslant 1$ and $W := j^\lambda_! \bar{W}$ for a $\bar{\Delta}_\lambda$-layer $\bar{W}$ of type $b \in \mathbf{B}_\lambda$. By the third point noted just before the statement of the theorem, i.e., the analog of Lemma 7.4 for $\blacktriangle$-flags, we have that

$$\mathrm{Ext}^n_A(\blacktriangle(b), W) \cong \mathrm{Ext}^n_{A_{\geqslant\lambda}}\left(i^*_{\geqslant\lambda}\blacktriangle(b), j^\lambda_! \bar{W}\right).$$

This is clearly zero if $\dot{b} \not\geqslant \lambda$. When $\dot{b} \geqslant \lambda$, we apply (6.1) to get that

$$\mathrm{Ext}^n_{A_{\geqslant\lambda}}\left(i^*_{\geqslant\lambda}\blacktriangle(b), j^\lambda_! \bar{W}\right) \cong \mathrm{Ext}^n_{A_\lambda}\left(j^\lambda\blacktriangle(b), \bar{W}\right).$$

This is clearly zero if $\dot{b} \neq \lambda$. Finally, when $\dot{b} = \lambda$, it is zero thanks to Theorem 7.5 applied in A_λ-gmod.
(2) This follows from (1) for A^{op} plus (2.6). □

Lemma 12.1 and Theorem 12.8 justify the following definitions for V with a $\blacktriangle$-flag or a $\bar{\blacktriangle}$-flag, respectively:

$$(V : \blacktriangle(b))_q := \overline{\dim_q \mathrm{Hom}_A(V, \bar{\blacktriangledown}(b))}, \qquad (V : \bar{\blacktriangle}(b))_q := \overline{\dim_q \mathrm{Hom}_A(V, \blacktriangledown(b))_q}. \tag{12.3}$$

These are analogous to (8.3) and (8.6). Now we can strengthen Corollary 12.7:

Corollary 12.9 (Pure BGG reciprocity). *For $a, b \in \mathbf{B}$, we have that $(P(b) : \blacktriangle(a))_q = \overline{[\bar{\blacktriangledown}(a) : L(b)]_q}$. If the graded triangular basis for A admits a duality τ such that for each $\lambda \in \Lambda$ the induced duality on A_λ-*gmod *satisfies $\nabla_\lambda(b)^{\circledast} \cong \Delta_\lambda(b)$ for all $b \in \mathbf{B}_\lambda$, this graded multiplicity is also equal to $[\bar{\blacktriangle}(a) : L(b)]_q$.*

Proof. We know already from Corollary 12.7 that $P(b)$ has a $\blacktriangle$-flag. We have that

$$(P(b) : \blacktriangle(a))_q = \overline{\dim_q \operatorname{Hom}_A(P(b), \bar{\blacktriangledown}(a))} = \overline{[\bar{\blacktriangledown}(a) : L(b)]_q}.$$

In the presence of the duality, for $a \in \mathbf{B}_\lambda$, we have that

$$\bar{\blacktriangledown}(a)^{\circledast} = (j_*^\lambda \nabla_\lambda(a))^{\circledast} \cong j_!^\lambda(\nabla_\lambda(a)^{\circledast}) \cong j_!^\lambda \Delta_\lambda(a) = \bar{\blacktriangle}(a).$$

So $\overline{[\bar{\blacktriangledown}(a) : L(b)]_q} = [\bar{\blacktriangle}(a) : L(b)]_q$. □

There are also analogs of Theorems 8.3 and 10.5 for $\blacktriangle$- and $\bar{\blacktriangle}$-flags. The statements are almost exactly the same as before, replacing the various standard, proper standard, costandard and proper costandard modules by their pure counterparts. The definitions of $\blacktriangle$- and $\bar{\blacktriangle}$-*supports* of $V \in A$-gmod needed for the modified statement of Theorem 8.3 are:

$$\operatorname{supp}_{\blacktriangle}(V) := \left\{\dot{b} \,\middle|\, b \in \mathbf{B} \text{ such that } \operatorname{Hom}_A(V, \bar{\blacktriangledown}(b)) \neq 0\right\}, \tag{12.4}$$

$$\operatorname{supp}_{\bar{\blacktriangle}}(V) := \left\{\dot{b} \,\middle|\, b \in \mathbf{B} \text{ such that } \operatorname{Hom}_A(V, \blacktriangledown(b)) \neq 0\right\}. \tag{12.5}$$

We have that $\operatorname{supp}_\Delta(V) \subseteq \operatorname{supp}_{\blacktriangle}(V) \subseteq \operatorname{supp}_{\bar{\blacktriangle}}(V) \subseteq \operatorname{supp}_{\bar{\Delta}}(V)$. All of these sets are finite when V is finitely generated by Lemma 8.1. We leave full proofs of the analogs of Theorems 8.3 and 10.5 to the reader, just recording one more lemma here which is the appropriate modification of the key Lemma 6.9 in the new setting—with this in hand, the other modifications to the earlier arguments are straightforward.

Lemma 12.10. *Suppose that $\lambda \in \Lambda$ is minimal and $V \in A$-*gmod *has the following properties:*

(1) V is locally finite-dimensional and bounded below;
(2) $V = Ae_\lambda V$;
(3) $\operatorname{Ext}_A^1(V, \bar{\blacktriangledown}(b)) = 0$ (resp., $\operatorname{Ext}_A^1(V, \blacktriangledown(b)) = 0$) for all $b \in \mathbf{B}$.

*Then V has a $\blacktriangle$-flag (resp., a $\bar{\blacktriangle}$-flag). More precisely, we have that $V \cong j_!^\lambda \bar{V}$ for $\bar{V} \in A_\lambda$-*gmod *with a Δ_λ-flag (resp., a $\bar{\Delta}_\lambda$-flag).*

Proof. The same arguments as given in the proof of Lemma 6.9 show that $V \cong j_!^\lambda j^\lambda V$. It then remains to show that $\bar{V} := j^\lambda V$ has a Δ_λ-flag (resp., a $\bar{\Delta}_\lambda$-flag). To see this, we can apply the homological criterion from Theorem 8.3 in A_λ-gmod. By (6.1), we have that

$$\operatorname{Ext}^1_{A_\lambda}(\bar{V}, W) \cong \operatorname{Ext}^1_{A_{\geqslant\lambda}}\left(V, j_*^\lambda W\right) \cong \operatorname{Ext}^1_A\left(V, j_*^\lambda W\right)$$

for any $W \in A_\lambda$-gmod. We apply this with $W = \bar{\nabla}_\lambda(c)$ (resp., $\nabla_\lambda(c)$) for $c \in \mathbf{B}_\lambda$ using (3) to complete the proof. □

Remark 12.11. The principles outlined in this section are sufficiently robust that they can be adapted to various similar situations. For example, there is an analogous theory if we instead have that the categories of finitely generated graded left A_λ-modules are *affine highest weight categories* in the sense of [Kle15a, Def. 5.2] with weight posets that are the opposites of the poset $(\mathbf{B}_\lambda, \leqslant_\lambda)$.

References

[AST18] Henning Haahr Andersen, Catharina Stroppel, and Daniel Tubbenhauer. Cellular structures using U_q-tilting modules. *Pacific J. Math.*, 292(1):21–59, 2018.

[BKM14] Jonathan Brundan, Alexander Kleshchev, and Peter J. McNamara. Homological properties of finite-type Khovanov-Lauda-Rouquier algebras. *Duke Math. J.*, 163(7):1353–1404, 2014.

[BS18] Jonathan Brundan and Catharina Stroppel. Semi-infinite highest weight categories. *Mem. Amer. Math. Soc.*, to appear, 2018.

[BWW23a] Jonathan Brundan, Ben Webster, and Weiqiang Wang. Categorification of PBW bases for modified quantum groups. In preparation, 2023.

[BWW23b] Jonathan Brundan, Ben Webster, and Weiqiang Wang. Nil-Brauer categorifies the split $\imath$-quantum group of rank one. Preprint, 2023.

[BWW23c] Jonathan Brundan, Ben Webster, and Weiqiang Wang. The nil-Brauer category. `arXiv:2305.03876`, 2023.

[CPS88] E. Cline, B. Parshall, and L. Scott. Finite-dimensional algebras and highest weight categories. *J. Reine Angew. Math.*, 391:85–99, 1988.

[DR98] Jie Du and Hebing Rui. Based algebras and standard bases for quasi-hereditary algebras. *Trans. Amer. Math. Soc.*, 350(8):3207–3235, 1998.

[EL16] Ben Elias and Aaron D. Lauda. Trace decategorification of the Hecke category. *J. Algebra*, 449:615–634, 2016.

[EW16] Ben Elias and Geordie Williamson. Soergel calculus. *Represent. Theory*, 20:295–374, 2016.

[GL96] J. J. Graham and G. I. Lehrer. Cellular algebras. *Invent. Math.*, 123(1):1–34, 1996.

[GRS23] Mengmeng Gao, Hebing Rui, and Linliang Song. Representations of weakly triangular categories. *J. Algebra*, 614:481–534, 2023.

[HM10] Jun Hu and Andrew Mathas. Graded cellular bases for the cyclotomic Khovanov-Lauda-Rouquier algebras of type *A*. *Adv. Math.*, 225(2):598–642, 2010.

[HMR21] Jun Hu, Andrew Mathas, and Salim Rostam. Skew cellularity of the Hecke algebras of type $G(\ell, p, n)$. `arXiv:2106.11459`, 2021.

[KL10] Mikhail Khovanov and Aaron D. Lauda. A categorification of quantum $\mathrm{sl}(n)$. *Quantum Topol.*, 1:1–92, 2010.

[Kle15a] Alexander S. Kleshchev. Affine highest weight categories and affine quasihereditary algebras. *Proc. Lond. Math. Soc. (3)*, 110(4):841–882, 2015.

[Kle15b] Alexander S. Kleshchev. Representation theory and cohomology of Khovanov-Lauda-Rouquier algebras. In *Modular representation theory of finite and p-adic groups*, volume 30 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 109–163. World Sci. Publ., Hackensack, NJ, 2015.

[KM20] Alexander Kleshchev and Robert Muth. Based quasi-hereditary algebras. *J. Algebra*, 558:504–522, 2020.

[KX12] Steffen Koenig and Changchang Xi. Affine cellular algebras. *Adv. Math.*, 229(1):139–182, 2012.

[LW15] Ivan Losev and Ben Webster. On uniqueness of tensor products of irreducible categorifications. *Selecta Math. (N.S.)*, 21(2):345–377, 2015.

[Rou08] Raphaël Rouquier. 2-Kac-Moody algebras. `arXiv:0812.5023`, 2008.

[SS22] Steven V. Sam and Andrew Snowden. The representation theory of Brauer categories I: triangular categories. *Appl. Categ. Structures*, 30(6):1203–1256, 2022.

[Tub22] Daniel Tubbenhauer. Sandwich cellularity and a version of cell theory. `arXiv:2206.06678`, 2022.

[Wan21] Weiqiang Wang. PBW basis for modified quantum groups. `arXiv:2109.00139`, 2021.

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[Wes09] Bruce W. Westbury. Invariant tensors and cellular categories. *J. Algebra*, 321(11):3563–3567, 2009.

Department of Mathematics, University of Oregon, Eugene, OR, USA
Email address: `brundan@uoregon.edu`