Algebraic Combinatorics
Draft

On the definition of Heisenberg category

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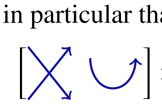
Abstract. We revisit the definition of the Heisenberg category of central charge $k \in \mathbb{Z}$. For central charge -1, this category was introduced originally by Khovanov, but with some additional cyclicity relations which we show here are unnecessary. For other negative central charges, the definition is due to Mackaay and Savage, also with some redundant relations, while central charge zero recovers the affine oriented Brauer category of Brundan, Comes, Nash and Reynolds. We also discuss cyclotomic quotients.

1. Introduction

In [14], Khovanov introduced a graphical calculus for the induction and restriction functors Ind_n^{n+1} and Res_{n-1}^n arising in the representation theory of the symmetric group S_n. This led him to the definition of a monoidal category $\mathcal{H}$, which he called the *Heisenberg category*. This category is monoidally generated by two objects $\uparrow$ and $\downarrow$ (corresponding to the induction and restriction functors) with morphisms defined in terms of equivalence classes of certain diagrams modulo Reidemeister-type relations plus a small number of additional relations. Khovanov's relations imply in particular that there is an *isomorphism*



$$: \uparrow \otimes \downarrow \oplus \mathbb{1} \xrightarrow{\sim} \downarrow \otimes \uparrow$$

in $\mathcal{H}$, mirroring the Mackey decomposition

$$\mathrm{Ind}_{n-1}^n \circ \mathrm{Res}_{n-1}^n \oplus \mathrm{Id}_n \cong \mathrm{Res}_n^{n+1} \circ \mathrm{Ind}_n^{n+1}$$

at the level of representation theory of the symmetric groups. There have been several subsequent generalizations of Khovanov's work, including a q-deformation [17], a version of Heisenberg category for wreath product algebras associated to finite subgroups of $SL_2(\mathbb{C})$ [10], and an odd analog incorporating a Clifford superalgebra [13].

To explain the name "Heisenberg category," let $\mathfrak{h}$ be the infinite-dimensional Heisenberg algebra, i.e., the complex Lie algebra with basis $\{c, p_n, q_n \mid n \geqslant 1\}$ and multiplication given by

$$[p_m, p_n] = [q_m, q_n] = [c, p_n] = [c, q_n] = 0, \qquad [p_m, q_n] = \delta_{m,n} mc.$$

Khovanov constructed an algebra homomorphism from $U(\mathfrak{h})$ specialized at central charge $c = -1$ to the complexified Grothendieck ring $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathrm{Kar}(\mathcal{H}))$ of the additive Karoubi

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envelope Kar($\mathcal{H}$) of $\mathcal{H}$. He proved that his map is injective, and conjectured that it is actually an isomorphism. This conjecture is still open.

We remark also that the trace of Khovanov's category and of its q-deformed version have recently been computed; see [9, 8].

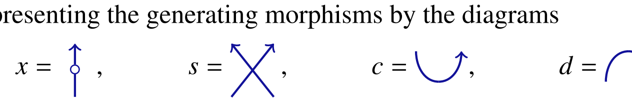
The group algebra of the symmetric group is the level one case of a family of finite-dimensional algebras: the cyclotomic quotients of degenerate affine Hecke algebras associated to symmetric groups. For cyclotomic quotients of level $\ell > 0$, the Mackey theorem instead takes the form

$$\mathrm{Ind}_{n-1}^{n} \circ \mathrm{Res}_{n-1}^{n} \oplus (\mathrm{Id}_n)^{\oplus \ell} \cong \mathrm{Res}_{n}^{n+1} \circ \mathrm{Ind}_{n}^{n+1},$$

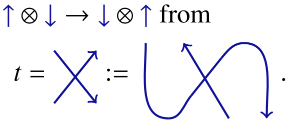
e.g., see [16, Theorem 7.6.2]. Mackaay and Savage [18] have extended Khovanov's construction to this setting, defining Heisenberg categories for all $\ell > 0$, with the case $\ell = 1$ recovering Khovanov's original category. They also constructed an injective homomorphism from $U(\mathfrak{h})$ specialized at central charge $c = -\ell$ to the complexified Grothendieck ring of the additive Karoubi envelope of their category, and conjectured that this map is an isomorphism. Again, this more general conjecture remains open.

In [4], motivated by quite different considerations, the author jointly with Comes, Nash and Reynolds introduced another diagrammatically-defined monoidal category we called the *affine oriented Brauer category* $\mathcal{AOB}$; the endomorphism algebras of objects in $\mathcal{AOB}$ are the *affine walled Brauer algebras* of [22]. In fact, the affine oriented Brauer category is the Heisenberg category for central charge zero. To make this connection explicit, and also to streamline the approach of Mackaay and Savage, we propose here a simplified definition of Heisenberg category for an arbitrary central charge $k \in \mathbb{Z}$. Our new formulation is similar in spirit to Rouquier's definition of Kac-Moody 2-category from [20] (as opposed to the Khovanov-Lauda definition from [15]); see also [3].

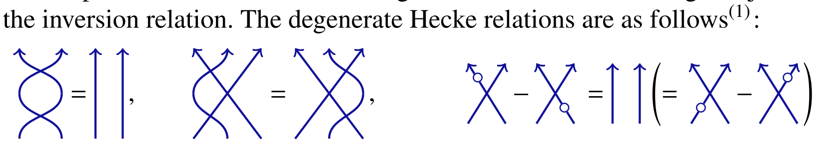
Definition 1.1. Fix a commutative ground ring $\Bbbk$. The *Heisenberg category* $\mathcal{H}eis_k$ of central charge $k \in \mathbb{Z}$ is the strict $\Bbbk$-linear monoidal category generated by objects $\uparrow$ and $\downarrow$, and morphisms $x : \uparrow \to \uparrow$, $s : \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow$, $c : \mathbb{1} \to \downarrow \otimes \uparrow$ and $d : \uparrow \otimes \downarrow \to \mathbb{1}$ subject to certain relations. To record these relations, we adopt the usual string calculus for strict monoidal categories, representing the generating morphisms by the diagrams



The horizontal composition $a \otimes b$ of two morphisms is a drawn to the left of b, and the vertical composition $a \circ b$ is a drawn above b (assuming this makes sense). We also denote the nth power $x^{\circ n}$ of x under vertical composition diagrammatically by labeling the dot with the multiplicity n, and define $t : \uparrow \otimes \downarrow \to \downarrow \otimes \uparrow$ from


(1.1)

Then we impose three sets of relations: degenerate Hecke relations, right adjunction relations, and the inversion relation. The degenerate Hecke relations are as follows[(1)]:


(1.2)

The right adjunction relations say that


(1.3)

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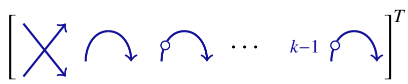
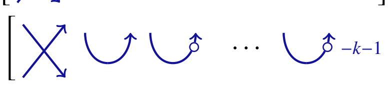
(1) The final one of these relations is in parentheses to indicate that it is a consequence of the other relations; we have included it just for convenience.

Finally, the inversion relation asserts that the following matrix of morphisms is an isomorphism in the additive envelope of $\mathcal{H}eis_k$:

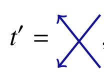


$$: \uparrow\otimes\downarrow \xrightarrow{\sim} \downarrow\otimes\uparrow\oplus\mathbb{1}^{\oplus k} \qquad \text{if } k \geqslant 0, \tag{1.4}$$



$$: \uparrow\otimes\downarrow\oplus\mathbb{1}^{\oplus(-k)} \xrightarrow{\sim} \downarrow\otimes\uparrow \qquad \text{if } k < 0. \tag{1.5}$$

In the special case $k = 0$, the inversion relation means that one should adjoin another generating morphism $t' : \downarrow\otimes\uparrow \to \uparrow\otimes\downarrow$, represented by



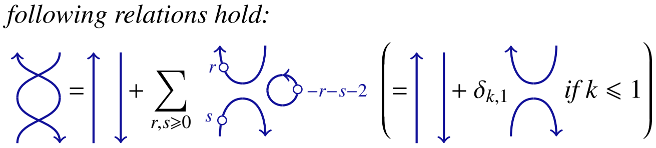
subject to the following relations asserting that t' is a two-sided inverse to t:

Up to reflecting diagrams in a vertical axis, this is exactly the definition of the affine oriented Brauer category $\mathcal{AOB}$ from [4]. Thus, there is a monoidal isomorphism $\mathcal{H}eis_0 \cong \mathcal{AOB}^{\mathrm{rev}}$.

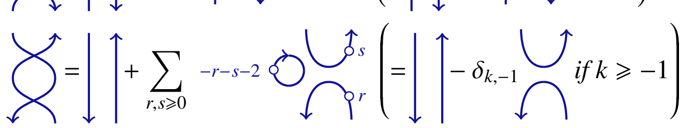
When $k \neq 0$, the inversion relation appearing in Definition 1.1 is much harder to interpret. We will analyze it systematically in the main part of this article. We summarize the situation with the following two theorems.

THEOREM 1.2. *There are unique morphisms $c' : \mathbb{1} \to \uparrow\otimes\downarrow$ and $d' : \downarrow\otimes\uparrow \to \mathbb{1}$ in $\mathcal{H}eis_k$, drawn as*

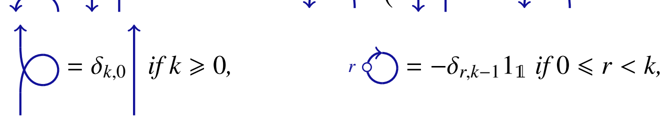
such that the following relations hold:



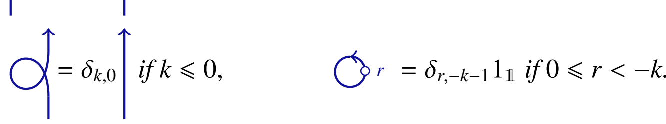
$$\text{if } k \leqslant 1 \tag{1.6}$$



$$\text{if } k \geqslant -1 \tag{1.7}$$

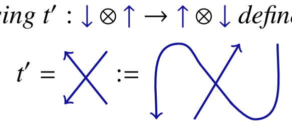


$$\text{if } k \geqslant 0, \qquad = -\delta_{r,k-1}1_{\mathbb{1}} \text{ if } 0 \leqslant r < k, \tag{1.8}$$



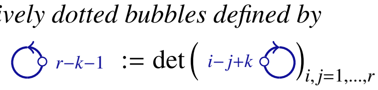
$$\text{if } k \leqslant 0, \qquad = \delta_{r,-k-1}1_{\mathbb{1}} \text{ if } 0 \leqslant r < -k. \tag{1.9}$$

Moreover, $\mathcal{H}eis_k$ can be presented equivalently as the strict $\Bbbk$-linear monoidal category generated by the objects $\uparrow, \downarrow$ and morphisms x, s, c, d, c', d' subject only to the relations (1.2)*–*(1.3) *and* (1.6)*–*(1.9)*. In these relations, as well as the rightward crossing t defined by* (1.1)*, we have used the leftward crossing $t' : \downarrow\otimes\uparrow \to \uparrow\otimes\downarrow$ defined by*

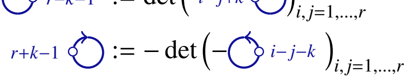


$$\tag{1.10}$$

and the negatively dotted bubbles defined by



$$\text{if } r \leqslant k, \tag{1.11}$$



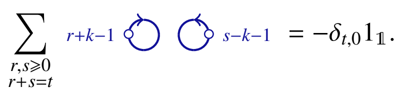
$$\text{if } r \leqslant -k, \tag{1.12}$$

interpreting the determinants as 1 *if* $r = 0$ *and* 0 *if* $r < 0$.

THEOREM 1.3. *Using the notation from Theorem* 1.2*, the following relations are consequences of the defining relations.*

(i) *("Infinite Grassmannian relations") For all* $r \in \mathbb{Z}$*:*

(1.13)

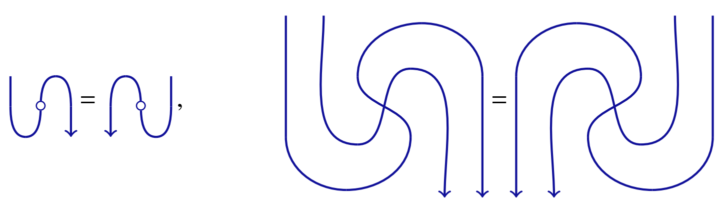


(1.14)

(ii) *("Left adjunction")*

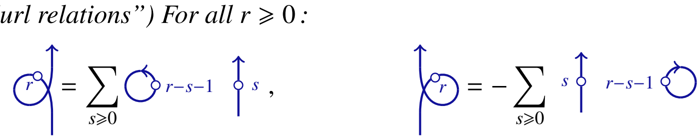
(1.15)

(iii) *("Cyclicity")*



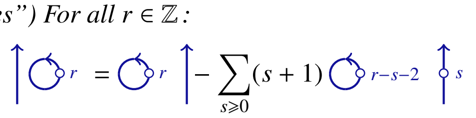
(1.16)

(iv) *("Curl relations") For all* $r \geqslant 0$*:*

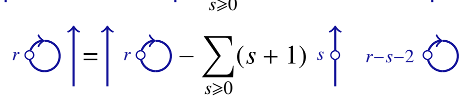


(1.17)

(v) *("Bubble slides") For all* $r \in \mathbb{Z}$*:*

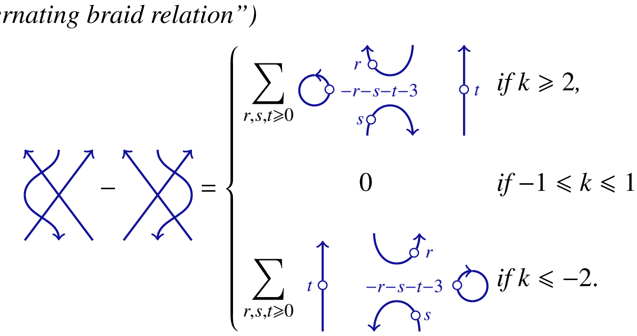


(1.18)



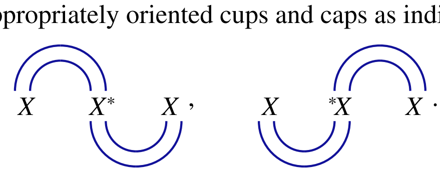
(1.19)

(vi) *("Alternating braid relation")*

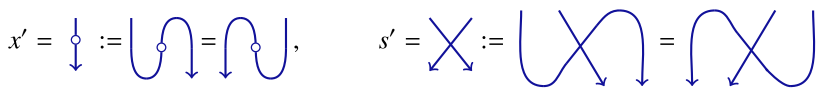


(1.20)

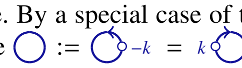
Part (ii) of Theorem 1.3 implies that the monoidal category $\mathcal{H}eis_k$ is *rigid*, i.e., any object X has both a right dual X^* (with its structure maps $X \otimes X^* \to \mathbb{1} \to X^* \otimes X$) and a left dual ${}^*X$ (with its structure maps ${}^*X \otimes X \to \mathbb{1} \to X \otimes {}^*X$). In fact, there is a canonical choice for both duals, by attaching the appropriately oriented cups and caps as indicated below:



Then part (iii) of the theorem shows that the right and left mates of x are equal, as are the right and left mates of s. We denote these by $x' : \downarrow \to \downarrow$ and $s' : \downarrow \otimes \downarrow \to \downarrow \otimes \downarrow$, respectively, and represent them diagrammatically by



It follows that the functors $(-)^*$ and ${}^*(-)$ defined by taking right and left duals/mates in the canonical way actually coincide; they are both defined by rotating diagrams through 180°. Thus, we have equipped $\mathcal{H}eis_k$ with a *strictly pivotal structure*.

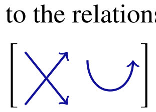
Now we can explain the relationship between the category $\mathcal{H}eis_k$ and the Heisenberg categories already appearing in the literature. By a special case of the bubble slide relations in Theorem 1.3(v), the lowest degree bubble  is *strictly central* in the sense that

This means that it is natural to specialize  to some scalar $\delta \in \Bbbk$. We denote the resulting monoidal category by $\mathcal{H}eis_k(\delta)$.

Theorem 1.4. *The Heisenberg category $\tilde{\mathcal{H}}^\lambda$ defined by Mackaay and Savage in* [18] *is isomorphic to the additive envelope of $\mathcal{H}eis_k(\delta)$, taking $k := -\sum_i \lambda_i$ and $\delta := \sum_i i\lambda_i$. In particular, the Heisenberg category $\mathcal{H}$ introduced originally by Khovanov in* [14] *is isomorphic to the additive envelope of $\mathcal{H}eis_{-1}(0)$.*

Remark 1.5. The above results give *two* new presentations for Khovanov's Heisenberg category $\mathcal{H}$, i.e., the additive envelope of our $\mathcal{H}eis_{-1}(0)$:

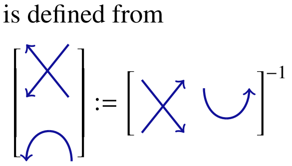
(1) The first presentation, which is essentially Definition 1.1, asserts that $\mathcal{H}$ is the strict additive $\Bbbk$-linear monoidal category generated by objects $\uparrow$ and $\downarrow$ and the morphisms x, s, c and d, subject to the relations (1.2) and (1.3), the relation (1.5) asserting that

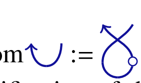
 $: \uparrow \otimes \downarrow \oplus \mathbb{1} \xrightarrow{\sim} \downarrow \otimes \uparrow$

is an isomorphism where the rightward crossing is defined by (1.1), and the relation

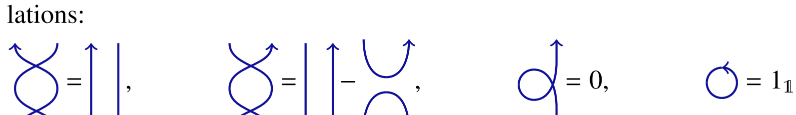
 $= 0$

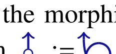
where the leftward cap is defined from



The leftward cup may also be recovered from  .

(2) The second presentation, which is a simplification of the presentation from Theorem 1.3, asserts that $\mathcal{H}$ is generated by objects $\uparrow$ and $\downarrow$ and the morphisms s, c, d, c', d' subject to the first two relations from (1.2), the relations (1.3), and four additional relations:



The rightward and leftward crossings used here are shorthands for the morphisms defined by (1.1) and (1.10), respectively. Then x may be defined from ; the third relation from (1.2) holds automatically.

The presentation (2) is almost the same as Khovanov's original definition. Khovanov's formulation also implicitly incorporated some additional cyclicity relations, which our results show are redundant, i.e., they are implied by the other relations.

Let Sym be the algebra of symmetric functions. Recall this is an infinite rank polynomial algebra generated freely by either the complete symmetric functions $\{h_r\}_{r\geqslant 1}$ or the elementary symmetric functions $\{e_r\}_{r\geqslant 1}$; we also let $h_0 = e_0 = 1$ and interpret h_r and e_r as 0 when $r < 0$. Let

$$\beta : \mathrm{Sym} \to \mathrm{End}_{\mathcal{H}eis_k}(\mathbb{1}) \tag{1.21}$$

be the algebra homomorphism defined by declaring that

$$\begin{cases} \beta(e_r) := -\ {\scriptstyle r+k-1}\!\bullet\!\!\bigcirc & \text{if } k \geqslant 0,\\ \beta(h_r) := (-1)^r \bigcirc\!\!\bullet {\scriptstyle r-k-1} & \text{if } k < 0.\end{cases}$$

Then the relations from Theorem 1.3(i) imply that

$$\begin{cases} \beta(h_r) = (-1)^r \bigcirc\!\!\bullet {\scriptstyle r-k-1} & \text{if } k \geqslant 0,\\ \beta(e_r) = -\ {\scriptstyle r+k-1}\!\bullet\!\!\bigcirc & \text{if } k < 0.\end{cases}$$

In fact, β is an isomorphism. This assertion is a consequence of the *basis theorem* for morphism spaces in $\mathcal{H}eis_k$, which we explain next.

Let $X = X_1 \otimes \cdots \otimes X_r$ and $Y = Y_1 \otimes \cdots \otimes Y_s$ be two words in the letters $\uparrow$ and $\downarrow$, representing two objects of $\mathcal{H}eis_k$. By an (X, Y)*-matching*, we mean a bijection

$$\{i \mid X_i = \uparrow\} \sqcup \{j \mid Y_j = \downarrow\} \xrightarrow{\sim} \{i \mid X_i = \downarrow\} \sqcup \{j \mid Y_j = \uparrow\}.$$

By a *reduced lift* of an (X, Y)-matching, we mean a diagram representing a morphism $X \to Y$ in $\mathcal{H}eis_k$ such that

- the endpoints of each strand in the diagram are paired under the matching;
- any two strands intersect at most once;
- there are no self-intersections;
- there are no dots or bubbles;
- each strand has at most one critical point coming from a cup or cap.

Let $B(X, Y)$ be a set consisting of a reduced lift for each of the (X, Y)-matchings. For each element of $B(X, Y)$, pick a distinguished point on each of its strands that is away from crossings and critical points. Then let $B_{\infty,\infty}(X, Y)$ be the set of all morphisms $\theta : X \to Y$ obtained from the elements of $B(X, Y)$ by adding zero or more dots to each strand at these distinguished points.

Theorem 1.6. *For any $k \in \mathbb{Z}$ and objects X, Y in $\mathcal{H}eis_k$, the space* $\mathrm{Hom}_{\mathcal{H}eis_k}(X, Y)$ *is a free right* Sym*-module with basis $B_{\infty,\infty}(X, Y)$. Here, the right action of* Sym *on morphisms is by $\theta \cdot p := \theta \otimes \beta(p)$ for $\theta : X \to Y$ and $p \in$* Sym.

In particular, this implies that $\mathcal{H}eis_k \cong \mathcal{H}eis_k(\delta) \otimes_{\mathbb{k}} \mathbb{k}[z]$ where z denotes the bubble $\bigcirc$. It follows that $K_0(\mathcal{H}eis_k) \cong K_0(\mathcal{H}eis_k(\delta))$ for any $\delta \in \mathbb{k}$. Combining this observation with Theorem 1.4, we then restate [18, Theorem 4.4] as follows: whenever $k \neq 0$ there is an algebra embedding

$$U(\mathfrak{h})/\langle c - k\rangle \hookrightarrow \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathrm{Kar}(\mathcal{H}eis_k)). \tag{1.22}$$

As we mentioned already above, this embedding is conjectured to be an isomorphism. There should be similar results when $k = 0$ too.

Theorem 1.6 was proved already in case $k = 0$ in [4, Theorem 1.2], by an argument based on the existence of a certain monoidal functor from $\mathcal{H}eis_0$ to the category of $\mathbb{k}$-linear endofunctors of the category of modules over the Lie algebra $\mathfrak{gl}_n(\mathbb{k})$. When $k \neq 0$, the theorem will instead be deduced from the basis theorems proved in [14, Proposition 5] and [18, Proposition 2.16]. The proofs in [14, 18] depend crucially on the action of $\mathcal{H}eis_k$ on the category of modules over the degenerate cyclotomic Hecke algebras mentioned earlier. Since it highlights the usefulness of Definition 1.1, we give a self-contained construction of this action in the next paragraph.

Fix a monic polynomial $f(u) = u^\ell + z_1 u^{\ell-1} + \cdots + z_\ell \in \Bbbk[u]$ of degree $\ell > 0$ and set $k := -\ell$. Let H_n be the degenerate affine Hecke algebra, that is, the tensor product $\Bbbk S_n \otimes \Bbbk[x_1, \dots, x_n]$ of the group algebra of the symmetric group with a polynomial algebra. Multiplication in H_n is defined so that $\Bbbk S_n$ and $\Bbbk[x_1, \dots, x_n]$ are subalgebras, and also

$$x_{i+1} s_i = s_i x_i + 1, \qquad x_i s_j = s_j x_i \ (i \neq j, j+1),$$

where s_j denotes the basic transposition $(j\, j{+}1)$. Let H_n^f be the quotient of H_n by the two-sided ideal generated by $f(x_1)$. There is a natural embedding $H_n^f \hookrightarrow H_{n+1}^f$ sending $x_i, s_j \in H_n^f$ to the same elements of H_{n+1}^f. Let

$$\begin{aligned} \mathrm{Ind}_n^{n+1} := H_{n+1}^f \otimes_{H_n^f} ? : H_n^f\text{-mod} \to H_{n+1}^f\text{-mod},\\ \mathrm{Res}_n^{n+1} : H_{n+1}^f\text{-mod} \to H_n^f\text{-mod} \end{aligned}$$

be the corresponding induction and restriction functors. The key assertion established in [14, 18] is that there is a strict $\Bbbk$-linear monoidal functor

$$\Psi_f : \mathcal{H}eis_k \to \mathcal{E}nd_\Bbbk\left(\bigoplus_{n \geqslant 0} H_n^f\text{-mod}\right) \tag{1.23}$$

sending $\uparrow$ (respectively, $\downarrow$) to the $\Bbbk$-linear endofunctor that takes an H_n^f-module M to the H_{n+1}^f-module $\mathrm{Ind}_n^{n+1} M$ (respectively, to the H_{n-1}^f-module $\mathrm{Res}_{n-1}^n M$, interpreted as zero in case $n = 0$). On generating morphisms, $\Psi_f(x), \Psi_f(s), \Psi_f(c)$ and $\Psi_f(d)$ are the natural transformations defined on an H_n^f-module M as follows:

- $\Psi_f(x)_M : \mathrm{Ind}_n^{n+1} M \to \mathrm{Ind}_n^{n+1} M,\ h \otimes m \mapsto h x_{n+1} \otimes m$;
- $\Psi_f(s)_M : \mathrm{Ind}_n^{n+2} M \to \mathrm{Ind}_n^{n+2} M,\ h \otimes m \mapsto h s_{n+1} \otimes m$, where we have identified $\mathrm{Ind}_{n+1}^{n+2} \circ \mathrm{Ind}_n^{n+1}$ with $\mathrm{Ind}_n^{n+2} := H_{n+2}^f \otimes_{H_n^f} ?$ in the obvious way;
- $\Psi_f(c)_M : M \to \mathrm{Res}_n^{n+1} \circ \mathrm{Ind}_n^{n+1} M,\ m \mapsto 1 \otimes m$;
- $\Psi_f(d)_M : \mathrm{Ind}_{n-1}^n \circ \mathrm{Res}_{n-1}^n M \to M,\ h \otimes m \mapsto hm$.

To prove this in our setting, we need to verify the three sets of relations from Definition 1.1. The first two are almost immediate. For the inversion relation, one calculates $\Psi_f(t)_M$ explicitly to see that it comes from the (H_n^f, H_n^f)-bimodule homomorphism $H_n^f \otimes_{H_{n-1}^f} H_n^f \to H_{n+1}^f, a \otimes b \mapsto a s_n b$. Thus, it suffices to show that the (H_n^f, H_n^f)-bimodule homomorphism

$$\begin{aligned} H_n^f \otimes_{H_{n-1}^f} H_n^f \oplus \bigoplus_{r=0}^{\ell-1} H_n^f \to H_{n+1}^f, \\ (a \otimes b, c_0, c_1, \dots, c_{\ell-1}) \mapsto a s_n b + \sum_{r=0}^{\ell-1} c_r x_{n+1}^r \end{aligned} \tag{1.24}$$

is an isomorphism, which is checked in the proof of [16, Lemma 7.6.1]. We remark further that Ψ_f maps the bubble $\bigcirc$ to the scalar $-z_1$, i.e., Ψ_f factors through the specialization $\mathcal{H}eis_k(\delta)$ where δ is the sum of the roots of the polynomial $f(u)$.

The natural transformations $\Psi_f(c)$ and $\Psi_f(d)$ in the previous paragraph come from the units and counits of the canonical adjunctions making $(\mathrm{Ind}_n^{n+1}, \mathrm{Res}_n^{n+1})$ into adjoint pairs. In view of Theorem 1.3(ii), we also get canonical adjunctions the other way around, with units and counits defined by $\Psi_f(c')$ and $\Psi_f(d')$, respectively. Thus, the induction and restriction functors Ind_n^{n+1} and Res_n^{n+1} are biadjoint; see also [16, Corollary 7.7.5] and [18, Proposition 5.13].

One reason that cyclotomic quotients of the degenerate affine Hecke algebra are important is that they can be used to realize the *minimal categorifications* of integrable lowest weight modules for the Lie algebra $\mathfrak{g} := \mathfrak{sl}_\infty$ (if $\Bbbk$ is a field of characteristic 0) or $\mathfrak{g} := \widehat{\mathfrak{sl}}_p$ (if $\Bbbk$ is a field of characteristic $p > 0$), e.g., see [1, 6]. The following theorem shows that these minimal categorifications can be realized instead as *cyclotomic quotients of Heisenberg categories*. All of this should be compared with [20, §5.1.2] (and [21, Theorem 4.25]), where the minimal categorification is realized as a cyclotomic quotient of the corresponding Kac-Moody 2-category. In the special case $\ell = 1$, some closely related constructions can be found in [12].

Theorem 1.7. *Fix $f(u) = u^\ell + z_1 u^{\ell-1} + \cdots + z_\ell \in \Bbbk[u]$ of degree $\ell = -k > 0$ as in (1.23). Let $\mathcal{I}_{f,1}$ be the $\Bbbk$-linear left tensor ideal of $\mathcal{H}eis_k$ generated by $f(x) : \uparrow \to \uparrow$; equivalently, by Lemma 1.8 below, $\mathcal{I}_{f,1}$ is the $\Bbbk$-linear left tensor ideal generated by $1_\downarrow : \downarrow \to \downarrow$ and $\overset{r+k-1}{\diamond}\!\!\circlearrowleft + z_r 1_{\mathbb{1}} : \mathbb{1} \to \mathbb{1}$ for $r = 1, \dots, \ell$. Let*

$$\mathrm{Ev} : \mathcal{E}nd_{\Bbbk}\left(\bigoplus_{n \geqslant 0} H_n^f\text{-mod}\right) \to \bigoplus_{n \geqslant 0} H_n^f\text{-mod}$$

be the functor defined by evaluating on the one-dimensional H_0^f-module. Then $\mathrm{Ev} \circ \Psi_f$ factors through the quotient category $\mathcal{H}eis_{f,1} := \mathcal{H}eis_k / \mathcal{I}_{f,1}$ to induce an equivalence of categories

$$\psi_f : \mathrm{Kar}(\mathcal{H}eis_{f,1}) \to \bigoplus_{n \geqslant 0} H_n^f\text{-pmod},$$

where Kar *denotes additive Karoubi envelope and* pmod *denotes finitely generated projectives.*

To get the full structure of a $\mathfrak{g}$-categorification on $\mathrm{Kar}(\mathcal{H}eis_{f,1})$ in the sense of [20, Definition 5.29], one also needs the endofunctors E and F defined by tensoring with $\uparrow$ and $\downarrow$, respectively. Under the equivalence in Theorem 1.7, these correspond to the induction and restriction functors on $\bigoplus_{n \geqslant 0} H_n^f$-pmod. It is immediate from the definition of $\mathcal{H}eis_{f,1}$ that E and F are biadjoint and that the powers of E admit the appropriate action of the degenerate affine Hecke algebra. It just remains to check that the complexified Grothendieck group $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathrm{Kar}(\mathcal{H}eis_{f,1}))$ is the appropriate integrable representation of $\mathfrak{g}$. This follows from [1, 6] using the equivalence in the theorem.

In [24], Webster introduced *generalized cyclotomic quotients of Kac-Moody 2-categories* which categorify lowest-tensored-highest weight representations; see also [5, §4.2]. For $\mathfrak{sl}_\infty$ or $\widehat{\mathfrak{sl}}_p$, Webster's categories can also be realized as *generalized cyclotomic quotients of Heisenberg categories*. This will be explained elsewhere, but we can at least formulate the definition of these generalized cyclotomic quotients here. Fix a pair of monic polynomials[(2)] $f(u), f'(u) \in \Bbbk[u]$ of degrees $\ell, \ell' \geqslant 0$, respectively, and define $k := \ell' - \ell$ and $\delta_r, \delta'_r \in \Bbbk$ so that

$$\delta(u) = \delta_0 + \delta_1 u^{-1} + \delta_2 u^{-2} + \cdots := u^{-k} f'(u)/f(u) \in \Bbbk[[u^{-1}]], \tag{1.25}$$

$$\delta'(u) = \delta'_0 + \delta'_1 u^{-1} + \delta'_2 u^{-2} + \cdots := -u^k f(u)/f'(u) \in \Bbbk[[u^{-1}]]. \tag{1.26}$$

Then the corresponding generalized cyclotomic quotient of $\mathcal{H}eis_k$ is the $\Bbbk$-linear category

$$\mathcal{H}eis_{f,f'} := \mathcal{H}eis_k / \mathcal{I}_{f,f'} \tag{1.27}$$

where $\mathcal{I}_{f,f'}$ is the $\Bbbk$-linear left tensor ideal of $\mathcal{H}eis_k$ generated by $f(x) : \uparrow \to \uparrow$ and $\circlearrowright\!\!\overset{r-k-1}{\diamond} - \delta_r 1_{\mathbb{1}} : \mathbb{1} \to \mathbb{1}$ for $r = 1, \dots, \ell'$. These categories were introduced already in the case that $\ell = \ell'$ in [4].

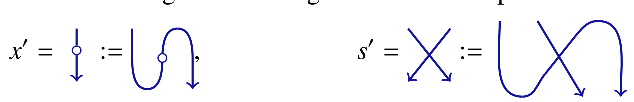
(2) We stress here that $f'(u)$ denotes a different polynomial; it is *not* the derivative of $f(u)$!

Lemma 1.8. *The ideal $\mathcal{I}_{f,f'}$ can be defined equivalently as the $\Bbbk$-linear left tensor ideal of $\mathcal{H}eis_k$ generated by $f'(x') : \downarrow \to \downarrow$ and ${}_{r+k-1}\circlearrowleft - \delta'_r 1_{\mathbb{1}} : \mathbb{1} \to \mathbb{1}$ for $r = 1, \dots, \ell$. It also contains $\circlearrowright_{r-k-1} - \delta_r 1_{\mathbb{1}}$ and ${}_{r+k-1}\circlearrowleft - \delta'_r 1_{\mathbb{1}}$ for all $r \geqslant 0$.*

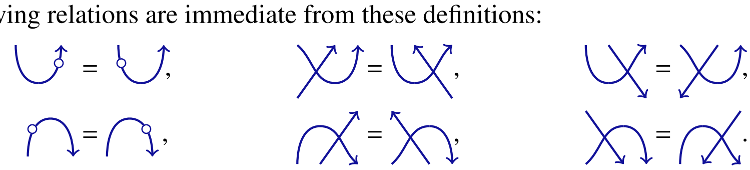
Let us finally mention that there is also a quantum analog $\mathcal{H}eis_k(z,t)$ of the Heisenberg category $\mathcal{H}eis_k(\delta)$. This will be defined in a sequel to this article [7]. Even in the case that $k = -1$, our approach is different to that of [17] as we require that the polynomial generator X is invertible, i.e., we incorporate the entire affine Hecke algebra into the definition (rather than the q-deformed degenerate affine Hecke algebra used in [17]). The quantum Heisenberg category $\mathcal{H}eis_0(z,t)$ of central charge zero is the *affine oriented skein category* $\mathcal{AOS}(z,t)$ from [2, §4]. Further generalizations incorporating Clifford and Frobenius superalgebras into the definition have also recently emerged building on the approach taken in this article; see [11] (which extends [13] to arbitrary central charge) and [23] (which extends [19]).

2. Analysis of the inversion relation

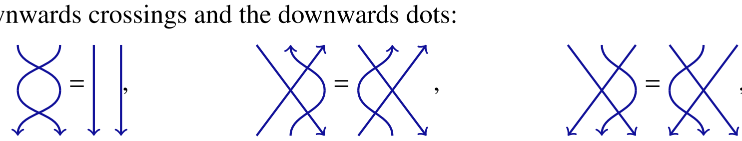
This section is the technical heart of the paper. The development is similar to that of [3]. Going back to the original definition of $\mathcal{H}eis_k$ from Definition 1.1, we begin our study by *defining* the downward dots and crossings to be the right mates of the upward dots and crossings:

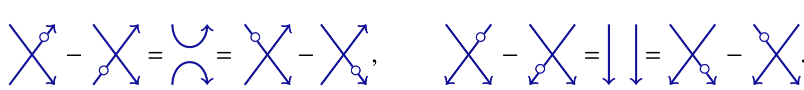

(2.1)

The following relations are immediate from these definitions:


(2.2)
(2.3)

Also, the following relations are easily deduced by attaching rightward cups and caps to the degenerate Hecke relations, then "rotating" the pictures using the definitions of the rightwards/downwards crossings and the downwards dots:


(2.4)

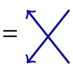

(2.5)

The important symmetry ω constructed in the next lemma is often useful since it reduces to the case that $k \geqslant 0$. In words, ω reflects in a horizontal axis then multiplies by $(-1)^x$, where x is the total number of crossings appearing in the diagram. This heuristic also holds for all of the other morphisms defined diagrammatically below, but in general the sign becomes $(-1)^{x+y}$ where x is the total number of crossings and y is the total number of leftward cups and caps (not counting the decorated caps and cups to be introduced shortly which are labelled with the symbol $\diamondsuit$).

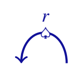
Lemma 2.1. *There is an isomorphism of monoidal categories $\omega : \mathcal{H}eis_k \xrightarrow{\sim} \mathcal{H}eis_{-k}^{\mathrm{op}}$ switching the objects $\uparrow$ and $\downarrow$, and defined on generating morphisms by $x \mapsto x'$, $s \mapsto -s'$, $c \mapsto d$ and $d \mapsto c$.*

Proof. The existence of ω follows by a straightforward relation check. Use (2.4)–(2.5) for the degenerate Hecke relations. The need to switch k and $-k$ comes from the inversion relations. To see that ω is an isomorphism, notice by the right adjunction relations that $\omega(x') = x$ and $\omega(s') = -s$, hence, $\omega^2 = \mathrm{Id}$. $\square$

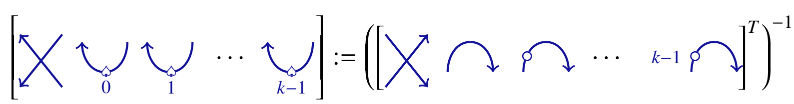
The inversion relation means that there are some as yet unnamed generating morphisms in $\mathcal{H}eis_k$ which are the matrix entries of two-sided inverses to the morphism (1.4)–(1.5). We next introduce notation for these matrix entries. First define

$t' =$  $: \downarrow \otimes \uparrow \to \uparrow \otimes \downarrow,$

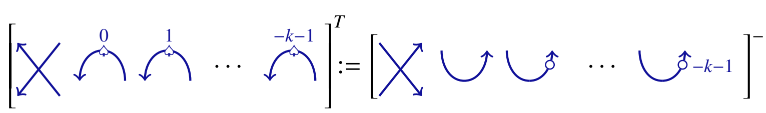
and the decorated leftward cups and caps

 $: \mathbb{1} \to \uparrow \otimes \downarrow,$  $: \downarrow \otimes \uparrow \to \mathbb{1}$

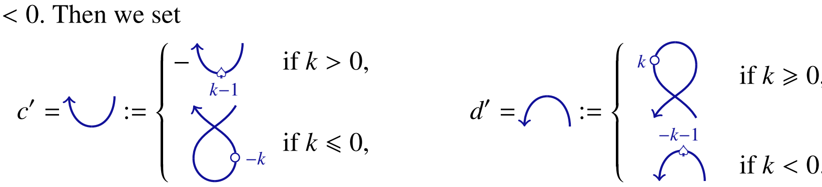
for $0 \leqslant r < k$ or $0 \leqslant r < -k$, respectively, by declaring that

 (2.6)

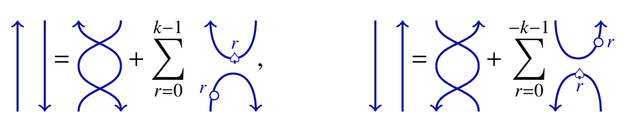
if $k \geqslant 0$, or

 (2.7)

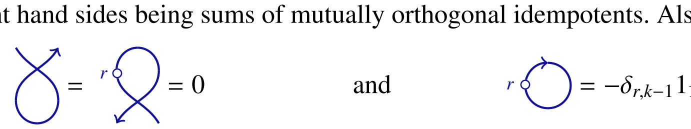
if $k < 0$. Then we set

 (2.8)

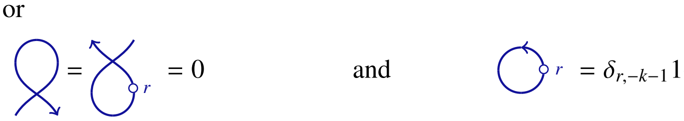
From these definitions, it follows that

 (2.9)

with the right hand sides being sums of mutually orthogonal idempotents. Also

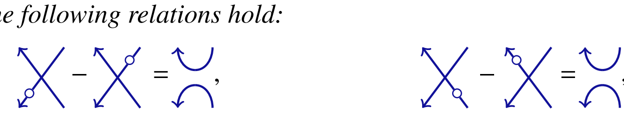
 (2.10)

if $0 \leqslant r < k$, or

 (2.11)

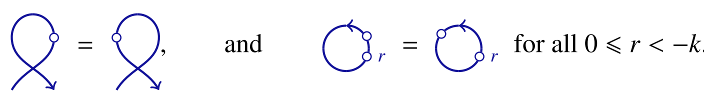
if $0 \leqslant r < -k$.

Lemma 2.2. *The following relations hold:*

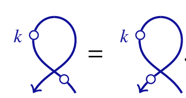
 (2.12)

 (2.13)

Proof. To prove (2.12), take the first equation from (2.5) describing how dots slide past rightward crossings, vertically compose on top and bottom with t', then simplify using (2.8)–(2.11). For (2.13), it suffices to prove the first equation, since the latter then follows on applying ω (recalling the heuristic for ω explained just before Lemma 2.1). If $k < 0$ we vertically compose on the bottom with the isomorphism $\uparrow \otimes \downarrow \oplus \mathbb{1}^{\oplus(-k)} \xrightarrow{\sim} \downarrow \otimes \uparrow$ from (1.5) to reduce to checking the following:

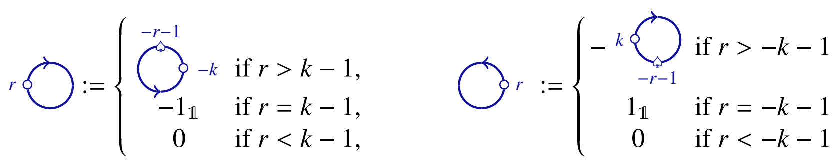


To establish the first identity here, commute the dot past the crossing on each side using (2.5), then use the vanishing of the curl from (2.11). The second identity follows using (2.2). Finally, we must prove the first equation from (2.13) when $k \geqslant 0$. In view of the definition of the leftward cap from (2.8), we must show equivalently that



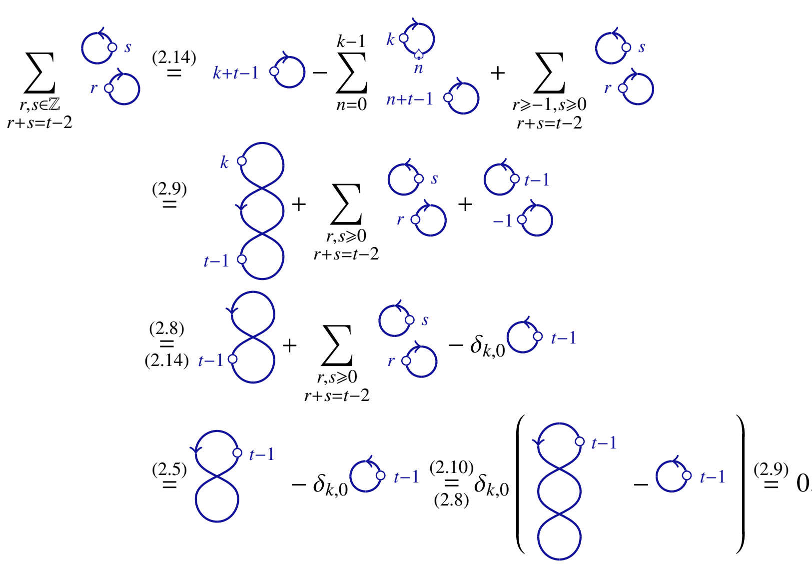
To see this, use (2.12) to commute the bottom dot past the crossing, then appeal to (2.3). ◻

We also give meaning to negatively dotted bubbles by making the following definitions for $r < 0$:


(2.14)

Lemma 2.3. *The infinite Grassmannian relations from Theorem 1.3(i) all hold.*

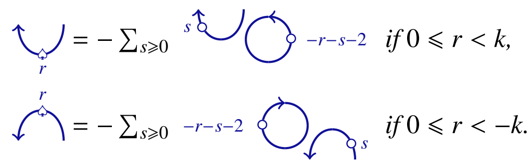
Proof. The equation (1.13) is implied by (2.10)–(2.11) and (2.14). For (1.14), we may assume using ω that $k \geqslant 0$. When $t = 0$ the result follows trivially using (1.13). When $t > 0$ we have:



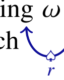
This implies (1.14). ◻

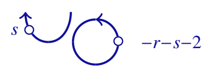
The next lemma expresses the decorated leftward cups and caps in terms of the undecorated ones. It means that we will not need to use the diamond notation again after this.

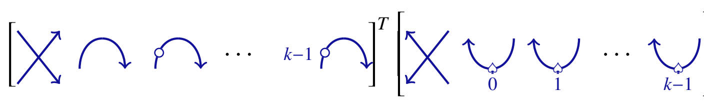
Lemma 2.4. *The following holds:*



(2.15)

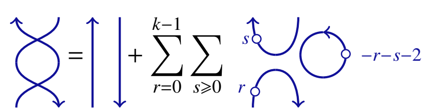
Proof. We explain the first equality; the second may then be deduced by applying ω using also (2.13). Remembering the definition (2.6), it suffices to show on replacing each  with

$-\sum_{s\geqslant 0}$  that the matrix product

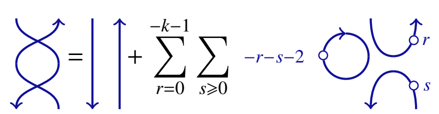


is the $(k+1)\times(k+1)$ identity matrix. This may be checked quite routinely using (2.9)–(2.10) and Lemma 2.3; cf. the proof of [3, Corollary 3.3] for a similar argument. □

If we substitute the formulae from Lemma 2.4 into (2.9), we obtain:



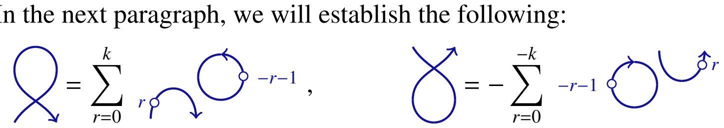
(2.16)



(2.17)

Lemma 2.5. *The curl relations from Theorem* 1.3*(iv) all hold.*

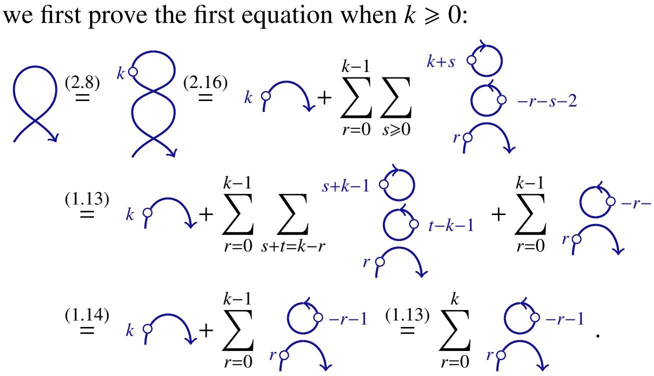
Proof. In the next paragraph, we will establish the following:



(2.18)

Then to obtain the curl relations in the form (1.17), take the dotted curls on the left hand side of those relations, use (1.2) to commute the dots past the upward crossing, convert the crossing to a rightward one using (1.3) and the definition of t, then apply (2.18).

For (2.18), we first prove the first equation when $k \geqslant 0$:



The first equation when $k < 0$ is immediate from (2.11). Then the second equation then follows by applying ω and using (2.2). □

The proofs of the next two lemmas are intertwined with each other.

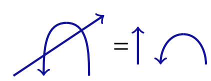
Lemma 2.6. *The following relations hold:*


(2.19)

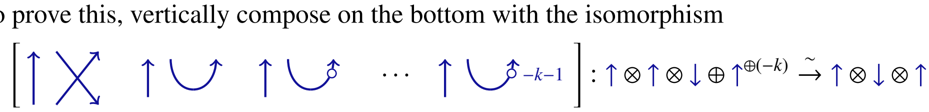

(2.20)

Proof. It suffices to prove the left hand equalities in (2.19)–(2.20); then the right hand ones follow by applying ω. In the next two paragraphs, we will prove the left hand equality in (2.19) assuming $k \leqslant 0$ and the left hand equality in (2.20) assuming $k > 0$.

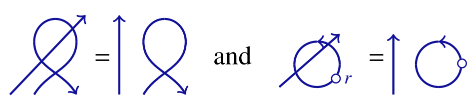
Consider (2.19) when $k \leqslant 0$. We claim that


(2.21)

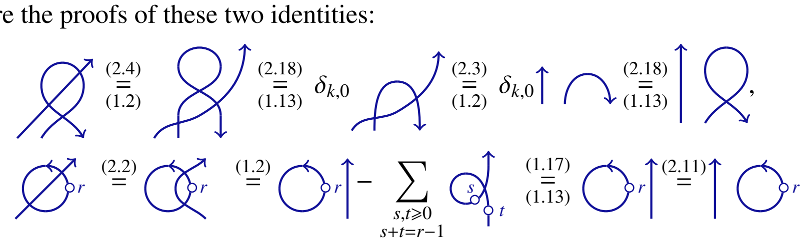
To prove this, vertically compose on the bottom with the isomorphism



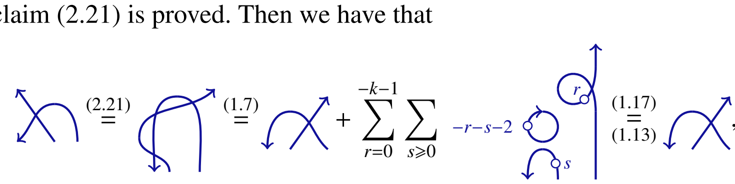
to reduce to showing equivalently that


for $0 \leqslant r < -k$.

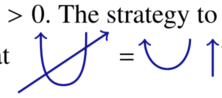
Here are the proofs of these two identities:

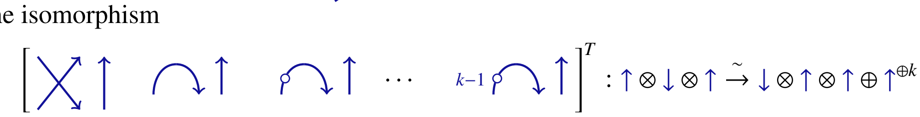


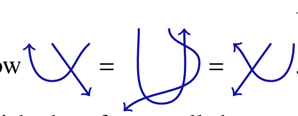
Thus, the claim (2.21) is proved. Then we have that



establishing (2.19).

Next consider (2.20) when $k > 0$. The strategy to prove this is the same as in the previous paragraph. One first verifies that  by vertically composing on the top with the isomorphism

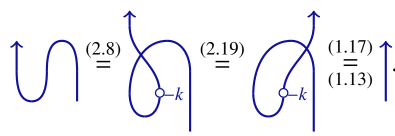


Then this can be used to show 

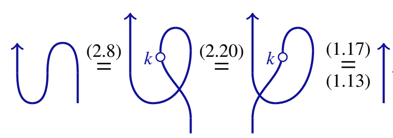
The partial results established so far are all that are needed to prove Lemma 2.7 below. To complete the proof of the present lemma, suppose first that $k > 0$. We take the left hand equality from (2.20) proved in the previous paragraph, attach leftward caps to the top left and top right strands, then simplify using the left adjunction relations to be established in Lemma 2.7. This establishes (2.19) for $k > 0$. Finally, (2.20) for $k \leqslant 0$ may be deduced from (2.19) by a similar procedure. ◻

Lemma 2.7. *The left adjunction relations from Theorem* 1.3*(ii) hold.*

Proof. As usual, it suffices to prove the first equality. If $k \leqslant 0$ then



If $k > 0$ then

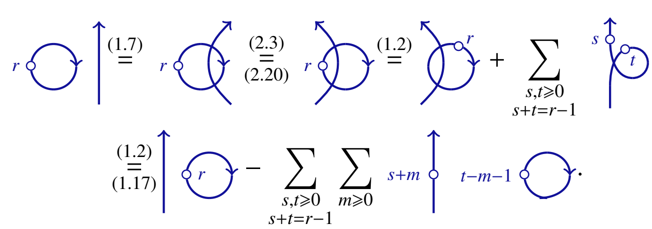


Note we have only used the parts of Lemma 2.6 that were already proved without forward reference to the present lemma. □

There are just two more relations to be checked; the arguments here are analogous to ones in [15, §3.1.2].

Lemma 2.8. *The bubble slide relations from Theorem* 1.3*(v) hold.*

Proof. We just explain the argument for $k \geqslant 0$; the case $k < 0$ is similar. We first prove (1.19). This is trivial for $r < 0$ due to (1.13), so we may assume that $r \geqslant 0$. Then we calculate:



This easily simplifies to the right hand side of (1.19).

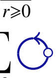
Now we deduce (1.18). Let u be an indeterminant and

$$e(u) := \sum_{r \geqslant 0} e_r u^{-r}, \qquad\qquad h(u) := \sum_{r \geqslant 0} h_r u^{-r} \tag{2.22}$$

be the generating functions for the elementary and complete symmetric functions. These are elements of $\mathrm{Sym}[[u^{-1}]]$ which satisfy $e(u)h(-u) = 1$. Lemma 2.3 implies that the homomorphism β defined after (1.21) satisfies

$$\beta(e(u)) = -\sum_{r \geqslant 0} \; r+k-1 \tag{2.23}$$

$$u^{-r},$$

$$\beta(h(-u)) = \sum_{r \geqslant 0}$$

$$r-k-1 \; u^{-r}. \tag{2.24}$$

Also let $p(u) := \sum_{r \geqslant 0} (r+1) x^r u^{-r-2}$, where x is the upward dot as usual. The identity (1.19) just proved asserts that

$$\beta(e(u)) \otimes 1_{\uparrow} = 1_{\uparrow} \otimes \beta(e(u)) - p(u) \otimes \beta(e(u)).$$

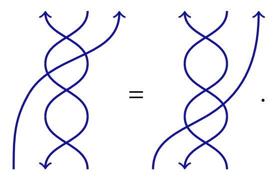
Multiplying on the left and right by $\beta(h(-u)) = \beta(e(u))^{-1}$, we deduce that

$$1_{\uparrow} \otimes \beta(h(-u)) = \beta(h(-u)) \otimes 1_{\uparrow} - \beta(h(-u)) \otimes p(u).$$

This is equivalent to (1.18). □

Lemma 2.9. *The alternating braid relation from Theorem* 1.3*(vi) holds.*

Proof. Again, we just sketch the argument when $k \geqslant 0$, since $k < 0$ is similar. The idea is to attach crossings to the top left and bottom right pairs of strands of the second equality of (2.4) to deduce that



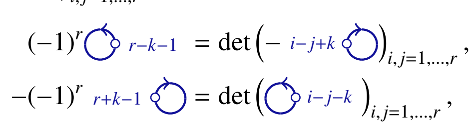
Now apply (1.6)–(1.7) to remove $t \circ t'$ and $t' \circ t$ on each side then simplify; along the way many bubbles and curls vanish thanks to (1.13) and (2.10). ◻

3. Proofs of Theorems

Proof of Theorem 1.2. We first establish the existence of c' and d' satisfying the relations (1.6)–(1.9). So let $\mathcal{H}eis_k$ be as in Definition 1.1. Define t' and the decorated leftward cups and caps from (2.6)–(2.7), then define c', d' and the negatively dotted bubbles by (2.8) and (2.14). We need to show that this t' and these negatively dotted bubbles are the same as the ones defined in the statement of Theorem 1.2. For t', this follows from (2.19) and the left adjunction relations (1.15) proved in Lemma 2.7. For the negatively dotted bubbles, the infinite Grassmannian relations (1.13)–(1.14) proved in Lemma 2.3 are all that are needed to construct the homomorphism β from (1.21). In the ring of symmetric functions, it is well known that

$$h_r = \det\left(e_{i-j+1}\right)_{i,j=1,\dots,r}. \tag{3.1}$$

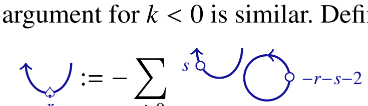
Hence, applying the automorphism of Sym that interchanges h_r and $(-1)^r e_r$, we get also $(-1)^r e_r = \det\left((-1)^{i-j+1} h_{i-j+1}\right)_{i,j=1,\dots,r}$. On applying β, this shows that



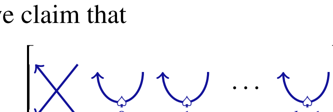
which easily simplify to produce the identities (1.11)–(1.12). Thus, we are indeed in the setup of Theorem 1.2. Now we get the relations (1.6)–(1.9) from (2.16)–(2.17), the infinite Grassmannian relations (1.13)–(1.14) proved in Lemma 2.3, and the curl relations (1.17) proved in Lemma 2.5.

Next let C be a strict monoidal category with generators x, s, c, d, c', d' subject to the relations (1.2)–(1.3) and (1.6)–(1.9). We have just demonstrated that all of these relations hold in $\mathcal{H}eis_k$, hence, there is a strict $\Bbbk$-linear monoidal functor $A : C \to \mathcal{H}eis_k$ taking objects $\uparrow, \downarrow$ and generating morphisms x, s, c, d, c', d' in C to the elements with the same names in $\mathcal{H}eis_k$.

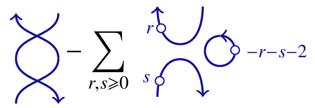
In the other direction, we claim that there is a strict $\Bbbk$-linear monoidal functor $B : \mathcal{H}eis_k \to C$ sending the generating objects $\uparrow, \downarrow$ and morphisms x, s, c, d in $\mathcal{H}eis_k$ to the elements with the same names in C; this will eventually turn out to be a two-sided inverse to A. To prove the claim, we must verify that the three sets of defining relations of $\mathcal{H}eis_k$ hold in C. It is immediate for (1.2) and (1.3), so we are left with checking the inversion relation. We just do this in case $k \geqslant 0$, since the argument for $k < 0$ is similar. Defining the new morphisms



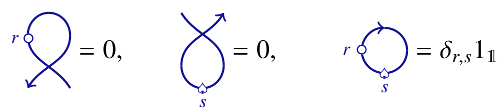
in C for $r = 0, 1, \dots, k-1$, we claim that



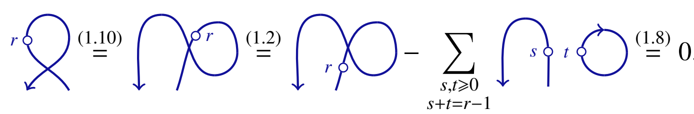
is the the two-sided inverse of the morphism (1.4). Composing one way round gives the morphism



which is the identity by the relation (1.6) in $\mathcal{C}$. The other way around, we get a $(k+1)\times(k+1)$-matrix. Its 1, 1-entry is the identity by (1.7). This is all that is needed when $k = 0$, but when $k > 0$ we also need to verifying the following for $r, s = 0, 1, \ldots, k-1$:

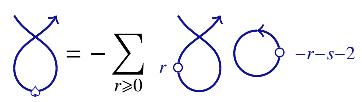


Here is the proof of the first of these for $r = 0, 1, \ldots, k-1$:



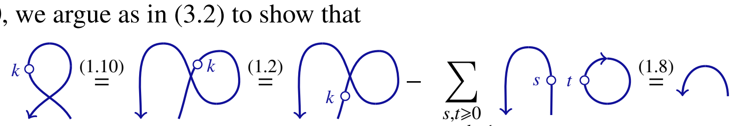
(3.2)

To prove the second, note by definition for $s = 0, 1, \ldots, k-1$ that

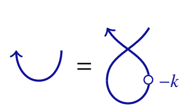


By the definition (1.11), the dotted bubble here is zero if $r \geqslant k$, while for $r = 0, 1, \ldots, k-1$ the dotted curl is zero by a similar argument to (3.2). For the final relation involving the decorated dotted bubble, define $\beta : \mathrm{Sym} \to \mathrm{End}_{\mathcal{C}}(\mathbb{1})$ by sending $e_r \mapsto -$ $r{+}k{-}1$ for each $r \geqslant 0$. Then by (3.1), we have that $\beta((-1)^r h_r) = \det\big($ $i{-}j{+}k$ $\big)_{i,j=1,\ldots,r}$. Assuming $r \leqslant k$, this is exactly the definition of $r{-}k{-}1$ from (1.11). Now suppose that $0 \leqslant r, s < k$. Applying β to the symmetric function identity $\sum_{t=0}^{k-s-1}(-1)^{k-s-1-t} e_{r-k+1+t} h_{k-s-1-t} = \delta_{r,s}$ and using (1.8) shows that $-\sum_{t=0}^{k-s-1}$ $r{+}t$ $-s{-}t{-}2$ $= \delta_{r,s} 1_{\mathbb{1}}$, which is exactly the identity we need. This proves the claim, so the functor B is well-defined.

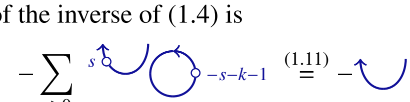
Next we check that c' and d' are the *unique* morphisms in $\mathcal{C}$ satisfying the relations (1.6)–(1.9). We do this by using the assumed relations to derive expressions for c' and d' in terms of the other generators. Note by the claim in the previous paragraph that the leftward crossing t' may be characterized as the first entry of the inverse of the morphism (1.4) when $k \geqslant 0$; similarly, it is the first entry of the inverse of the morphism (1.5) when $k < 0$. This shows that t' does not depend on the values of c' and d' (despite being defined in terms of them). Then, when $k \geqslant 0$, we argue as in (3.2) to show that



This establishes the uniqueness of d' when $k \geqslant 0$. Similarly, using (1.9) in place of (1.8), one gets that



when $k \leqslant 0$, hence, c' is unique when $k \leqslant 0$. It remains to prove the uniqueness of c' when $k > 0$ and of d' when $k < 0$. In the case that $k > 0$, the claim from the previous paragraph shows that the last entry of the inverse of (1.4) is



Hence, c' is unique when $k > 0$. The uniqueness of d' when $k < 0$ is proved similarly.

Now we can complete the proof of the theorem. First we show that C and $\mathcal{H}eis_k$ are isomorphic, thereby establishing the equivalent presentation from the statement of the theorem. To see this, we check that the functors A and B are two-sided inverses. We have that $A \circ B = \mathrm{Id}_{\mathcal{H}eis_k}$ obviously. To see that $B \circ A = \mathrm{Id}_C$, it is clear that $B \circ A$ is the identity on the generating morphisms x, s, c, d, and follows on the morphisms c', d' by the uniqueness established in the previous paragraph. Finally, since $\mathcal{H}eis_k \cong C$, the uniqueness of c' and d' established in the previous paragraph implies they are also the unique morphisms in $\mathcal{H}eis_k$ satisfying (1.6)–(1.9), and we are done. $\square$

Proof of Theorem 1.3. Parts (i), (ii), (iv), (v) and (vi) are proved in Lemmas 2.3, 2.7, 2.5, 2.8 and 2.9, respectively. Part (iii) for dots follows from (2.13), while for crossings it is an easy consequence of the "pitchfork relations" from Lemma 2.6 (combined with the adjunction relations). $\square$

Proof of Theorem 1.4. We first explain the identification with Khovanov's category $\mathcal{H}$ from [14, §2.1]; this also follows from the more general identification with the Mackaay-Savage category made in the next paragraph together with [18, Remark 2.10] but it seems helpful to treat this important special case independently. So assume that $k = -1$. Theorem 1.2 gives a presentation of $\mathcal{H}eis_{-1}$ with generating morphisms x, s, c, d, c' and d'. Comparing the relations (1.2)–(1.3) and (1.6)–(1.9) with the local relations in Khovanov's definition, we see that there is a strict monoidal functor $\mathcal{H}eis_{-1} \to \mathcal{H}$ sending $\uparrow$ and $\downarrow$ to Khovanov's objects $\uparrow = Q_+$ and $\downarrow = Q_-$, s, c, d, c' and d' to the morphisms in Khovanov's category represented by the same diagrams, and x to the right curl. This functor sends $=$ to the figure-of-eight, which is zero since it involves a left curl. Hence, our functor factors through the specialization to induce a functor from the additive envelope of $\mathcal{H}eis_{-1}(0)$ to $\mathcal{H}$. To see that this functor is an isomorphism, we construct its two-sided inverse. This sends any diagram representing a morphism in Khovanov's category to the morphism in the additive envelope of $\mathcal{H}eis_{-1}(0)$ encoded by the same diagram. It is well-defined since all of Khovanov's local relations hold in $\mathcal{H}eis_{-1}(0)$, and also we have shown in Theorem 1.3 that $\mathcal{H}eis_{-1}(0)$ is strictly pivotal (something which is required implicitly in Khovanov's definition).

For arbitrary $k \leqslant -1$, the identification of $\mathcal{H}eis_k(\delta)$ with the Mackaay-Savage category $\tilde{\mathcal{H}}^\lambda$ follows by a very similar argument. Let $\lambda = \sum_i \lambda_i \omega_i$ be a dominant weight (in the notation of [18]), and set $k := -\sum_i \lambda_i$ and $\delta := \sum_i i\lambda_i$. In one direction, the monoidal isomorphism from the additive envelope of $\mathcal{H}eis_k(\delta)$ to $\tilde{\mathcal{H}}^\lambda$ sends our x, s, c, d, c' and d' to the morphisms in [18] denoted by the same diagrams. The morphism denoted c_n in [18, (2.1)] for $0 \leqslant n \leqslant -k$ is our $-\ {\scriptstyle n+k-1}$, thanks to the definition of negatively dotted clockwise bubble at the end of Theorem 1.2. Using this, it is straightforward to check that the local relations in [18, (2.2)–(2.9)] agree with the defining relations for $\mathcal{H}eis_k(\delta)$ from (1.2)–(1.3) and (1.6)–(1.9). Finally, $\mathcal{H}eis_k$ is strictly pivotal, which again is required implicitly in the approach of [18]. $\square$

Proof of Theorem 1.6. By induction on the number of crossings, one checks using the relations established in §2 that any diagram representing a morphism $\theta \in \mathrm{Hom}_{\mathcal{H}eis_k}(X, Y)$ can be written as a Sym-linear combination of morphisms in $B_{\infty,\infty}(X, Y)$ with the same or fewer crossings. So $B_{\infty,\infty}(X, Y)$ spans $\mathrm{Hom}_{\mathcal{H}eis_k}(X, Y)$. The problem is to prove it is also linearly independent. This is done already in the case $k = 0$ in [4, Theorem 1.2]. When $k < 0$, we will explain how to deduce it from [18, Proposition 2.16] in the next paragraph. Then it follows for $k > 0$ by applying the isomorphism ω from Lemma 2.1.

So assume henceforth that $k < 0$. In order to make an observation about base change, let us add a superscript $\mathcal{H}eis_k^{\Bbbk}$ to indicate the ground ring: it suffices to establish linear independence for $\mathcal{H}eis_k^{\mathbb{Z}}$; then one can obtain the linear independence for arbitrary $\Bbbk$ by using the obvious functor $\mathcal{H}eis_k^{\Bbbk} \to \mathcal{H}eis_k^{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$. Thus we are reduced to the case that $\Bbbk = \mathbb{Z}$. Suppose we are given some linear relation

$$\sum_{\theta \in B_{\infty,\infty}(X,Y)} p_\theta \theta = 0$$

for $p_\theta \in \mathrm{Sym}$. Take any dominant integral weight λ for $\mathfrak{sl}_\infty$ with $k = -\sum_i \lambda_i$, and set $\delta := \sum_i i\lambda_i$. By Theorem 1.4, the specialized category $\mathcal{H}eis_k(\delta)$ embeds into the Mackaay-Savage category $\tilde{\mathcal{H}}^\lambda$ over ground ring $\mathbb{Z}$. So we can appeal to [18, Proposition 2.16] to deduce that $B_{\infty,\infty}(X,Y)$ is a basis for $\mathrm{Hom}_{\mathcal{H}eis_k(\delta)}(X,Y)$ as a free right module over Sym specialized at $e_1 = -\delta$. We deduce that $p_\theta|_{e_1=-\delta} = 0$ for each θ. Since there are infinitely many possibilities for δ as λ varies (keeping $k < 0$ fixed), this is enough to show that all p_θ are zero. □

Proof of Theorem 1.7. Noting that $H_0^f \cong \Bbbk$, we denote the one-dimensional H_0^f-module also by $\Bbbk$. As $f(x_1) = 0$ in H_1^f, the functor $\mathrm{Ev} \circ \Psi_f$ sends $f(x)$ to zero, hence, it factors through the quotient category $\mathcal{H}eis_{f,1}$ of $\mathcal{H}eis_k$. Since $\Bbbk$ is a projective H_0^f-module and the induction and restriction functors are biadjoint, it follows that $\mathrm{Ev} \circ \Psi_f$ has image contained in the full subcategory $\bigoplus_{n \geqslant 0} H_n^f$-pmod of $\bigoplus_{n \geqslant 0} H_n^f$-mod. This subcategory is additive and Karoubian, hence, the functor $\mathcal{H}eis_{f,1} \to \bigoplus_{n \geqslant 0} H_n^f$-pmod constructed so far extends to the functor ψ_f on $\mathrm{Kar}(\mathcal{H}eis_{f,1})$ from the statement of the theorem.

Now take $n \geqslant 0$. The functor ψ_f maps $\uparrow^{\otimes n}$ to $(\mathrm{Ind}_{n-1}^n \circ \cdots \circ \mathrm{Ind}_0^1)\Bbbk = H_n^f$, hence, it defines an algebra homomorphism

$$\psi_n : \mathrm{End}_{\mathcal{H}eis_{f,1}}(\uparrow^{\otimes n})^{\mathrm{op}} \to \mathrm{End}_{H_n^f}(H_n^f)^{\mathrm{op}} \equiv H_n^f. \tag{3.3}$$

We claim that ψ_n is actually an algebra isomorphism. To see this, note by the relations that there is a homomorphism

$$\begin{aligned} \phi_n : H_n^f &\to \mathrm{End}_{\mathcal{H}eis_{f,1}}(\uparrow^{\otimes n})^{\mathrm{op}}, \\ x_i &\mapsto (1_\uparrow)^{\otimes(n-i)} \otimes x \otimes (1_\uparrow)^{\otimes(i-1)}, \\ s_j &\mapsto (1_\uparrow)^{\otimes(n-j-1)} \otimes s \otimes (1_\uparrow)^{\otimes(j-1)}. \end{aligned} \tag{3.4}$$

Now we observe that bubbles on the right edge are scalars in $\mathrm{End}_{\mathcal{H}eis_{f,1}}(\uparrow^{\otimes n})^{\mathrm{op}}$. This is straightforward to prove directly at this point, but it also follows from the more general statement made in the last part of Lemma 1.8; the proof of that given below is independent of the present theorem. Hence, the easy spanning part of Theorem 1.6 implies that ϕ_n is surjective. Also $\psi_n \circ \phi_n = \mathrm{Id}_{H_n^f}$ as the two sides agree on generators. These two facts combined show that ψ_n and ϕ_n are two-sided inverses, and we have proved the claim.

By the claim, for any primitive idempotent $e \in H_n^f$, there is a corresponding idempotent $e \in \mathrm{End}_{\mathcal{H}eis_{f,1}}(\uparrow^{\otimes n})$ defining an object $(\uparrow^{\otimes n}, e) \in \mathrm{Kar}(\mathcal{H}eis_{f,1})$ which maps to $H_n^f e$ under the functor ψ_f. This shows that the functor ψ_f is dense. It remains to show that it is full and faithful. To see this, it suffices to take words $X = X_1 \otimes \cdots \otimes X_r$ and $Y = Y_1 \otimes \cdots \otimes Y_r$ in the letters $\uparrow$ and $\downarrow$ representing objects of $\mathcal{H}eis_{f,1}$ such that

$$n := \#\{i \mid X_i = \uparrow\} - \#\{i \mid X_i = \downarrow\} = \#\{j \mid Y_j = \uparrow\} - \#\{j \mid Y_j = \downarrow\},$$

and show that $\psi_f : \mathrm{Hom}_{\mathcal{H}eis_{f,1}}(X,Y) \to \mathrm{Hom}_{H_n^f}(\psi_f(X), \psi_f(Y))$ is an isomorphism. To prove this, we first reduce to that case that $X = \mathbb{1}$ using the following commutative diagram, whose horizontal maps are the canonical isomorphisms coming from adjunction/duality:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{H}eis_{f,1}}(X,Y) \arrow[r, "\sim"] \arrow[d, "\psi_f"'] & \mathrm{Hom}_{\mathcal{H}eis_{f,1}}(\mathbb{1}, X^* \otimes Y) \arrow[d, "\psi_f"] \\ \mathrm{Hom}_{H_n^f}(\psi_f(X), \psi_f(Y)) \arrow[r, "\sim"] & \mathrm{Hom}_{H_0^f}(\Bbbk, \psi_f(X^* \otimes Y)). \end{tikzcd} \tag{3.5}$$

Assume henceforth that $X = \mathbb{1}$. We then proceed by induction on the length s of Y, the case $s = 0$ following since ψ_0 is an isomorphism. If $s > 0$, then at least one letter Y_i of Y must equal $\downarrow$. If $i = s$, i.e., the letter $\downarrow$ is on the right, then $Y \cong \mathbf{0}$ as $1_\downarrow = 0$ in $\mathcal{H}eis_{f,1}$, and the conclusion is trivial. Otherwise, we may assume that $Y_i = \downarrow$ and $Y_{i+1} = \uparrow$ for some $i < s$. Let Y' be Y with these two letters interchanged and Y'' be Y with these two letters removed. Using the induction hypothesis and the following commutative diagram, whose horizontal maps are the canonical isomorphisms coming from (1.5), we see that the conclusion follows for Y if we can prove it for Y':

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{H}eis_{f,1}}(\mathbb{1}, Y) \arrow[r, "\sim"] \arrow[d, "\psi_f"'] & \mathrm{Hom}_{\mathcal{H}eis_{f,1}}(\mathbb{1}, Y' \oplus Y''^{\oplus(-k)}) \arrow[d, "\psi_f"] \\ \mathrm{Hom}_{H_0^f}(\mathbb{k}, \psi_f(Y)) \arrow[r, "\sim"] & \mathrm{Hom}_{H_0^f}(\mathbb{1}, \psi_f(Y') \oplus \psi_f(Y'')^{\oplus -k}). \end{tikzcd} \tag{3.6}$$

Repeating in this way, we can move the letter $\downarrow$ of Y to the right, and then we are done as before. □

Proof of Lemma 1.8. Suppose that

$$f(u) = u^\ell + z_1 u^{\ell-1} + \cdots + z_\ell, \qquad f'(u) = u^{\ell'} + z'_1 u^{\ell'-1} + \cdots + z'_{\ell'},$$

for $z_1, \dots, z_\ell, z'_1, \dots, z'_{\ell'} \in \mathbb{k}$. Also set $z_0 = z'_0 := 1$.

We first show that $\mathcal{I}_{f,f'}$ contains $\circlearrowleft^{r-k-1} - \delta_r 1_{\mathbb{1}}$ for all $r \geqslant 0$. Proceed by induction on r. If $r \leqslant \ell'$, we are done by the definition of $\mathcal{I}_{f,f'}$, so assume that $r > \ell'$. By (1.25), $u^k \delta(u) f(u) = f'(u)$, which is a polynomial in u. Hence, its $u^{\ell'-r}$-coefficient is zero. This shows that

$$\sum_{s=0}^{\ell} z_s \delta_{r-s} = 0. \tag{3.7}$$

Since $r - k - 1 = \ell + r - \ell' - 1 \geqslant \ell$, we can use $x^\ell + z_1 x^{\ell-1} + \cdots + z_\ell \in \mathcal{I}_{f,f'}$ to deduce that $\sum_{s=0}^{\ell} z_s \circlearrowleft^{r-s-k-1} \in \mathcal{I}_{f,f'}$. Then by induction we get that

$$\circlearrowleft^{r-k-1} - \delta_r 1_{\mathbb{1}} = \circlearrowleft^{r-k-1} + \sum_{s=1}^{\ell} z_s \delta_{r-s} 1_{\mathbb{1}} \equiv \sum_{s=0}^{\ell} z_s \circlearrowleft^{r-s-k-1} \equiv 0$$

modulo $\mathcal{I}_{f,f'}$, as required.

Next, let $e(u), h(u) \in \mathrm{Sym}[[u^{-1}]]$ be the power series from (2.22). The previous paragraph and (2.24) shows that $\beta(h(-u)) \equiv \delta(u) 1_{\mathbb{1}} \pmod{\mathcal{I}_{f,f'}}$. Since $e(u) = h(-u)^{-1}$ and $\delta'(u) = -\delta(u)^{-1}$, it follows that $\beta(e(u)) \equiv -\delta'(u) 1_{\mathbb{1}} \pmod{\mathcal{I}_{f,f'}}$. In vew of (2.23), this shows that $\mathcal{I}_{f,f'}$ contains ${}^{r+k-1}\circlearrowright - \delta'_r 1_{\mathbb{1}}$ for all $r \geqslant 0$.

Now we can show that $f'(x') \in \mathcal{I}_{f,f'}$. By (1.25), $z'_r = \sum_{s=0}^{\ell} z_s \delta_{r-s}$. So

$$f'(x') = \sum_{r=0}^{\ell'} z'_r \downarrow^{\ell'-r} = \sum_{r=0}^{\ell'} \sum_{s=0}^{\ell} z_s \delta_{r-s} \downarrow^{\ell'-r} \equiv \sum_{s=0}^{\ell} z_s \sum_{r=0}^{\ell'} \downarrow^{\ell'-r} \circlearrowleft^{r-s-k-1}$$

$$\overset{(2.14)}{=} \sum_{s=0}^{\ell} z_s \sum_{r \geqslant 0} \downarrow^{r} \circlearrowleft^{\ell-s-1-r} \overset{(1.17)}{=} \sum_{s=0}^{\ell} z_s \downarrow\!\circlearrowleft^{\ell-s} \equiv 0 \pmod{\mathcal{I}_{f,f'}}.$$

So far, we have shown that the left tensor ideal generated by $f(x)$ and $\circlearrowleft^{r-k-1} - \delta_r 1_{\mathbb{1}}$ for $r = 1, \dots, \ell'$ contains $f'(x')$ and ${}^{r+k-1}\circlearrowright - \delta'_r 1_{\mathbb{1}}$ for $r = 1, \dots \ell$. Similar argument shows that the left tensor ideal generated by the latter elements contains the former elements. This proves the lemma. □

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