ODD GRASSMANNIAN BIMODULES AND DERIVED EQUIVALENCES FOR SPIN SYMMETRIC GROUPS

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Abstract. We prove odd analogs of results of Chuang and Rouquier on $\mathfrak{sl}_2$-categorification. Combined also with recent work of the second author with Livesey, this allows us to complete the proof of Broué’s Abelian Defect Conjecture for the double covers of symmetric groups. The article also develops the theory of odd symmetric functions initiated a decade ago by Ellis, Khovanov and Lauda. A key role in our approach is played by a 2-supercategory consisting of odd Grassmannian bimodules over superalgebras which are odd analogs of equivariant cohomology algebras of Grassmannians. This is the odd analog of the category of Grassmannian bimodules which was at the heart of Lauda’s independent approach to categorification of $\mathfrak{sl}_2$.

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1. Introduction

This paper establishes “odd” analogs of results of Chuang and Rouquier [CR]. The motivating problem is to prove Broué’s Abelian Defect Group Conjecture for the double covers of symmetric groups. In the ordinary even theory, Broué’s conjecture for symmetric groups was proved in two steps. First came the work of Chuang and Kessar [CK], which established a Morita equivalence reducing the proof of Broué’s conjecture to proving that all blocks of symmetric groups in characteristic $p > 0$ with the same defect are derived equivalent. Then the second part of the proof came in Chuang and Rouquier’s work which deduced this assertion from a special case of a remarkable general theory of $\mathfrak{sl}_2$-categorification. Their theory has had many other significant applications and generalizations, especially following the

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works of Rouquier [R1] and Khovanov and Lauda [KL], which upgraded from \( \mathfrak{sl}_2 \) to an arbitrary symmetrizable Kac-Moody algebra \( \mathfrak{g} \).

The analogous story for the double covers of symmetric groups has an equally long history, being initiated of course by Schur soon after the ordinary representation theory of symmetric groups was worked out. In [BK], we uncovered a connection in the same spirit as Grojnowski’s work [G]—an important predecessor of [CR]—between modular representations of spin symmetric groups in odd characteristic \( p = 2l + 1 \) and the Kac-Moody group of type \( A^{(2)}_{2l} \). A few years later, the odd theory was given new life by work of Ellis, Khovanov and Lauda [EK, EKL, EL, E], whose motivation came from the completely different direction of the categorification program related to odd Khovanov homology. They developed a substantial theory of odd symmetric functions which plays a key role in this article. Soon after the work of Ellis, Khovanov and Lauda, a major breakthrough was made in work of Kang, Kashiwara, Oh and Tsuchioka [KKT, KKO1, KKO2]. They introduced so-called quiver Hecke superalgebras, which are the odd analogs of the Khovanov-Lauda-Rouquier algebras that underpin all current approaches to categorification of Kac-Moody algebras. In fact, quiver Hecke superalgebras categorify the positive part of the so-called covering quantum groups, which were defined and studied separately by Clark, Wang and Hill [CW, CHW, CHW2, C]. Then there was a lull in activity, until work of the first author with Ellis [BE2] which simplified the definition of the odd analog of the 2-category associated to \( \mathfrak{sl}_2 \) made originally by Ellis and Lauda [EL] and extended it to other odd Kac-Moody types. Recently, Dupont, Ebert and Lauda [DEL] have proved that the odd \( \mathfrak{sl}_2 \) 2-category from [BE2] is non-degenerate, but this is still an open problem for other odd types.

It has in fact been expected for long time that there should exist a comprehensive odd analog of the Chuang and Rouquier theory, and that this should play a role in constructing the derived equivalences required to prove Broué Conjecture for spin symmetric groups. However, due in part to the lack of an appropriate analog of the first part of the proof for symmetric groups—the part provided by the work of Chuang and Kessar—it was not investigated seriously until now. This analog has recently been established, in work of the second author with Livesey [KLi], and is in fact highly non-trivial. The arguments in [KLi] depend essentially on the Morita equivalence between cyclotomic quiver Hecke superalgebras and group algebras of spin symmetric groups constructed in [KKT], and also rely on the new approach to the study of RoCK blocks developed by the second author and Evseev in [EvK].

This article completes the second step of this program for spin symmetric groups. In order to do this, one needs to be able to compute explicitly with some realization of the categorification of the analog \( V(-\ell) \) of the \( \mathfrak{sl}_2 \)-module of highest weight \( \ell \) for the covering quantum group \( U_{q,p}(\mathfrak{sl}_2) \). We do this in this article by developing a non-trivial theory of odd Grassmannian bimodules. These are bimodules over pairs of algebras which we refer to as the odd equivariant Grassmannian cohomology algebras since they are analogous to the \( GL_n(\mathbb{C}) \)-equivariant cohomology algebras of the usual Grassmannian of \( n \)-dimensional subspaces of \( \mathbb{C}^n \). The specialized versions of these algebras with the word “equivariant” removed were worked out already by Ellis, Khovanov and Lauda [EKL], but the generalization to the equivariant setting is not obvious due to the non-commutativity of the algebra \( O\text{Sym} \) of odd symmetric functions. The definition of odd equivariant cohomology algebras—which are purely algebraic in nature rather than coming from any known cohomology theory—is given in Definition 7.1, and then the all-important category \( O\mathfrak{g}\text{Bim}_\ell \) of bimodules over these algebras is introduced in Definition 9.8. The most important fact about this, its rigidity, is established in Theorem 9.11 and Corollary 9.13.

With this theory in place, in Definition 10.2, we are able to write down the odd analog of the singular Rouquier complex in the category \( O\mathfrak{g}\text{Bim}_\ell \), proving the necessary homological properties of this needed to be able to obtain derived equivalences between the module categories over Grassmannian cohomology algebras. After that, we digress to explain the relationship between the odd \( \mathfrak{sl}_2 \) 2-category \( \hat{\mathfrak{U}}(\mathfrak{sl}_2) \) from [EL, BE2] (Definition 11.1) and the category \( O\mathfrak{g}\text{Bim}_\ell \), namely, there is a 2-functor from the
former to the latter. This is the odd analog of the main result about the ordinary \( \mathfrak{sl}_2 \) 2-category obtained by Lauda in [L1, L2]. We use this 2-functor to give a self-contained proof of the non-degeneracy of the odd \( \mathfrak{sl}_2 \) 2-category, as has recently been proved via “rewriting theory” in [DEL]. However, for reasons related to Remark 7.9, we found it was necessary at this point to pass to a reduced version \( \hat{\mathfrak{U}}(\mathfrak{sl}_2) \) of the odd \( \mathfrak{sl}_2 \) 2-category in which the odd bubble is specialized to zero; see Definition 11.2. This is perfectly adequate for all applications that we are aware of, but it means we actually prove a slightly weaker version of non-degeneracy than the result established already in [DEL].

In Section 12, we develop some of the basic theory of 2-representations of the odd \( \mathfrak{sl}_2 \) 2-category, following [R1] quite closely. This is applied in the next section to prove Theorem 13.5, which may be paraphrased as follows:

**Theorem.** The bounded homotopy category \( K^b(V) \) of any integrable Karoubian 2-representation \( V \) of \( \hat{\mathfrak{U}}(\mathfrak{sl}_2) \) admits an auto-equivalence categorifying the action of the simple reflection in the associated Weyl group.

In the final Section 14, we apply this to establish the key derived equivalences between blocks of double covers of symmetric and alternating groups predicted by Kessar and Schaps [KS]. In fact, we establish derived equivalences between the corresponding cyclotomic quiver Hecke superalgebras of type \( A_n^{(2)} \), which were shown to be Morita equivalent to spin blocks of symmetric groups up to Clifford twists in [KKT]. Our arguments here also rest crucially on the results of [KKO1, KKO2] in order to check that representations of cyclotomic quiver Hecke superalgebras do admit the structure of a super Kac-Moody 2-representation. Combined with the results in [KLi], this is sufficient to complete the proof of the Broué Conjecture for double covers of symmetric and alternating groups.

We would finally like to discuss some significant overlaps between the results of this article and the independent work of Ebert, Lauda and Vera [ELV]. Their work also introduces the odd equivariant Grassmannian cohomology algebras studied here, relating them to deformed odd cyclotomic nilHecke algebras in exactly the same way as in Theorem 8.2 and Remark 8.10 below, and they also establish the derived equivalences necessary to complete the proof of Broué’s Conjecture for spin symmetric groups. In fact, they prove a more general result showing that the odd Rickard complex, which induces the auto-equivalences in the above theorem, is invertible in a formal completion of the homotopy category of the graded super Karoubi envelope of \( \hat{\mathfrak{U}}(\mathfrak{sl}_2) \) itself. We view their general approach as complementary to ours, and it is reassuring to have an independent proof of this difficult place in the theory.

The strategy adopted by Ebert, Lauda and Vera is modelled on Vera’s new treatment of derived equivalences in the ordinary even case developed in [V]. In particular, it uses a version of the results of Kang, Kashiwara and Oh [KKO1, KKO2] to construct the universal categorification of the \( U_{q,\pi}(\mathfrak{sl}_2) \)-module \( V(-\ell) \) in terms of representations of odd deformed cyclotomic nil-Hecke algebras. This is the place where we use instead the theory of odd Grassmannian bimodules developed in this article, making our article more self-contained but also considerably more technical.

It is also interesting to compare our proof of Theorem 10.3, which establishes the key homological result about the odd singular Rouquier complex, with the analogous result in their paper. In fact, we consider the exact analog of the Rickard complex studied in [CR], whereas Ebert, Lauda and Vera work with the dual complex which defines the inverse equivalence as in [R1]. The argument in [ELV] is based on an explicit computation in the odd deformed cyclotomic nil-Hecke algebra, where the necessary adjunctions are closer to the surface compared to the ones we construct in Theorem 9.11. Although equivalent, their method turns out to be less involved combinatorially than ours, which uses some quite subtle arguments involving Pieri formulae and Littlewood-Richardson coefficients for odd Schur polynomials.
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General conventions. We will work always over an algebraically closed field $\mathbb{F}$ of characteristic different from 2. All categories, functors, etc., are assumed to be $\mathbb{F}$-linear without further comment. The symbol $\otimes$ with no additional subscript denotes tensor product over $\mathbb{F}$. We use the shorthand $X \in C$ to indicate that $X$ is an element of the object set of a category $C$.

Let $\Lambda^+$ be the set of all partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$. We adopt standard notations such as $\lambda^T = (\lambda_1^T, \lambda_2^T, \ldots)$ for the transpose partition and $ht(\lambda)$ for the number $\lambda_1^T$ of non-zero parts. The usual dominance ordering is denoted $\leq$. The lexicographic ordering $\leq_{\text{lex}}$ is a total order refining $\leq$. We use the English convention for Young diagrams and tableaux, so rows and columns are indexed like for $\lambda$.

The longest element of $S_n$ is an element of the object set of a category $S$. Let $\Lambda^+_{\max} = \{ \lambda \in \Lambda^+ \mid ht(\lambda) \leq n \}$ be the set of partitions of height at most $n$ and $\Lambda^+_{\max}$ be the set of partitions $\lambda$ whose Young diagram fits into an $m \times n$ rectangle, i.e. $ht(\lambda) \leq m$ and $\lambda_1 \leq n$. Note that $|\Lambda^+_{\max}| = \binom{m+n}{n}$.

For $\lambda \in \Lambda^+$, the following will be needed in various formulae for signs, following [E, Sec. 2.2]:

- $N(\lambda)$ is the number of pairs $(A, B)$ such that $B$ is north of $A$ (strictly above and strictly to the right);
- $NE(\lambda)$ is the number of pairs of boxes $(A, B)$ such that $B$ is northeast of $A$ (strictly above and strictly to the right);
- $dN(\lambda)$ is the number of pairs $(A, B)$ of boxes in the Young diagram of $\lambda$ such that $B$ is due north of $A$ (strictly above in the same column);
- $dW(\lambda)$ is the number of pairs of boxes $(A, B)$ in the Young diagram such that $B$ is due west of $A$ (strictly to the left in the same row).

Some equivalent definitions: $N(\lambda) = \sum_{1 \leq i < j \leq ht(\lambda)} \lambda_i \lambda_j$; $dN(\lambda) = \sum_{1 \leq i < j \leq ht(\lambda)} (i-1) \lambda_i$; $dW(\lambda) = \sum_{i \geq 1} \binom{i}{2}$.

For $\lambda \in \Lambda^+$, the following will be needed in various formulae for signs, following [E, Sec. 2.2]:

- $N(\lambda)$ is the number of pairs $(A, B)$ such that $B$ is north of $A$ (strictly above and strictly to the right);
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- $dN(\lambda)$ is the number of pairs $(A, B)$ of boxes in the Young diagram of $\lambda$ such that $B$ is due north of $A$ (strictly above in the same column);
- $dW(\lambda)$ is the number of pairs of boxes $(A, B)$ in the Young diagram such that $B$ is due west of $A$ (strictly to the left in the same row).

Let $\Lambda(k, n)$ be the set of all compositions of $n$ with $k$ parts, that is, $k$-tuples $a = (a_1, \ldots, a_k)$ of non-negative integers such that $|a| := a_1 + \cdots + a_k = n$. Let $N(a) := \sum_{1 \leq i < j < k} a_i a_j$ (like for partitions). The reversed composition is $a^\leq := (a_k, \ldots, a_1)$. Also $a \sqcup b$ denotes concatenation of compositions $a$ and $b$.

We denote the symmetric group by $S_n$ acting on the left on $[1, \ldots, n]$. The $i$th basic transposition is $s_i = (i \ i+1)$ and $\ell : S_n \rightarrow \mathbb{N}$ is the associated length function. We use the notation $w_n$ to denote the longest element of $S_n$ of length $\ell(w_n) = \binom{n}{2}$. We will often use the identities

$$\binom{r+s}{2} = \binom{r}{2} + \binom{s}{2} + rs, \quad \binom{-r+1}{2} = \binom{r+1}{2}$$

for $r, s \in \mathbb{Z}$. For $a \in \Lambda(k, n)$, there is a corresponding parabolic subgroup $S_a$ of $S_n$; it is the subgroup generated by all $s_i$ for $i \in \{1, \ldots, n\} - \{a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_k\}$. We use $[S_n/S_a]_{\min}$ and $[S_n \setminus S_a]_{\min}$ to denote the sets of minimal length left and right coset representatives, respectively. Also let $w_0$ be the longest element of $S_a$ and $w^a$ be the longest element of $[S_n/S_a]_{\min}$, so that $w_n = w^a w_0$. For example, $S_{(n-1,1)}$ is $S_{n-1}$ embedded into $S_n$ as the permutations that fix $n$. These natural embeddings define a tower of subgroups

$$S_0 < S_1 < S_2 < \cdots .$$

There is also the “unnatural embedding” $\sigma_n : S_{n'} \rightarrow S_{n+n'}$, $s_j \mapsto s_{j+n}$ for $n, n' \geq 0$.

We may implicitly identify $\lambda \in \Lambda^+$ with the “dominant” composition $(\lambda_1, \ldots, \lambda_k) \in \Lambda(k, n)$ where $n := |\lambda|$ and $k := ht(\lambda)$. Note then that $NE(\lambda)$ defined combinatorially above is the length of the longest of the minimal length $S_{\lambda^T} \setminus S_n/\Sigma_\alpha$-double coset representatives. We let

$$n \# r := n + (n+1) + \cdots + (n+r-1) = nr + \binom{r}{2},$$

for any $n$ and $r \geq 0$. 
2. Graded superalgebra

In this section, we review some basic language, referring the reader to the exposition in the introduction of [BE1] for more details; see also [BE1, Sec. 6] which discusses gradings. A graded superalgebra is an associative, unital \( \mathbb{Z} \times \mathbb{Z}/2 \)-graded \( \mathbb{F} \)-algebra \( A = \bigoplus_{d \in \mathbb{Z}, p \in \mathbb{Z}/2} A_{d,p} \). We may also write \( A_p \) for \( \bigoplus_{d \in \mathbb{Z}} A_{d,p} \), so \( A_0 \) is the even part and \( A_1 \) is the odd part of \( A \). For a homogeneous element \( a \in A_{d,p} \), we write \( \text{deg}(a) \) for its degree \( d \) (the \( \mathbb{Z} \)-grading) and \( \text{par}(a) \) for its parity \( p \) (the \( \mathbb{Z}/2 \)-grading). Any graded superalgebra has the parity involution

\[
p : A \to A, \quad a \mapsto (-1)^{\text{par}(a)} a.
\]

This equation only makes sense if \( a \) is homogeneous, but we adopt the usual abuse of notation by suppressing this assumption. We also write \( A^{\text{op}} \) for the opposite superalgebra, whose multiplication is defined from \( a \cdot b := (-1)^{\text{par}(a) \text{par}(b)} ba \). As in [BE1], we use the notation \( \Pi \) for the parity switch functor and \( Q \) for the upward grading shift functor, using \( \pi \) and \( q \) for the induced maps at the decategorified level of Grothendieck groups. We let

\[
Z^\pi := \mathbb{Z}[\pi]/(\pi^2 - 1), \quad Q^\pi := Q[\pi]/(\pi^2 - 1).
\]

We will systematically write \( \cong \) to denote the existence of an isomorphism that is not necessarily homogeneous, and \( \approx \) to denote the existence of an isomorphism that preserves parities and degrees.

We write \( \text{gsVec} \) for the symmetric monoidal category of graded vector superspaces with morphisms that preserve both degree and parity of vectors. It possesses a symmetric braiding which is defined on vector superspaces \( V \) and \( W \) by

\[
B_{V,W} : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto (-1)^{\text{par}(v) \text{par}(w)} w \otimes v.
\]

A graded supercategory is a category enriched in \( \text{gsVec} \), and a graded superfunctor is an enriched functor. In particular, graded superfunctors preserve degrees and parities of morphisms. We use the notation \( \text{gsHom}(\mathcal{A}, \mathcal{B}) \) for the graded supercategory of graded superfunctors and graded supernatural transformations in the sense of [BE1, Ex. 1.1]. In particular, \( \text{gsEnd}(\mathcal{A}) := \text{gsHom}(\mathcal{A}, \mathcal{A}) \) is a strict graded monoidal supercategory; see [BE1, Ex. 1.5(ii)]. For superfunctors \( F, G : \mathcal{A} \to \mathcal{B} \), we denote the morphism space \( \text{Hom}_{\text{gsHom}(\mathcal{A}, \mathcal{B})}(F, G) \), that is, the graded superspace consisting of all graded supernatural transformations from \( F \) to \( G \), simply by \( \text{gsHom}(F, G) \). If \( F = G \) we denote it by \( \text{gsEnd}(F) \) if \( F = G \), this being a graded superalgebra.

For any graded supercategory \( \mathcal{A} \), we denote the underlying ordinary category consisting of the same objects and the morphisms that are even of degree 0 by \( \overline{\mathcal{A}} \). The category \( \text{gsVec} \) is the underlying ordinary category of a graded monoidal supercategory \( \text{gsEnd} \). In \( \text{gsVec} \), a linear map \( f : V \to W \) is homogeneous of degree \( d \in \mathbb{Z} \) and parity \( p \in \mathbb{Z}/2 \) if \( f(V_{d',p'}) \subseteq W_{d+d',p+p'} \) for all \( d' \in \mathbb{Z}, p' \in \mathbb{Z}/2 \). Then an arbitrary morphism in \( \text{gsVec} \) is a graded linear map, that is, a linear map with the property that it can be written as a finite sum of homogeneous linear maps of different degrees. In fact, \( \text{gsVec} \) is a graded monoidal \((\mathcal{Q}, \Pi)\)-supercategory; see [BE1, Def. 1.12, Def. 6.5] for the formal definition. The unit object is the field \( \mathbb{F} \) viewed as a graded superspace so that it is even in degree zero, the parity shift functor is \( \Pi \mathbb{F} \otimes - \) where \( \mathbb{F} \otimes \mathbb{F} = \mathbb{F} \) in degree zero and odd parity, and grading shift functor is \( Q\mathbb{F} \otimes - \) where \( Q\mathbb{F} \) is \( \mathbb{F} \) in degree one and even parity. For any graded \((\mathcal{Q}, \Pi)\)-supercategory \( \mathcal{V} \), its underlying ordinary category is a \((\mathcal{Q}, \Pi)\)-category in the sense of [BE1, Def. 6.12]. Thus, \( \text{gsVec} \) is a \((\mathcal{Q}, \Pi)\)-category.

For graded superalgebras \( A \) and \( B \), their tensor product \( A \otimes B \) is the tensor product of the underlying graded vector superspaces viewed as a graded superalgebra so that

\[
(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\text{Par}(b_1) \text{Par}(a_2)} a_1 a_2 \otimes b_1 b_2
\]
for \(a_1, a_2 \in A, b_1, b_2 \in B\). Very important in this article is the graded superalgebra \(OPol_n\) of odd polynomials. By definition, this is the tensor product
\[
OPol_n := \bigotimes_{i=1}^{n} OPol_1,
\]
where \(OPol_1 := \mathbb{F}[x]\) is the usual commutative polynomial algebra in an indeterminate \(x\), viewed as a graded superalgebra so that \(x\) is odd of degree 2. We let \(x_i := 1 \otimes \cdots \otimes x \otimes \cdots \otimes 1\) where \(x\) is in the \(i\)th tensor position from the left. Here are some further observations.

- For \(a \in \Lambda(k, n)\), the tensor product \(OPol_{a_1} \otimes \cdots \otimes OPol_{a_k}\) is canonically identified with \(OPol_n\) so that \(1^\otimes(j-1) \otimes x_j \otimes 1^\otimes(k-j) \equiv x_{a_1+\cdots+a_j-1+i}\).
- The elements \(x_1, \ldots, x_n\) generate \(OPol_n\) subject to the relations \(x_j x_i = -x_i x_j\) for \(1 \leq i < j \leq n\).
- There are no (non-zero) zero divisors in \(OPol_n\), although it does not have unique factorization, e.g., \((x_1 - x_2)^2 = (x_1 + x_2)^2\).
- The monomials \(x^k := x_1^{k_1} \cdots x_n^{k_n} \in OPol_n\) for \(k = (k_1, \ldots, k_n) \in \mathbb{N}^n\) give a linear basis for \(OPol_n\). From the last point, it follows that
\[
dim_{q,p} OPol_n = \frac{1}{(1 - q^2)^n} \in \mathbb{Z}[\lbrack q\rbrack],
\]
meaning that the coefficient of \(q^d p^n\) in this generating function is the dimension of the homogeneous component in degree \(d \in \mathbb{Z}\) and parity \(p \in \mathbb{Z}/2\).

Given two graded superalgebras \(A\) and \(B\), we write \(A\)-\(gsMod-B\) for the graded supercategory of graded \((A, B)\)-superbimodules and graded \((A, B)\)-superbimodule homomorphisms; such a homomorphism is a finite sum of homogeneous \((A, B)\)-superbimodule homomorphisms of different degrees and parities. We adopt the usual sign convention for morphisms as in [BE1, Ex. 1.8]. So a morphism \(f : V \rightarrow W\) in \(A\)-\(gsMod-B\) satisfies \(f(abv) = (-1)^{\text{par}(f)\text{par}(a)}af(b)\) for \(a \in A, b \in B, v \in M\). The category \(A\)-\(gsMod-B\) is a graded \((Q, \Pi)\)-supercategory in the sense of [BE1, Def. 1.7, Def. 6.4]. We use the notation \(\text{Hom}_{A,B}(V, W)\) to denote a morphism space in this category, which is a graded vector superspace. The parity switching functor \(\Pi\) takes a graded \((A, B)\)-superbimodule \(V\) to the same underlying graded vector space viewed as a superspace with the opposite parities and actions of \(a \in A\) and \(b \in B\) on \(v \in IV\) defined in terms of the original action so that
\[
a \cdot v \cdot b := (-1)^{\text{par}(a)} avb.
\]
On a morphism \(f : V \rightarrow W, \Pi f : IV \rightarrow IW\) is defined so that \((\Pi f)(v) = (-1)^{\text{par}(f)} f(v)\). The grading shift functor \(Q\) takes \(V\) to the same underlying superbimodule with the new grading \((QV)_d := V_{d-1}\).

This is less delicate since it does not introduce any additional signs.

The graded \((Q, \Pi)\)-supercategories \(A\)-\(gsMod\) and \(gsMod-B\) of graded left \(A\)-supermodules and right \(B\)-supermodules are \(A\)-\(gsMod-\mathbb{F}\) and \(\mathbb{F}\)-\(gsMod-B\), respectively. We write \(\text{Hom}_{A,B}(V, W)\) and \(\text{Hom}_{A,B}(V, W)\) for the morphism spaces in these categories. We use \(A\)-\(psgsm\) (resp., \(psgsm-B\)) to denote the full subcategory of \(A\)-\(gsMod\) (resp., \(gsMod-B\)) consisting of the finitely generated projective graded superbimodules.

By the graded super Karoubi envelope \(gsKar(\mathcal{A})\) of a graded supercategory \(\mathcal{A}\) we mean the graded \((Q, \Pi)\)-supercategory obtained by first passing to its \((Q, \Pi)\)-envelope as defined in [BE1, Def. 6.8], then to its additive envelope, then finally to its completion at all even idempotents. The underlying ordinary category is then an additive and idempotent complete \((Q, \Pi)\)-category. For example, if \(\mathcal{A}\) is the graded supercategory with one object whose endomorphisms are given by some graded superalgebra \(A\) then, by Yoneda Lemma, \(gsKar(\mathcal{A})\) is equivalent to \(\text{pgsm}-A\). It also makes sense to take the graded super Karoubi envelope \(gsKar(\mathfrak{A})\) of a graded 2-supercategory \(\mathfrak{A}\), which is a graded \((Q, \Pi)\)-2-supercategory.
The non-trivial step here is the the construction of \((Q, \Pi)\)-envelope of a graded 2-supercategory. This is explained in [BE1, Def. 6.10].

For graded supercategories \(\mathcal{A}, \mathcal{B}\), an adjacent pair \((E, F)\) of graded superfunctors \(E : \mathcal{A} \to \mathcal{B}\) and \(F : \mathcal{B} \to \mathcal{A}\) means an adjacent pair of functors in the usual sense of \(\mathbb{F}\)-linear categories, such that in addition the unit and the counit of the adjunction are both even supernatural transformations of degree zero. It follows that the restrictions of \(E\) and \(F\) to the underlying ordinary categories also form an adjacent pair.

Now suppose that \(A\) and \(B\) are graded superalgebras such that \(A\) is a (unital) subalgebra of \(B\). Then we have the usual adjacent triple of functors \((\text{Ind}_A^B, \text{Res}_A^B, \text{Coind}_A^B)\) defined from

\[
\text{Ind}_A^B := B \otimes_A - : \text{A-gsMod} \to \text{B-gsMod}, \quad \text{Res}_A^B := \text{Hom}_{B_\ast}(B, -) = B \otimes_B - : \text{B-gsMod} \to \text{A-gsMod}, \quad \text{Coind}_A^B := \text{Hom}_{A_\ast}(B, -) : \text{A-gsMod} \to \text{B-gsMod}. \tag{2.7}
\]

Following [PS] (which explicitly treats graded superalgebras), we say that \(B\) is a Frobenius extension of \(A\) of degree \(d \in \mathbb{Z}\) and parity \(p \in \mathbb{Z}/2\) if there exists a trace map \(\text{tr} : B \to A\) that is a homogeneous graded \((A, A)\)-superbimodule homomorphism of degree \(-d\) and parity \(p\), together with homogeneous elements \(b_1, \ldots, b_m, b_1^\vee, \ldots, b_m^\vee \) of \(B\) such that

\[
\deg(b_i) + \deg(b_i^\vee) = d, \quad \text{par}(b_i) + \text{par}(b_i^\vee) = p, \quad \sum_{i=1}^m b_i \text{tr}(b_i^\vee b) = b, \quad \sum_{i=1}^m (-1)^{p \text{par}(b_i^\vee) + p \text{par}(b_i)} \text{tr}(b b_i) b_i^\vee = b \tag{2.10}
\]

for any \(b \in B\). We will only apply this in situations where \(B_{0,0} = \mathbb{F}\), in which case the trace map is unique up to multiplication by a non-zero scalar; see [PS, Prop. 4.7] for a more general uniqueness statement. When this holds, the elements \(b_1, \ldots, b_m\) give a basis for \(B\) as a free right \(A\)-supermodule, the elements \(b_1^\vee, \ldots, b_m^\vee\) give a basis for \(B\) as a free left \(A\)-supermodule, and \(\text{tr}(b_i^\vee b_j) = \delta_{i,j}\). Moreover, there is a unique homogeneous \((B, B)\)-superbimodule homomorphism

\[
\delta : B \to B \otimes_A B, \quad 1 \mapsto \sum_{j=1}^m (-1)^{p \text{par}(b_j^\vee)} b_j \otimes b_j^\vee \tag{2.12}
\]

of degree \(d\) and parity \(p\). The unit and counit of the canonical adjunction making \((\text{Ind}_A^B, \text{Res}_A^B)\) into an adjacent pair are induced by the superbimodule homomorphisms \(\eta : A \to B\) and \(\mu : B \otimes_A B \to B\) given by the canonical inclusion and multiplication, respectively. There is also an adjunction making \((\text{Res}_A^B, \Pi^p Q^{-d} \text{Ind}_A^B)\) into an adjacent pair. The unit and counit of this adjunction are induced by the superbimodule homomorphisms \(\text{tr}\) and \(\delta\) viewed now as homogeneous graded superbimodule homomorphisms \(\text{tr} : \Pi^p Q^{-d}B \to A\) and \(\delta : B \to \Pi^p Q^dB \otimes_A B\) that are even of degree 0. This adjunction induces a canonical isomorphism \(\Pi^p Q^{-d} \text{Ind}_A^B \simeq \text{Coind}_A^B\).

Suppose that \(A\) is a commutative graded algebra viewed as a purely even commutative graded superalgebra. We say that an \(A\)-superalgebra \(B\) is a graded Frobenius superalgebra over \(A\) of degree \(d\) and parity \(p\) if the canonical map \(\eta : A \to B\), \(a \mapsto a 1_B\) is injective, and \(B\) is a Frobenius extension of \(\eta(A)\) of degree \(d\) and parity \(p\) in the sense from the previous paragraph.

3. Odd quantum \(sl_2\)

In this section, we recall briefly the definition of the enveloping algebra of “odd quantum \(sl_2\)” discovered by Clark and Wang [CW] and developed in much greater generality in [CHW]. We work initially
over $\mathbb{Q}(q)$, adopting the same conventions as in [BE2, Sec. 9]; the most significant difference compared to [CHW, C] is that our $q$ is $q^{-1}$ in [CHW] and $\nu^{-1}$ in [C]. We define the $(q, \pi)$-integers

$$[n]_{q,\pi} := \frac{(\pi q)^n - q^{-n}}{\pi q - q^{-1}} = \begin{cases} q^{-n} + \pi q^{3-n} + \cdots + \pi^{n-1}q^{\pi-1} & \text{if } n \geq 0, \\ -\pi^n(q^{\pi+1} + \pi q^{\pi+3} + \cdots + \pi^{n-1}q^{\pi-n}) & \text{if } n \leq 0 \end{cases}$$

for any $n \in \mathbb{Z}$.) This is exactly the same as the definition given in [CHW, Sec. 1.6]—but because our $q$ is the $q^{-1}$ in [CHW] it is actually a different convention! Note that

$$[-n]_{q,\pi} = -\pi^n[n]_{q,\pi}.$$  

(3.2)

There are corresponding $(q, \pi)$-factorials $[n]!,_q,\pi$ for $n \geq 0$:

$$[n]!,_q,\pi := [n]_{q,\pi}[n-1]_{q,\pi} \cdots [1]_{q,\pi} = q^{-\binom{n}{2}} \sum_{\omega \in S_n} (\pi q^2)^{\ell(\omega)},$$

(3.3)

where the last equality is a consequence of the well-known factorization of the Poincaré polynomial for the symmetric group. Then we have the $(q, \pi)$-binomial coefficients $\binom{n}{r},_q,\pi$, which make sense as written for any $n \in \mathbb{Z}$ and $r \geq 0$:

$$\binom{n}{r},_q,\pi := \frac{[n]_{q,\pi}[n-1]_{q,\pi} \cdots [n-r+1]_{q,\pi}}{[r]!,_q,\pi}.$$  

(3.4)

We also adopt the convention that $\binom{n}{r},_q,\pi = 0$ for any $n \in \mathbb{Z}$ and $r < 0$. Note by (3.2) that

$$\binom{-n}{r},_q,\pi = (-1)^r \pi^{n+r}(\pi q_2)^{\ell(n)}.$$  

(3.5)

We also need quantum trinomial coefficients for $n \in \mathbb{Z}$ and $r, s \geq 0$:

$$\binom{n}{r,s},_q,\pi := \frac{[n]_{q,\pi}[n-1]_{q,\pi} \cdots [n-r-s+1]_{q,\pi}}{[r]!,_q,\pi[s]!,_q,\pi} = \frac{[n]_{q,\pi}[n-r-s]}{[r]!,_q,\pi[s],_q,\pi}.$$  

(3.6)

Again we interpret $\binom{n}{r,s},_q,\pi$ as zero if $r < 0$ or $s < 0$. More generally, for $\alpha \in \Lambda(k, n)$, let

$$\binom{n}{\alpha},_q,\pi := \frac{[n]!,_q,\pi}{[\alpha_1]!,_q,\pi \cdots [\alpha_k]!,_q,\pi}$$

(3.7)

be the $(q, \pi)$-multinomial coefficient. The identity (3.3) implies that

$$\binom{n}{\alpha},_q,\pi = q^{-N(\alpha)} \sum_{w \in [S_k]/S_0} (\pi q^2)^{\ell(\omega)},$$

(3.8)

We let $\sim: \mathbb{Q}(q) \to \mathbb{Q}(q)$ be the $\mathbb{Q}(q)$-algebra involution with $\bar{q} = q^{-1}$. This is not the bar involution introduced in [CHW, C]; the latter is the $\mathbb{Q}(q)$-algebra involution $\bar{\cdot}: \mathbb{Q}(q) \to \mathbb{Q}(q)$ with $\bar{q} = \pi q^{-1}$, which fixes all of the $(q, \pi)$-integers, $(q, \pi)$-factorials and $(q, \pi)$-binomial coefficients. This is definitely not the case for $\sim$, which will be useful to us for exactly this reason. In fact, we have that

$$\overline{[n]_{q,\pi}} = \pi^{n-1}[n]_{q,\pi}, \quad \overline{[n]!,_q,\pi} = \pi^{\binom{n}{2}}[n]!,_q,\pi,$$

$$\overline{\binom{n}{r},_q,\pi} = \pi^{(n-r)}\binom{n}{r},_q,\pi, \quad \overline{\binom{n}{r,s},_q,\pi} = \pi^{(n-r)(s+1)}\binom{n}{r,s},_q,\pi.$$  

(3.9)

(3.10)

Some further properties of $(q, \pi)$-binomial and trinomial coefficients are proved in the next two lemmas.

**Lemma 3.1.** The $(q, \pi)$-binomial and trinomial coefficients have the following properties.
(1) For \( n \in \mathbb{Z} \) and \( r \geq 0 \), we have that
\[
\binom{n}{r}_{q,\pi} = q^{-r} \binom{n-1}{r}_{q,\pi} + (\pi q)^{n-r} \binom{n-1}{r-1}_{q,\pi} = (\pi q)^{r} \binom{n-1}{r}_{q,\pi} + q^{r-n} \binom{n-1}{r-1}_{q,\pi}.
\]

(2) For \( n \in \mathbb{Z} \) and \( r, s \geq 0 \), we have that
\[
\binom{n}{r,s}_{q,\pi} = \pi^t q^{r-s} \binom{n-1}{r,s}_{q,\pi} + (\pi q)^{n-m} \binom{n-1}{r-1,s}_{q,\pi} + q^{s-n} \binom{n-1}{r,s-1}_{q,\pi}.
\]

(3) For \( n \in \mathbb{Z} \) and \( r \geq 0 \), we have that
\[
\sum_{s+t=r} \pi^{(3)}(-q)^{s} \binom{n+s}{s,t}_{q,\pi} = (\pi q)^{nr}.
\]

**Proof.**
(1) The first equality follows from the definition of \((q,\pi)\)-binomial coefficient by replacing the \([n]_{q,\pi}\) in the numerator with \(q^{-r}[n-r]_{q,\pi} + (\pi q)^{n-r}[r]_{q,\pi}\) and then splitting the result into two fractions. We just prove the first equality. Then the second equality can be deduced by applying the involution \(\sim\).

(2) Using (1) twice, we have that
\[
\pi^t q^{r-s} \binom{n-1}{r,s}_{q,\pi} + q^{s-n} \binom{n-1}{r,s-1}_{q,\pi} = q^{-r} \binom{n-1}{r}_{q,\pi} \left( (\pi q)^{r} \binom{n-1}{s}_{q,\pi} + q^{r-s-n} \binom{n-1}{r-1,s}_{q,\pi} \right)
\]
\[
= q^{-r} \binom{n-1}{r}_{q,\pi} \binom{n-r}{s}_{q,\pi} = \binom{n}{r}_{q,\pi} - (\pi q)^{r} \binom{n-1}{r-1,s}_{q,\pi} \binom{n-r}{s}_{q,\pi}.
\]

(3) Let \(c_n(r) := \sum_{s+t=r} \pi^{(3)}(-q)^{s} \binom{n+s}{s,t}_{q,\pi}\) and \(\tilde{c}_n(r)\) be its image under the involution \(\sim\). The goal is to show that \(c_n(r) = (\pi q)^{nr}\). It is easy to check that this is true when \(nr = 0\). This gives the base of an induction. For the induction step, we also need the identity
\[
c_n(r) = \pi^{nr} (\pi q)^{r} \tilde{c}_{n-1}(r) + (\pi q)^{n} c_n(r-1) - \pi^{nr} (\pi q)^{1-n-r} \tilde{c}_{n-1}(r-1),
\]
which will be verified in the next paragraph. Using this, it is easy to complete the proof for all \(n \geq 0\) and \(r \geq 0\) by induction on \(n + r\). The proof for \(n \leq 0\) and \(r \geq 0\) goes instead by induction on \(r - n\) using the following:
\[
c_n(r) = \pi^{nr} q^{r} \tilde{c}_{n+1}(r) - \pi^{nr} (\pi q)^{n-1} q^{-r} \tilde{c}_{n+1}(r-1) + (\pi q)^{n} c_n(r-1)
\]
This follows by applying \(\sim\) to the previous identity, then replacing \(n\) by \(n + 1\) and rearranging.

It remains to prove the first identity. Using (2), we have that
\[
c_n(r) = \sum_{s+t=r} \pi^{(3)}(-q)^{s} \left( \pi^t q^{r-s} \binom{n+s-1}{s,t}_{q,\pi} + (\pi q)^{n} \binom{n+s-1}{s-1,t}_{q,\pi} + q^{t-s-n} \binom{n+s-1}{s,t-1}_{q,\pi} \right).
\]

Moving the sum inside the parentheses produces three terms which we simplify separately, reindexing the second sum by replacing \(s\) by \(s + 1\) and the third sum by replacing \(t\) by \(t + 1\). We also use
\[
\tilde{c}_n(r) = \pi^{nr} \sum_{s+t=r} \pi^{(3)}(-q)^{s} \binom{n+s}{s,t}_{q,\pi},
\]
which follows by (3.10). In this way, the three terms become:
\[
q^{-r} \sum_{s+t=r} \pi^{(3)}(-q)^{s} \binom{n+s-1}{s,t}_{q,\pi} = \pi^{(n-1)r} q^{-r} \tilde{c}_{n-1}(r),
\]
Corollary 3.2. For $0 \leq r \leq n$, we have that
\[
q^{(n-r)} \prod_{r \in \mathbb{N}_{>n-r}} \left( \sum_{r \in \mathbb{N}_{>n-r}} \right).
\]

Proof. This is an induction exercise using Lemma 3.1(1). □

Lemma 3.3. Recall from the General conventions that $n\# r$ denotes $n + (n + 1) + \cdots + (n + r - 1)$. For $m, n \in \mathbb{Z}$ and $r \geq 0$, we let
\[
b_{m,n}(r) := (\pi q^{2-1})^{n-r} \sum_{s=0}^{r-1} \left( \sum_{s=0}^{r-1} \right) q^{(m-r-s+1)(n-r)+s} \prod_{s=0}^{r-1} \left( n + s \right) \prod_{s=0}^{r-1} \left( n + s + 1 \right) \prod_{s=0}^{r-1} \left( n + s + 2 \right).
\]

Then we have that $c_{m,n}(r) = b_{m,n}(r) + b_{m,n}(r+1)$ for any $m, n \in \mathbb{Z}$ and $r \geq 0$.

Proof. Proceed by induction on $r$. The base case $r = 0$ is easily checked. For the induction step, take $r > 0$. We have that
\[
b_{m,n}(r) = (\pi q^{2-1})^{n-r} b_{m,n}(r-1) + (\pi q^{2-1})^{n-r} \prod_{s=0}^{r-1} \left( n + s \right) \prod_{s=0}^{r-1} \left( n + s + 1 \right) \prod_{s=0}^{r-1} \left( n + s + 2 \right).
\]

Note that $(n-r)\#(r-1) = (n-r)\# r - n + 1$ and $(n-r-1)\#(r+1) - n + r + 1 = (n-r)\# r$. Adding the above equations and using the induction hypothesis gives that
\[
b_{m,n}(r) + b_{m,n}(r+1) = (\pi q^{2-1})^{n-r} \prod_{s=0}^{r-1} \left( n + s \right) \prod_{s=0}^{r-1} \left( n + s + 1 \right) \prod_{s=0}^{r-1} \left( n + s + 2 \right).
\]

Using the identity $(\pi q)^{n-r} \prod_{s=0}^{r-1} \left( n + s \right) \prod_{s=0}^{r-1} \left( n + s + 1 \right) \prod_{s=0}^{r-1} \left( n + s + 2 \right)$ from Lemma 3.1(1), the first two terms combine into one leaving us with
\[
(\pi q^{2-1})^{n-r} \prod_{s=0}^{r-1} \left( n + s \right) \prod_{s=0}^{r-1} \left( n + s + 1 \right) \prod_{s=0}^{r-1} \left( n + s + 2 \right).
\]

Then we use the identity $\prod_{s=0}^{r-1} \left( n + s \right) \prod_{s=0}^{r-1} \left( n + s + 1 \right) \prod_{s=0}^{r-1} \left( n + s + 2 \right)$ to see finally that this is equal to $c_{m,n}(r)$. □
Corollary 3.4. \( q^{(n-r)q} \left\lfloor \frac{n}{r} \right\rfloor_{q, \pi} = \sum_{s=0}^{r} (\pi q^2)^{(n-r)(r-s)} q^{(n-r)(r-s)} \left\lfloor \frac{n-r+s-1}{s} \right\rfloor_{q, \pi} \) for \( 0 \leq r \leq n \).

Proof. Take \( m = n - r \) in Lemma 3.3. \( \square \)

Let \( U_{q, \pi}(sl_2) \) denote the locally unital \( \mathbb{Q} \)-algebra with distinguished idempotents \( \{ 1_k \mid k \in \mathbb{Z} \} \) and generators \( E 1_k = 1_{k+2}, F 1_k = 1_{k-2} \) subject to the relations

\[
EF 1_k - \pi FE 1_k = [k]_{q, \pi} 1_k
\]

for all \( k \in \mathbb{Z} \). Note there is some flexibility in writing the idempotents \( 1_k \)—in any given monomial one just needs to include one idempotent somewhere in the word for the notation to be unambiguous. Let

\[
E^{(d)} 1_k := \frac{E^d 1_k}{[d]_{q, \pi}}, \quad \overline{E}^{(d)} 1_k := \frac{E^d 1_k}{[d]_{q, \pi}}, \quad F^{(d)} 1_k := \frac{F^d 1_k}{[d]_{q, \pi}}, \quad \overline{F}^{(d)} 1_k := \frac{F^d 1_k}{[d]_{q, \pi}}.
\]

There is a \( \sim \)-antilinear involution

\[
\varpi : U_{q, \pi}(sl_2) \to U_{q, \pi}(sl_2), \quad 1_k \mapsto 1_{-k}, \ E 1_k \mapsto F 1_{-k}, \ F 1_k \mapsto E 1_{-k}.
\]

This sends \( E^{(d)} 1_k \mapsto \overline{F}^{(d)} 1_{-k} \) and \( F^{(d)} 1_k \mapsto \overline{E}^{(d)} 1_{-k} \). We warn the reader that this is different from the involution \( \omega \) in [CHW]; the latter is linear rather than \( \sim \)-antilinear.

By a \( U_{q, \pi}(sl_2) \)-module we mean a locally unital left module \( V = \bigoplus_{k \in \mathbb{Z}} 1_k V \). We call \( 1_k V \) the \( k \)-weight space of \( V \). We say that \( V \) is integrable if any weight vector \( v \in 1_k V \) for \( k \in \mathbb{Z} \) is annihilated by \( E^n 1_k \) and \( F^n 1_k \) for \( n \gg 0 \) (depending on \( v \)). For \( \ell \in \mathbb{N} \)—a dominant weight for \( sl_2 \)—there is a \( U_{q, \pi}(sl_2) \)-module \( V(-\ell) \) which is free as a \( \mathbb{Q} \)-module with basis \( \{ b_n^{\ell} \mid 0 \leq n \leq \ell \} \) such that

- \( b_n^{\ell} \) is of weight \( 2n - \ell \), i.e., \( 1_{2n-\ell} b_n^{\ell} = b_n^{\ell} \);
- \( b_{\ell}^{\ell} \) is a highest weight vector and \( b_0^{\ell} \) is a lowest weight vector, i.e., \( E b_{\ell}^{\ell} = F b_{\ell}^{\ell} = 0 \);
- for \( 0 \leq n < \ell \), we have that \( E b_n^{\ell} = [n+1]_{q, \pi} b_{n+1}^{\ell} \) and \( F b_n^{\ell} = [\pi(\ell - n)]_{q, \pi} b_n^{\ell} \).

We visualize the action with the familiar \( sl_2 \)-type picture showing how the operators \( E \) and \( F \) raise and lower basis vectors to multiples of basis vectors:

\[
\begin{array}{c}
E \downarrow \\
\{1\}_{q, \pi} \downarrow \pi^{-1} \{1\}_{q, \pi} \\
{b_{\ell}^{\ell}} \downarrow \pi^{-(\ell-1)} {1}_{q, \pi} \downarrow \\
\{2\}_{q, \pi} \downarrow \pi^{-(\ell-2)} \{2\}_{q, \pi} \\
\vdots \\
\{\ell-1\}_{q, \pi} \downarrow \pi^{-(\ell-2)} \{\ell-1\}_{q, \pi} \\
{b_{\ell}^{\ell-1}} \downarrow \pi {1}_{q, \pi} \\
\{\ell\}_{q, \pi} \downarrow \pi {1}_{q, \pi} \\
{b_0^{\ell}} \\
\end{array}
\]

(3.15)
For $0 \leq n \leq \ell - d$, we have that

$$E^{(d)}b_n^\ell = \left[ \frac{n + d}{d} \right] b_{n+d}^\ell,$$

$$F^{(d)}b_{n+d}^\ell = \pi(n)^{\ell+n} \left[ \frac{n - d}{d} \right] b_n^\ell. \quad (3.16)$$

These are easily checked starting from the observation that $b_n^\ell = \bar{E}(n)b_n^\ell = F^{(\ell-n)}b_n^\ell$. There is a ~- antilinear involution

$$\bar{\sigma} : V(-\ell) \rightarrow V(-\ell), \quad b_n^\ell \mapsto \pi^{n(\ell-n)}b_n^\ell. \quad (3.17)$$

This has the key property that

$$\bar{\sigma}(uv) = \bar{\sigma}(u)\bar{\sigma}(v) \quad (3.18)$$

for all $u \in U_{q,\pi}(\mathfrak{sl}_2)$, $v \in V(-\ell)$.

Let $V_\pm(\ell) := \frac{1}{2}(1 \pm \pi)V(-\ell)$. These are irreducible $U_{q,\pi}(\mathfrak{sl}_2)$-modules generated by the highest weight vectors $\frac{1}{2}(1 \pm \pi)b_n^\ell$ of weight $\ell$ on which $\pi$ acts by the scalar $\pm 1$. In particular, these modules are not isomorphic for different $\ell$ or different choices of sign.

**Theorem 3.5** ([CHW, Cor. 3.3.3]). Any integrable $U_{q,\pi}(\mathfrak{sl}_2)$-module decomposes as a direct sum of the modules $V_\pm(\ell)$ for $\ell \in \mathbb{N}$.

Now we can prove the main result of the section.

**Theorem 3.6.** Let $V$ be an integrable $U_{q,\pi}(\mathfrak{sl}_2)$-module. There is a linear automorphism $T : V \rightarrow V$ sending $1_k V$ to $1_{k'} V$ for each $k \in \mathbb{Z}$ such that

$$T(v) = \sum_{d \geq \max(0,-k)} (-q)^d E^{(k+d)}F^{(d)}v$$

on a vector $v \in 1_k V$. The inverse is given explicitly by the formula

$$T^{-1}(v_k) = \sum_{d \geq \max(0,-k)} (-q)^{-d} \bar{F}^{(k+d)}F^{(d)}v$$

on a vector $v \in 1_k V$.

**Proof.** In view of Theorem 3.5, it suffices to check this when $V = V(-\ell)$ for $\ell \in \mathbb{N}$. Take $-\ell \leq k \leq \ell$ with $k \equiv \ell \pmod{2}$ and set $n := \frac{-\ell + k}{2}$ and $n' := \frac{\ell - k}{2}$, so $n' + n = \ell$ and $n' - n = k$. The space $1_k V(-\ell)$ is spanned by $b_{n'}^\ell$ and $1_k V(-\ell)$ is spanned by $b_{n'}^\ell$. Since $F^{(d)}b_n^\ell = 0$ for $d > n$, we have by the definition in the statement of the theorem that $T(b_n^\ell) = ub_n^\ell$ where

$$u := \sum_{d = \max(0,-k)}^n (-q)^d E^{(k+d)}F^{(d)} \in U_{q,\pi}(\mathfrak{sl}_2).$$

In the next paragraph, we show that

$$ub_n^\ell = (-1)^n \pi(\ell)^{n+n'} q^{n+n'} b_{n'}^\ell. \quad (3.19)$$

Assuming this, the proof can be completed as follows. Applying $\bar{\sigma}$ to (3.19) using (3.17) and (3.18), we also have that

$$\bar{\sigma}(u)b_n^\ell = (-1)^n \pi(\ell)^{n+n'} q^{-n-n'} b_n^\ell. \quad (3.20)$$

From (3.19) and (3.20), it follows that $\bar{\sigma}(u)ub_n^\ell = b_n^\ell$ and $u\bar{\sigma}(u)b_n^\ell' = b_n^\ell$. Hence, $T : 1_k V(-\ell) \rightarrow 1_k V(-\ell)$ is an isomorphism with inverse $T^{-1}$ defined by multiplication by $\bar{\sigma}(u)$. Finally we observe that $\bar{E}(d)b_n^\ell = 0$ for $d > n$ so

$$\bar{\sigma}(u)b_n^\ell' = \sum_{d \geq \max(0,-k)} (-q)^{-d} \bar{F}^{(k+d)}E^{(d)}b_n^\ell'.$$
which agrees with the formula for $T^{-1}(b'_{n'})$ in the statement of the theorem.

It remains to prove (3.19). First we make some elementary computations using (3.16):

$$ub'_{n'} = \sum_{d=\max(0,n-n')}^{n} (-q)^d E^{(n'\cdot d)}F^{(d)} b'_{n'}$$

$$= \sum_{d=\max(0,n-n')}^{n} \pi^{(d)}_{(n'\cdot d)}(-q)^d \left[ n' \atop q, \pi \right] \left[ \pi \right]_{n-d} b'_{n'}$$

$$= \sum_{d=0}^{n} \pi^{(d)}_{(n'\cdot d)}(-q)^d \left[ n' \atop q, \pi \right] \left[ n' \atop q, \pi \right]_{n-d} b'_{n'},$$

noting in the last step that $\left[ n' \atop n-d \right] = 0$ if $d < n' - n$ so that we can remove the restriction on the summation. Then we switch to another variable $s := n - d$ and sum instead over $d, s \geq 0$ with $d + s = n$ to get that

$$ub'_{n'} = \sum_{d+s=n} \pi^{(n-s)}_{(s-n)}(-q)^{n-s} \left[ n' + d \atop q, \pi \right] \left[ n' \atop q, \pi \right]_{s} b'_{n'} = \pi^{(s)}_{(n-d)}(-q)^{n} \sum_{d+s=n} \pi^{(s)}_{(n-d)}(-q)^{s} \left[ n' + d \atop d, s \right] v_{n', s}.$$ 

Now an application of Lemma 3.1(3) completes the proof of (3.19). □

**Remark 3.7.** The specialization of $U_{q, \pi}(sl_2)$ at $\pi = 1$, that is, the algebra $U_{q}(sl_2) := U_{q, \pi}(sl_2) \otimes_{\mathbb{Q}} \mathbb{Q}$ where $\mathbb{Q}$ is viewed here as a $Q^\pi$-module so that $\pi$ acts as 1, is the usual quantized enveloping algebra of $SL_2$. Theorem 3.6 is well known in this case. The specialization at $\pi = -1$, i.e. the same tensor product with $\pi$ acting as $-1$, is the quantized enveloping algebra $U_{q}(osp_{1|2})$ of Clark and Wang [CW].

The algebra $U_{q, \pi}(sl_2)$ has a $\mathbb{Z}[q, q^{-1}]$-form we denote by $U_{q, \pi}(sl_2)$, namely, the $\mathbb{Z}[q, q^{-1}]$-algebra generated by the divided powers $E^{(r)}1_k, F^{(r)}1_k$ for $r \geq 1, k \in \mathbb{Z}$. The module $V(-\ell)$ is also defined over $\mathbb{Z}[q, q^{-1}]$, with its integral form $V(-\ell)$ being the $\mathbb{Z}[q, q^{-1}]$-submodule of $V(-\ell)$ generated by the basis vectors chosen above.

**Theorem 3.8 ([C, Lem. 3.5]).** The algebra $U_{q, \pi}(sl_2)$ is free as a $\mathbb{Z}[q, q^{-1}]$-module with basis given by the monomials $\{F^{(r)}E^{(s)}1_k \mid r, s \geq 0, k \in \mathbb{Z}\}$.

We say that a $\mathbb{Z}[q, q^{-1}]$-free $U_{q, \pi}(sl_2)$-module $V$ is integrable if any weight vector $v \in 1_k V$ is annihilated by $E^{(n)}1_k$ and $F^{(n)}1_k$ for $n \gg 0$ (depending on $v$). Equivalently, the $\mathbb{Q}[q]$-free $U_{q, \pi}(sl_2)$-module $\mathbb{Q}[q] \otimes_{\mathbb{Z}[q, q^{-1}]} V$ is integrable in the earlier sense. It is clear that the automorphism $T$ from Theorem 3.6 descends to an automorphism of any integrable $U_{q, \pi}(sl_2)$-module that is free as a $\mathbb{Z}[q, q^{-1}]$-module.

4. Odd Symmetric Functions

This section is largely an exposition of results from [EKL], and assumes the reader is already familiar with the classical theory of symmetric functions as in [Mac]. However, we have made one substantial modification to the setup: instead of the elementary odd symmetric functions denoted $e_r$ in [EKL], we usually prefer to work with the renormalized odd elementary symmetric functions $e_r := (-1)^{[r]} e_r$. We will explain the implications of this more thoroughly as we proceed. We also warn the reader that in [EK] the notation $e_r$ is used for the same thing as the element denoted $e_r$ in [EKL], so the $e_r$ of [EK] is not the one here.
The algebra $\text{OSym}$ of odd symmetric functions is the graded superalgebra generated by elements $h_r (r \geq 1)$ of degree $2r$ and parity $r \pmod{2}$ subject to the relations of [EK, Cor. 2.13]:
\begin{align}
h_r h_s &= h_s h_r & \text{if } r \equiv s \pmod{2} \quad (4.1) \\
h_r h_s + (-1)^r h_s h_r &= (-1)^{\delta} h_{r+1} h_{s-1} + h_{s-1} h_{r+1} & \text{if } r \not\equiv s \pmod{2} \quad (4.2)
\end{align}
for $r \geq 0, s \geq 1$, interpreting $h_0$ as 1. We also define elements $e_r (r \geq 0)$ so that the infinite Grassmannian relation
\begin{equation}
\sum_{s=0}^{r} (-1)^s h_{r-s} = \delta_{r,0} \quad (4.3)
\end{equation}
holds for all $r \geq 0$. The element $h_r$ is exactly the $r$th complete odd symmetric function from [EK]. We call $e_r$ the $r$th elementary odd symmetric function.

In [EK, Cor. 2.13, Prop. 2.10], it is shown that their elements $\{e_r \mid r \geq 1\}$ generate $\text{OSym}$ subject to exactly the same relations as the $h_r$. Noting that \(\binom{r}{2} + \binom{s}{2} \equiv \binom{r+1}{2} + \binom{s-1}{2} \pmod{2}\) when $r \not\equiv s \pmod{2}$, this means that our elements $\{e_r \mid r \geq 1\}$ also generate $\text{OSym}$ subject to the same relations
\begin{align}
e_r e_s &= e_s e_r & \text{if } r \equiv s \pmod{2} \quad (4.4) \\
e_r e_s + (-1)^r e_s e_r &= (-1)^r e_{s+1} e_{r-1} + e_{s-1} e_{r+1} & \text{if } r \not\equiv s \pmod{2} \quad (4.5)
\end{align}
for $r \geq 0, s \geq 1$, again interpreting $e_0$ as 1. There are also mixed relations, which are derived in [EK, Prop. 2.11]. These look slightly different with our modified odd elementary symmetric functions:
\begin{align}
e_r h_s &= h_s e_r & \text{if } r \equiv s \pmod{2} \quad (4.6) \\
e_r h_s + (-1)^r h_s e_r &= e_{r+1} h_{s-1} + (-1)^r h_{s-1} e_{r+1} & \text{if } r \not\equiv s \pmod{2} \quad (4.7)
\end{align}
for $r \geq 0, s \geq 1$. The following is equivalent to [EK, (2.6)]:
\begin{equation}
e_r = \det \left( h_{[r-1]} \right)_{i,j=1,...,r} \quad (4.8)
\end{equation}
where det should be interpreted as the usual Laplace expansion of determinant ordering monomials in the same way as the elements appear in the rows of the matrix. For example:
\begin{align*}
e_0 &= 1, \\
e_1 &= h_1, \\
e_2 &= h_1^2 - h_2, \\
e_3 &= h_1^3 - h_1 h_2 - h_2 h_1 + h_3 = h_3^3 - h_3.
\end{align*}
In fact, (4.8) is a formal consequence of the infinite Grassmannian relation which does not require any commutativity. The same thing holds for ordinary symmetric functions, indeed, (4.3) is the same relation as for the algebra $\text{Sym}$ of symmetric functions from [Mac, (I.2.6')] and (4.8) is [Mac, Ex. I.2.8].

It is often useful to work with the generating functions
\begin{equation}
e(t) = \sum_{r \geq 0} e_r t^{-r}, \quad h(t) = \sum_{r \geq 0} h_r t^{-r}, \quad (4.9)
\end{equation}
which are elements of $\text{OSym}[\![t^{-1}]\!]$ for a formal even variable $t$. Now the infinite Grassmannian relation becomes the first of the following:
\begin{align}
e(-t)h(t) &= 1, & h(t)e(-t) &= 1. \quad (4.10)
\end{align}
Since $h(t)$ is invertible in the formal power series ring, its left inverse $e(-t)$ is also its right inverse, proving the second equality. In other words, we have that
\begin{equation}
\sum_{s=0}^{r} (-1)^s h_s e_{r-s} = \delta_{r,0} \quad (4.11)
\end{equation}
for all $r \geq 0$. Consequently,
\begin{equation}
h_r = \det \left( e_{[r+1]} \right)_{i,j=1,...,r}. \quad (4.12)
\end{equation}
The evident symmetry between complete and elementary odd symmetric functions is best expressed in terms of the algebra automorphism

\[ \psi : \text{OSym} \rightarrow \text{OSym}, \quad h_r \mapsto (-1)^r e_r. \]  

(4.13)

Extending \( \psi \) trivially to \( \text{OSym} \oplus \mathbb{R}^1 \), we have that \( \psi(h(t)) = e(-t) \). As \( e(-t) \) is left inverse to \( h(t) \) by (4.10), it follows that \( \psi(e(-t)) = \psi^2(h(t)) \) is left inverse to \( \psi(h(t)) = e(-t) \). Since \( h(t) \) is also left inverse to \( e(-t) \), this shows that \( \psi^2(h(t)) = h(t) \). Thus, we have shown that \( \psi \) is an involution, so we also have that \( \psi(e_r) = (-1)^r h_r \).

For \( \lambda \in \Lambda^+ \), we let

\[ h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots, \quad e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots. \]  

(4.14)

Similarly, we define \( h_\alpha \) and \( e_\alpha \) for a composition \( \alpha \in \Lambda(k, n) \). As in \([\text{EKL}, (2.25)]\), the relations (4.1) and (4.2) imply for \( r < s \) that

\[ h_r h_s = \begin{cases} 
    h_s h_r & \text{if } r \text{ and } s \text{ have the same parity} \\
    h_s h_r + 2 \sum_{i=1}^{r} (-1)^r h_{s+i} h_{r-i} & \text{if } r \text{ is even and } s \text{ is odd} \\
    -h_s h_r - 2 \sum_{i=1}^{r} (-1)^r h_{s+i} h_{r-i} & \text{if } r \text{ is odd and } s \text{ is even}
\end{cases} \]  

(4.15)

Similarly, by (4.4) and (4.5), we have for \( r < s \) that

\[ e_r e_s = \begin{cases} 
    e_s e_r & \text{if } r \text{ and } s \text{ have the same parity} \\
    e_s e_r + 2 \sum_{i=1}^{r} (-1)^r e_{s+i} e_{r-i} & \text{if } r \text{ is even and } s \text{ is odd} \\
    -e_s e_r - 2 \sum_{i=1}^{r} (-1)^r e_{s+i} e_{r-i} & \text{if } r \text{ is odd and } s \text{ is even}
\end{cases} \]  

(4.16)

Consequently, any monomial \( h_\alpha \) or \( e_\alpha \) for \( \alpha \in \Lambda(k, n) \) can be rearranged into decreasing order modulo a linear combination of lexicographically greater monomials of the same degree. This proves the easy spanning part of the next theorem.

**Theorem 4.1** ([EK, Cor. 2.12]). The set \( \{h_\lambda \mid \lambda \in \Lambda^+\} \) is a linear basis for \( \text{OSym} \). Equivalently, applying \( \psi \), the set \( \{e_\lambda \mid \lambda \in \Lambda^+\} \) is a basis.

There is a comultiplication \( \Delta^- : \text{OSym} \rightarrow \text{OSym} \otimes \text{OSym} \) making \( \text{OSym} \) into a graded Hopf superalgebra such that

\[ \Delta^-(h_r) = \sum_{s=0}^{r} h_s \otimes h_{r-s} \]  

(4.17)

for all \( r \geq 0 \). This can be written more concisely in terms of generating functions as

\[ \Delta^-(h(t)) = h(t) \otimes h(t). \]  

(4.18)

By [EK, Prop. 2.17], the antipode \( S^- : \text{OSym} \rightarrow \text{OSym} \), which we remind the reader is both a superalgebra anti-automorphism and a cosuperalgebra anti-automorphism, satisfies \( S^-(h_1) = -h_1 \) and \( S^-(h_2) = h_1^2 - h_2 \), hence, \( S^-(h_2) = (-1)^n(h_2 - nh_1^2) \) for any \( n \geq 0 \). Another important point is that \( \text{OSym} \) is not a cocommutative
cosuperalgebra, e.g., \( \Delta^-(h_2) = h_2 \otimes 1 + h_1 \otimes h_1 + 1 \otimes h_2 \) is not invariant under the braiding \( B \) on \( gsVec \). So the opposite comultiplication
\[
\Delta^+ := B_{OSym,OSym} \circ \Delta^- : OSym \to OSym \otimes OSym
\]
gives a second graded Hopf superalgebra structure on \( OSym \) (the multiplication is the same as before). Remembering that our \( e_r \) is \((-1)^{\binom{r}{2}} e_r\), [EK, Prop. 2.5] implies that
\[
\Delta^+(e_r) = \sum_{s=0}^{r} e_s \otimes e_{r-s}
\]
or, equivalently,
\[
\Delta^+(e(t)) = e(t) \otimes e(t).
\]
It follows that
\[
\Delta^- \circ \psi = (\psi \otimes \psi) \circ \Delta^+,
\]
\[
\Delta^+ \circ \psi = (\psi \otimes \psi) \circ \Delta^-,
\]
because both sides of the left hand equation agree on \( e(t) \) and both sides of the right hand equation agree on \( h(t) \). This shows that \( \psi \) is a cosuperalgebra anti-involution. The antipode \( S^+ \) for the second Hopf superalgebra structure is the inverse of \( S^- \), so it takes \( e_r \) to \((-1)^r h_r \).

We will use two more useful symmetries
\[
e : OSym \to OSym,
\]
\[
* : OSym \to OSym,
\]
the first of which is an algebra involution, and the second is a superalgebra anti-involution. It is a routine check using (4.4) and (4.5) to see that these make sense. Note in particular that \( e \) takes our \( e_r \) to the \( e_r \) of [EK, EKL]. The symmetries \( e \) and \( * \) obviously commute. Neither \( e \) nor \( * \) commutes with \( \psi \), but it is still true that \( * \circ e \) commutes with \( \psi \); see Lemma 4.9 below for the proof of this. We have that
\[
\Delta^- \circ e = (e \otimes e) \circ \Delta^+,
\]
\[
\Delta^+ \circ e = (e \otimes e) \circ \Delta^-,
\]
\[
\Delta^- \circ * = (* \otimes *) \circ \Delta^+,
\]
\[
\Delta^+ \circ * = (* \otimes *) \circ \Delta^-.
\]
To justify these, it suffices to check the left hand equations, then the right hand ones follow because \( B_{OSym,OSym} \circ (e \otimes e) = (e \otimes e) \circ R \) and \( B_{OSym,OSym} \circ (\psi \otimes \psi) = (\psi \otimes \psi) \circ R \). To check the left hand equation from (4.25), one instead shows that \( B_{OSym,OSym} \circ \Delta^+ \circ e = (e \otimes e) \circ \Delta^+ \) by checking that both sides do the same on \( e_r \). The left hand equation form (4.26) holds because both sides take \( e(t) \) to \( e(t) \otimes e(t) \).

**Lemma 4.2.** For \( \lambda \in \Lambda^+ \), \( e(e_{\lambda})^* = (-1)^{dN(\lambda)+dW(\lambda)} e_{\lambda}^* \) (a \( \mathbb{Z} \)-linear combination of \( e_\mu \) for \( \mu >_{lex} \lambda \)).

**Proof.** Let \( k := ht(\lambda) \). By the definitions, we have that \( e(e_{\lambda})^* = (-1)^{\binom{\lambda}{2}} e_{\lambda_1} \cdots e_{\lambda_k} \). Then we use (4.16) to rewrite \( e_{\lambda_1} \cdots e_{\lambda_k} \) as \( \pm e_{\lambda} \) plus a sum of lexicographically higher \( e_\mu \). It remains to compute the sign. We get a sign change each time we commute \( e_{\lambda_j} \) with \( e_{\lambda_i} \) for \( 1 \leq i < j \leq k \). We use (4.14) to rewrite \( e_{\lambda_j} \) as \( \pm e_{\lambda} \) plus a sum of lexicographically higher \( e_\mu \). It remains to compute the sign. We get a sign change each time we commute \( e_{\lambda_j} \) with \( e_{\lambda_i} \) for \( 1 \leq i < j \leq k \) such that \( \lambda_j \) is odd and \( \lambda_i \) is even. So the overall sign is \((-1)^{\binom{\lambda}{2} + \sum_{1 \leq i < j \leq k} (\lambda_i - 1) \lambda_j} \). This simplifies to \((-1)^{dN(\lambda)+dW(\lambda)} \). \( \square \)

**Remark 4.3.** In [EK, Sec. 2.3], symmetries denoted \( \psi_1, \psi_2 \) and \( \psi_3 \) are introduced. These are related to our \( \psi, e \) and \( * \) by \( \psi_1 = e \circ p \circ \psi \) (because the latter takes \( h_r \) to \( e_r = (-1)^{\binom{r}{2}} e_r \)), \( \psi_2 = p \circ e \circ \psi \) (because the latter sends \( h_r \) to \((-1)^{\binom{r}{2}} h_r \)), and \( \psi_3 = e \circ * \circ \psi \) (because the latter is a superalgebra anti-involution taking \( h_r \) to \( h_r \)). We emphasize that our \( \psi_1 = \psi_1 \circ \psi_2 \) is an involution, whereas \( \psi_1 \) is not.

In [EK], the definition of \( OSym \) is motivated by the definition of a non-degenerate symmetric\(^1\) bilinear form \( (\cdot, \cdot)^* : OSym \otimes OSym \to \mathbb{F} \). Extending the bilinear form \((\cdot, \cdot)^*\) on \( OSym \) to \( OSym \otimes OSym \) so that
\[\text{We really do mean symmetric rather than supersymmetric here!}\]
(a_1 \otimes a_2, b_1 \otimes b_2)^- = (a_1, b_1)^- (a_2, b_2)^-\), the form is characterized uniquely by the following properties:

\[ (h_r, h_s)^- = \delta_{r,s}, \quad (ab, c)^- = (a \otimes b, \Delta^*(c))^+ \]  

for \( r, s \geq 0, a, b, c \in \text{OSym} \). For symmetry’s sake, one can also consider a form \((\cdot, \cdot)^+\) which is defined in a similar way so that

\[ (e_r, e_s)^+ = \delta_{r,s}, \quad (ab, c)^+ = (a \otimes b, \Delta^+(c))^+ \]  

for \( r, s \geq 0, a, b, c \in \text{OSym} \). The forms \((\cdot, \cdot)_\pm\) are related by the first of the following properties:

\[ \psi(a), \psi(b)^\pm = (a, b)^\mp, \quad \epsilon(a^\ast), \epsilon(b^\ast)^\pm = (a, b)^\mp \]  

for any \( a, b \in \text{OSym} \). This and the second property are both checked by induction on degree, using (4.13) and (4.22) to (4.26). The first of the next two properties follows from [EK, (2.10)], then the second follows by applying \( \psi \):

\[ (e_r, e_s)^\pm = (-1)^{(r)} \delta_{r,s}, \quad (h_r, h_s)^\pm = (-1)^{(r)} \delta_{r,s}. \]  

The following allows \((h_r, e_\mu)^\pm\) to be computed:

\[ (h_r, e_\lambda)^- = (-1)^{(r)} \delta_{\lambda, (\lambda')^+}, \quad (h_r, e_\lambda)^+ = (-1)^{(r)} \delta_{\lambda, (\lambda')^+}. \]  

The first equality here is established in [EK, Prop. 2.5], again remembering that our normalization of the elements \(e_r\) is different; then the second follows on applying \( \psi \). In particular, (4.31) can be used to show that

\[ (h_\lambda, e_\mu)^- = \begin{cases} (-1)^{NE(\lambda)+dN(\mu)} & \text{if } \lambda = \mu^\ast \\ 0 & \text{if } \lambda \not\leq_{\text{lex}} \mu^\ast \end{cases} \]  

\[ (h_\lambda, e_\mu)^+ = \begin{cases} (-1)^{NE(\mu)+dN(\mu)} & \text{if } \lambda = \mu^\ast \\ 0 & \text{if } \lambda \not\leq_{\text{lex}} \mu^\ast \end{cases} \]  

for \( \lambda, \mu \in \Lambda^\ast \); see [EK, Prop. 2.14] for the first equality. This “semi-orthogonality” is used to complete the proof of Theorem 4.1 in [EK].

Recall that \( \text{OPol}_n \) is the algebra of odd polynomials from (2.4). Another way to motivate the definition of \( \text{OSym} \) is explained in [EKL]. It is based on the following observation.

**Theorem 4.4.** There is a graded superalgebra homomorphism \( \pi_n : \text{OSym} \to \text{OPol}_n \) taking \( e_r \) and \( h_r \) to the polynomials

\[ e_r(x_1, \ldots, x_n) := \sum_{1 \leq i_1 < \cdots < i_r \leq n} x_{i_1} \cdots x_{i_r} \]  

\[ h_r(x_0, \ldots, x_1) := \sum_{n \geq i_1 \geq \cdots \geq i_r \geq 1} x_{i_1} \cdots x_{i_r}, \]  

respectively. Moreover, \( \pi_n \) intertwines the involution \( e \) of \( \text{OSym} \) with the algebra involution \( e_0 \) of \( \text{OPol}_n \) defined by \( e_0(x_i) := x_{n+1-i} \), and it intertwines the anti-involution \( * \) of \( \text{OSym} \) with the superalgebra anti-involution \( * \) of \( \text{OPol}_n \) defined by \( x_i^* := x_i \). Finally, if \( n = a + b \), the diagram

\[ \begin{array}{ccc} \text{OSym} & \xrightarrow{\Lambda^\ast} & \text{OSym} \otimes \text{OSym} \\ \pi_n \downarrow & & \pi_a \otimes \pi_b \\ \text{OPol}_n & = & \text{OPol}_a \otimes \text{OPol}_b \end{array} \]  

commutes, where the identification at the bottom is as explained after (2.4).

**Proof.** Note our \( x_i \) is the variable \( \tilde{x}_i = (-1)^{i-1} x_i \) in the notation of [EKL], hence, our \( e_r(x_1, \ldots, x_n) \) is the polynomial denoted \( e_r(x_1, \ldots, x_n) \) in [EKL]. With this in mind, [EKL, Lem. 2.3] checks that the polynomials \( e_r(x_1, \ldots, x_n) \in \text{OPol}_n \) satisfy the defining relations of \( \text{OSym} \) from (4.4) and (4.5). Hence, there is a unique homomorphism \( \pi_n : \text{OSym} \to \text{OPol}_n \) such that \( \pi_n(e_r) = e_r(x_1, \ldots, x_n) \) for all \( r \geq 0 \).
The involution $\epsilon$ of $OSym$ takes $e_r$ to $(-1)^{[\lambda]} e_r$. The involution $\epsilon_n$ of $OPol_n$ defined in the statement of the theorem takes $e_r(x_1, \ldots, x_n)$ to
\[ e_r(x_1, \ldots, x_n) = \sum_{n \geq i_2 > \cdots > i_1 \geq 1} x_{i_1} \cdots x_{i_2} x_{i_1}. \]
Rearranging these monomials into increasing order of $x_i$ produces a sign of $(-1)^{[\lambda]}$. Hence, $\epsilon_n$ takes $e_r(x_1, \ldots, x_n)$ to $(-1)^{[\lambda]} e_r(x_1, \ldots, x_n)$. This checks that $\pi_n \circ \epsilon = \epsilon_n \circ \pi_n$. Similarly, we see that $\pi_n \circ * = * \circ \pi_n$ because $*$ on $OSym$ fixes $e_r$ and $*$ on $OPol_n$ fixes $e_r(x_1, \ldots, x_n)$.

In [EKL, Lem. 2.8], again using that our $x_i$ is $\bar{x}_i$ in [EKL], it is checked that the polynomials
\[ h_r(x_1, \ldots, x_n) = \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r} \]
satisfy $\sum_{r=0}^{\lambda} (-1)^{[\lambda]} e_s(x_1, \ldots, x_n) h_{r-s}(x_1, \ldots, x_n) = \delta_{r,s}$ for all $r \geq 0$. Applying $\epsilon_n$, it follows that $\sum_{r=0}^{\lambda} (-1)^{[\lambda]} e_s(x_1, \ldots, x_1) h_{r-s}(x_1, \ldots, x_1) = \delta_{r,s}$. We already know that $e_s(x_n, \ldots, x_1) = (-1)^{[\lambda]} \pi_n(e_s)$, so this shows that $\sum_{r=0}^{\lambda} (-1)^{[\lambda]} \pi_n(e_s) h_{r-s}(x_n, \ldots, x_1) = \delta_{r,s}$. Comparing with (4.3), this proves that $\pi_n(h_r) = h_r(x_n, \ldots, x_1)$.

Finally, to see that (4.35) commutes, use (4.20) and the definition of $e_r(x_1, \ldots, x_n)$. \qed

Now we define $OSym_n$, the algebra of odd symmetric polynomials, to be the subalgebra of $OPol_n$ that is the image of the homomorphism $\pi_n$ from Theorem 4.4. For any $a \in OSym$, we use the notation $a^{(n)}$ to denote its canonical image in $OSym_n$. For $\lambda \in \Lambda^+$, we have that
\[ e^{(n)}_{\lambda} = \begin{cases} (-1)^{NE(\lambda)} x^\lambda + \text{(a $\mathbb{Z}$-linear combination of $x^\kappa$ for $\kappa < \lambda$)} & \text{if } \operatorname{ht}(\lambda) \leq n, \\ 0 & \text{if } \operatorname{ht}(\lambda) > n, \end{cases} \tag{4.36} \]
where $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ and $x^\lambda$ is defined similarly, identifying $\lambda \in \Lambda^+$ with $\operatorname{ht}(\lambda) \leq n$ with $(\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$. This is easily checked from the definition and gives the clearest explanation for the sign $NE(\lambda)$.

**Theorem 4.5.** Both of the sets $\{h_\lambda^{(n)} \mid \lambda \in \Lambda_+^n\}$ and $\{e^{(n)}_{\lambda} \mid \lambda \in \Lambda^+\}$ are bases for $OSym_n$. Also, the bilinear forms $(\cdot, \cdot)^{\pm}$ on $OSym$ induce non-degenerate forms on the quotient $OSym_n$.

**Proof.** The set $\{e^{(n)}_{\lambda} \mid \lambda \in \Lambda_+^n\}$ is a basis for $OSym_n$ thanks to (4.36), since this establishes the linear independence of these elements, and also the images of all other $e_\mu$ in the basis for $OSym$ from Theorem 4.1 are zero. To deduce the result for $\{h_\lambda^{(n)} \mid \lambda \in \Lambda_+^n\}$, the ideal ker $\pi_n$ has linear basis $\{e_\mu \mid \mu \in \Lambda^+, \mu_1 > n\}$. So by (4.32) the orthogonal complement (ker $\pi_n$)$^\perp$ with respect to either of the forms $(\cdot, \cdot)^{\pm}$ has basis $\{h_\lambda \mid \lambda \in \Lambda^+, \operatorname{ht}(\lambda) \leq n\}$. Hence, $\{h_\lambda^{(n)} \mid \lambda \in \Lambda^+, \operatorname{ht}(\lambda) \leq n\}$ is a basis for the quotient $OSym_n \cong (\ker \pi_n)^{\perp}$. It is also clear from this argument that both bilinear forms induce non-degenerate forms on $OSym_n$ so that $(\pi_n(a), \pi_n(b))^\pm = (a, b)^\pm$ for all $a, b \in OSym$. \qed

**Corollary 4.6.** The quotient maps $\pi_n : OSym \to OSym_n$ induce an isomorphism
\[ OSym \xrightarrow{\sim} \varprojlim OSym_n, \]
where on the right we have the inverse limit of the inverse system $\cdots \to OSym_1 \to OSym_0$ taken in the category of graded superalgebras, with the map $OSym_{n+1} \to OSym_n$ taking $e_r^{(n+1)}$ to $e_r^{(n)}$ if $r \leq n$ and to zero otherwise. Moreover, $OSym_n$ may be identified with the quotient $OSym / \langle e_r \mid r > n \rangle$.

**Corollary 4.7** ([EKL, (2.20)]), $\dim_{n, \pi} OSym_n = \prod_{i=1}^{n-1} \frac{1}{1 - (\pi q^2)^i}$. 

Proof. Theorem 4.5 shows that $O\text{Sym}_n$ has the same graded superdimension as a commutative polynomial algebra with generators $x_1, \ldots, x_n$ such that $x_r$ is of degree $2r$ and parity $r \pmod{2}$. □

Corollary 4.8. $\dim_{q,\pi} O\text{Pol}_n = \dim_{q,\pi} O\text{Sym}_n \times q^{(2)} [n]_{q,\pi}^1 = \dim_{q,\pi} O\text{Sym}_n \times \sum_{w \in S_n} (\pi q^2)^{(w)}$.

Proof. The second equality follows from the first by (3.3). To obtain the first equality, we use Corollary 4.7 to see that

$$\dim_{q,\pi} O\text{Sym}_n \times q^{(2)} [n]_{q,\pi}^1 = q^{(2)} [n]_{q,\pi}^1 \prod_{r=1}^{n} \frac{1}{1 - (\pi q^2)^r} = \frac{q^{(2)} [n]_{q,\pi}^1}{(1 - \pi q^2)^n} \prod_{r=1}^{n} \frac{1 - \pi q^2}{1 - (\pi q^2)^r} = \frac{[n]_{q,\pi}^1}{(1 - \pi q^2)^n} \prod_{r=1}^{n} \frac{\pi q - q^{-1}}{\pi q^r - q^{-r}} = \frac{1}{(1 - \pi q^2)^n} \dim_{q,\pi} O\text{Pol}_n.$$ □

It is time to say a little more about the various signed variants of the odd complete and elementary symmetric functions. The following lemma, which is another application of Theorem 4.4, is helpful to understand the possibilities.

Lemma 4.9. We have that $e(h_r) = (-1)^{(2)} h_r^*$ and $e(e_r) = (-1)^{(2)} e_r^*$ for any $r \geq 0$. Hence, $\psi \circ \epsilon \circ \psi = \psi \circ \epsilon \circ \epsilon$. Proof. It is immediate from the definitions that $e(e_r) = (-1)^{(2)} e_r^*$. To see the analogous thing for $h_r$, it suffices to show that $e(h_r^{(n)}) = (-1)^{(2)} h_r^{n}$. This follows from the explicit descriptions of these polynomials and maps given in Theorem 4.4. To deduce finally that $\psi \circ \epsilon$ and $\psi$ commute, it suffices to check that $(\psi \circ \epsilon \circ \psi)(e_r) = (\psi \circ \epsilon \circ \epsilon)(e_r)$ for all $r \geq 0$, which is clear at this point. □

As we have said before, our odd complete symmetric function $h_r$ is the same as $h_r$ in [EK, EKL], but our odd elementary symmetric function $e_r$ is different from the one there, which is

$$e_r := e(e_r) = (-1)^{(2)} e_r^* = (-1)^{(2)} e_r,$$

where the non-trivial equality follows by Lemma 4.9. There is also a natural variant on the odd complete symmetric function $h_r$, namely,

$$h_r := e(h_r) = (-1)^{(2)} h_r^*.$$ (4.38)

Since $h_r^* \neq h_r$ for $r > 1$, it is not the case that $h_r = (-1)^{(2)} h_r$. We call $e_r$ and $h_r$ the dual odd elementary and complete symmetric functions, respectively. Applying $\epsilon$ to (4.10) gives that

$$\epsilon(-t)\gamma(t) = 1, \quad \gamma(t)\epsilon(-t) = 1.$$ (4.39)

These should make it clear that $\epsilon$ and $h$ belong together as do $\epsilon$ and $h$—it is not so easy to relate $\epsilon$ to $\gamma$ or $\epsilon$ to $h$ in terms of generating functions. We try as far as possible to work in terms of $e_r$ and $h_r$, but are forced to use the “re-signed” generating functions $\gamma(t)$ and $\epsilon(t)$ in formulating Lemma 9.10 below.

Consider again the truncation $O\text{Sym}_n$. Let $e_r^{(n)}$ and $h_r^{(n)}$ be the images $e_r$ and $h_r$ in $O\text{Sym}_n$. This actually gives an explanation for the existence of the four basic families of odd symmetric functions $e_r, h_r, e_r, h_r$: from Theorem 4.4, it is clear that

$$e_r^{(n)} = \sum_{1 \leq i_1 < \cdots < i_r \leq n} x_{i_1} \cdots x_{i_r}, \quad \gamma_r^{(n)} = \sum_{1 \leq i_1 < \cdots < i_r \leq n} x_{i_1} \cdots x_{i_r},$$

$$h_r^{(n)} = \sum_{n \geq i_1 \geq \cdots \geq i_r \geq 1} x_{i_1} \cdots x_{i_r}, \quad h_r^{(n)} = \sum_{n \geq i_1 \geq \cdots \geq i_r \geq 1} x_{i_1} \cdots x_{i_r}.$$ (4.41)
When working in $O\text{Sym}_n$ with generating functions, we prefer to modify the definitions slightly, working not exactly with their images in $O\text{Sym}[t^{-1}]$, but incorporating a shift in $t$:

$$
e^{(n)}(t) := \sum_{r=0}^{n} e_r^{(n)} t^{-r}, \\
h^{(n)}(t) := \sum_{r=0}^{n} h_r^{(n)} t^{-n-r}, \\
e^{(n)}(t) := \sum_{r=0}^{n} e_r^{(n)} t^{-r}, \\
g^{(n)}(t) := \sum_{r=0}^{n} g_r^{(n)} t^{-n-r}.
$$

(4.42)

(4.43)

The advantage of this is that $e^{(n)}(t)$ and $e^{(n)}(t)$ are polynomials $O\text{Sym}_n[t]$, indeed, we have that

$$
e^{(n)}(t) = (t + x_1) \cdots (t + x_n), \\
h^{(n)}(t) = (t - x_n)^{-1} \cdots (t - x_1)^{-1},
$$

(4.44)

$$
e^{(n)}(t) = (t + x_n) \cdots (t + x_1), \\
g^{(n)}(t) = (t - x_1)^{-1} \cdots (t - x_n)^{-1}.
$$

(4.45)

A disadvantage is that the infinite Grassmannian relation in the polynomial setting requires an additional sign:

$$
h^{(n)}(t)^{-1} = (-1)^n e^{(n)}(-t), \\
g^{(n)}(t)^{-1} = (-1)^n e^{(n)}(-t),
$$

(4.46)

equality in the ring $O\text{Sym}_n[[t^{-1}]]$ of formal Laurent series in $t^{-1}$.

The next result is elementary but does not appear in the existing literature. Observe by (4.3) that

$$z_{2r} := \sum_{s=0}^{r} e_2 h_{2r-2s} = \sum_{s=0}^{r-1} e_{2s+1} h_{2r-2s-1}.
$$

(4.47)

The element $z_{2r}$ is central: it commutes with all even $e_r$ by the first form of the definition, and it commutes with all odd $e_r$ by the second one. Also let omicron be the special element

$$
o := e_1 = h_1,
$$

(4.48)

noting that $z_2 = o_2$. The relations (4.2) and (4.5) imply that

$$
e_{2r+1} = \frac{1}{2}(oe_{2r} + e_{2r} o), \\
h_{2r+1} = \frac{1}{2}(oh_{2r} + h_{2r} o),
$$

(4.49)

so that $O\text{Sym}$ is generated already by $o$ and all even $e_{2r}$ ($r \geq 1$).

**Theorem 4.10.** The graded superalgebra $O\text{Sym}$ is generated by $o$ and $e_{2r}$ ($r \geq 1$) subject only to the relations

$$
[e_{2r}, e_{2s}] = 0 \\
[o^2, e_{2r}] = 0 \\
[o, e_{2r+2}] = \left[\frac{1}{2}(oe_{2r} + e_{2r} o), e_2\right]
$$

(4.50)

(4.51)

(4.52)

for $r, s \geq 1$.

**Proof.** Let $A$ be the superalgebra generated by an odd element $o$ and even elements $e_{2r}$ ($r \geq 1$) subject to the relations (4.50) to (4.52). For $r \geq 0$, we set $e_1 := o$ and $e_{2r+1} := \frac{1}{2}(oe_{2r} + e_{2r} o) \in A$ for $r \geq 1$; cf. (4.47).

We first construct a homomorphism $\alpha : A \to O\text{Sym}$ by mapping $o \mapsto o$ and $e_r \mapsto e_r$. To see that this makes sense, we need to check that the relations (4.50) to (4.52) hold in $O\text{Sym}$. The first is immediate, and the second follows because we have observed already that $z_2 = o_2^2$ is central in $O\text{Sym}$. For the third, in $O\text{Sym}$, we have that $[o, e_{2r+2}] = [e_{2r+1}, e_2]$ by (4.5), and also $\frac{1}{2}(oe_{2r} + e_{2r} o) = e_{2r+1}$ by (4.49). Now the relation is clear.

Next we construct a homomorphism $\beta : O\text{Sym} \to A$ in the other direction so that on generators it sends $e_r \mapsto e_r$ for each $r \geq 1$. To show this is well defined, we must again check relations, this time
showing that (4.4) and (4.5) hold in \(A\). The first one is immediate if \(r\) and \(s\) are both even. When \(r\) is odd and \(s\) is even, (4.5) is equivalent to the relation 

\[
[e_{2r-1}, e_{2s}] = [e_{2s-1}, e_{2r}]
\]  

(4.53)

for \(r, s \geq 1\). To check it holds in \(OSym\), we must show that the expression \([e_{2r-1}, e_{2s}] \in A\) is symmetric in \(r\) and \(s\). We have that 

\[
4[e_{2r-1}, e_{2s}] = 2[e_{2r-2} + e_{2r-2}o, e_{2s}]
\]

\[
= 2oe_{2r-2}e_{2s} + 2e_{2r-2}oe_{2s} - 2e_{2r}oe_{2r-2} - 2e_{2r}e_{2r-2}o
\]

\[
= 2(e_{2s} - e_{2s}o)e_{2r-2} + 2e_{2r-2}(oe_{2s} - e_{2s}o)
\]

\[
= 2[e_{2r-2} + 2e_{2r-2}o, e_{2s}]
\]

\[
= [oe_{2s-2} + e_{2s-2}o, e_2]e_{2r-2} + e_{2r-2}[oe_{2s-2} + e_{2s-2}o, e_2]
\]

\[
= [o, e_2]e_{2s-2}e_{2r-2} + e_{2s-2}[o, e_2]e_{2r-2} + e_{2r-2}[o, e_2]e_{2s-2} + e_{2r-2}e_{2s-2}[o, e_2],
\]

which is indeed symmetric in \(r\) and \(s\). When \(r\) is even and \(s\) is odd, (4.5) is equivalent to the relation 

\[
[e_{2r}, e_{2s+1}] = [e_{2s}, e_{2r+1}]
\]  

(4.54)

for \(r, s \geq 0\), where \([x, y]\) here denotes \(xy + yx\). So again we must show that \([e_{2r-2}, e_{2s-1}] \in A\) is symmetric in \(r\) and \(s\). We have that 

\[
2[e_{2r}, e_{2s+1}] = e_{2r}oe_{2s} + e_{2r}e_{2s}o + oe_{2s}e_{2r} + e_{2s}oe_{2r}.
\]

This is symmetric in \(r\) and \(s\) because \(e_{2r}e_{2s} = e_{2s}e_{2r}\). It remains to check the relation (4.4) when \(r\) and \(s\) are both odd. Equivalently, we show that \([e_{2r+1}, e_{2s+1}] = 0\) for \(r, s \geq 0\) by induction on \(s\). The base case \(s = 0\) follows because 

\[
2[e_{2r+1}, o] = [oe_{2r+1} + e_{2r+1}o, o] = oe_{2r+1}o + e_{2r+1}o^2 - o^2e_{2r+1} - oe_{2r}o = -[o^2, e_{2r+1}] = 0.
\]

The following establishes the induction step: for \(s > 0\) we have in \(A\) that 

\[
2[e_{2r+1}, e_{2s+1}] = [e_{2r+1}, oe_{2s} + e_{2s}o] = o[e_{2r+1}, e_{2s}] + [e_{2r+1}, e_{2s}o]
\]

\[
= o[e_{2r+1}, e_{2r+2}] + [e_{2s-1}, e_{2r+2}]o = [e_{2s-1}, oe_{2r+2} + e_{2r+2}o]
\]

\[
= 2[e_{2s-1}, e_{2r+3}] = -2[e_{2r+3}, e_{2s-1}] = 0,
\]

using the \(s = 0\) case for the second and fourth equalities, (4.53) for the third equality, and the induction hypothesis for the final equality.

It remains to observe that \(\alpha\) and \(\beta\) are two-sided inverses. This is obviously the case on generators by the way we have defined the maps. \(\square\)

As an application, we can describe the largest supercommutative quotient of \(OSym\). To formulate the result, let \(R\) be the usual commutative algebra of symmetric functions. We will denote its generators usually called \(e_r\) and \(h_r\) by \((-1)^r\hat{e}_2r\) and \(\hat{h}_2r\) in order to better match the notation being used in \(OSym\). Thus, \(R\) is the polynomial algebra generated freely by either of the sets \(\{\hat{e}_2r \mid r \geq 1\}\) or \(\{\hat{h}_2r \mid r \geq 1\}\), and the two sets of generators are related by 

\[
\sum_{s=0}^{r} \hat{e}_{2s} \hat{h}_{2r-2s} = \delta_{r,0}
\]  

(4.55)

where \(\hat{e}_0 = \hat{h}_0 := 1\). There is no sign in this relation since we have incorporated it into the definition of \(\hat{e}_2r\). We view \(R\) as a purely even graded superalgebra so that \(\deg \hat{e}_2r = \deg \hat{h}_2r = 4\). Also let \(R[\hat{o}]\) be the supercommutative graded superalgebra obtained by adjoining an odd central element \(\hat{o}\) of degree 2. We necessarily have that \(\hat{o}^2 = 2\) due to supercommutativity.
Theorem 4.11. The largest supercommutative quotient of $\text{OSym}$ is $\tilde{R} := \text{OSym}/I$ where $I$ is the two-sided ideal generated by $o^2$ and $[o, e_2]$. Moreover, denoting the image of $a \in \text{OSym}$ in $R$ by $\hat{a}$, there is a unique isomorphism of graded superalgebras

$$\gamma : \tilde{R} \sim R[\hat{o}]$$

such that $\gamma(\hat{o}) = \hat{o}$, $\gamma(\hat{e}_{2r}) = \hat{e}_{2r}$ and $\gamma(\hat{h}_{2r}) = \hat{h}_{2r}$ for all $r \geq 0$. Also we have that $\hat{e}_{2r+1} = \hat{e}_{2r} \hat{o}$, $\hat{h}_{2r+1} = \hat{h}_{2r} \hat{o}$ and $\hat{z}_{2r} = \delta_{r,0}$ for all $r \geq 0$.

Proof. In any supercommutative quotient of $\text{OSym}$, we must have that $o^2 = 0$ and $[o, e_2] = 0$. Now we let $I$ be the two-sided ideal of $\text{OSym}$ generated by $o^2$ and $[o, e_2]$ and show that $\text{OSym}/I$ is supercommutative. Since $\text{OSym}$ is generated by the elements $e_{2r}$ ($r \geq 1$) and $o$, the proof of this reduces at once to checking that $[o, e_{2r}] \in I$ for all $r \geq 1$, which holds because

$$2[o, e_{2r}] = 2[e_{2r-1}, e_2] = [e_{2r-2}o + oe_{2r-2}, e_2] = e_{2r-2}[o, e_2] + [o, e_2]e_{2r-2} \in I.$$ 

Thus, we have shown that $\tilde{R} := \text{OSym}/I$ is the largest supercommutative quotient of $\text{OSym}$.

Next we observe that $\hat{e}_{2r+1} = \hat{e}_{2r} \hat{o}$, $\hat{h}_{2r+1} = \hat{h}_{2r} \hat{o}$ and $\hat{z}_{2r} = \delta_{r,0}$ for all $r \geq 0$. The first two of these follow from (4.49) and the supercommutativity of $\text{OSym}/I$, then the final equality follows using the first two together with the second form of the definition of $z_{2r}$ in (4.47).

Finally, we construct the isomorphism $\gamma$. There is a unique homomorphism $\hat{\gamma} : \text{OSym} \rightarrow R[\hat{o}]$ taking $o \mapsto \hat{o}$ and $e_{2r} \mapsto \hat{e}_{2r}$ for $r \geq 1$. To see this, we apply Theorem 4.10 to reduce to checking that the relations (4.50) to (4.52) all hold in $R[\hat{o}]$, which is clear because it is supercommutative. The homomorphism $\hat{\gamma}$ factors through the quotient to induce $\gamma : \tilde{R} \rightarrow R[\hat{o}]$. Moreover, $\text{OSym}/I$ is spanned by the monomials $\hat{e}_\lambda$ and $\hat{e}_\lambda \hat{o}$ for partitions $\lambda$ with all parts even. The images under $\gamma$ of these elements give a linear basis for $R[\hat{o}]$. This shows that $\gamma$ is an isomorphism.

It just remains to check that $\gamma(\hat{h}_{2r}) = \hat{h}_{2r}$. In view of (4.55), this follows if we can show that

$$\sum_{s=0}^{r} \hat{e}_{2s} \hat{h}_{2r-2s} = \delta_{r,0} \quad (4.56)$$

for all $r \geq 0$. This is true because the sum on the left hand side is $\hat{z}_{2r}$ by the first form of the definition (4.47), which we have already shown is zero for $r \geq 1$. \hfill \Box

Corollary 4.12. The largest commutative quotient of $\text{OSym}$ is $\text{OSym}/J$ where $J$ is the two-sided ideal generated by $o$. Moreover, the homomorphism $\text{OSym} \rightarrow R$ mapping $e_{2r} \mapsto \hat{e}_{2r}, h_{2r} \mapsto \hat{h}_{2r}, e_{2r+1} \mapsto 0, h_{2r+1} \mapsto 0$ for all $r \geq 0$ induces an isomorphism $\text{OSym}/J \rightarrow R$ of graded superalgebras.

Henceforth, we will simply identify $\text{OSym}/J$ with $R$ via the isomorphism in Corollary 4.12. More generally, we will use the notation $\hat{c}$ to denote the image of $c \in \text{OSym}$ under the homomorphism $\text{OSym} \rightarrow R$ defined in the corollary. In particular, we have that $\hat{e}_{2r+1} = \hat{h}_{2r+1} = 0$ for all $r \geq 0$. With this convention, the relations (4.55) are equivalent to the more complicated relations

$$\sum_{s=0}^{r} (-1)^s \hat{e}_{r-s} \hat{h}_{r-s} = \delta_{r,0} \quad (4.57)$$

for all $r \geq 0$. This exactly matches the infinite Grassmannian relation in $\text{OSym}$. Finally we let

$$R_n := R/\langle \hat{e}_r \mid r > n \rangle. \quad (4.58)$$

This is just the usual ring of symmetric polynomials in $[n/2]$ variables.

Corollary 4.13. The largest commutative quotient of $\text{OSym}_n$ is $\text{OSym}_n/\langle o^n \rangle$. Moreover, this ring is canonically isomorphic to $R_n$. 

Proof. This follows from the previous corollary since a commutative quotient of $O\text{Sym}_n$ is a commutative quotient of $O\text{Sym}$ in which the images of all $e_r (r > n)$ are zero. 

We will also use the notation $\tilde{c}$ to denote the canonical image of $c \in O\text{Sym}_n$ in $R_n$. Note also that

$$\dim_{q,\pi} R_n = \prod_{r=1}^{\lfloor n/2 \rfloor} \frac{1}{1 - q^{4r}}. \quad (4.59)$$

Later in the article, we will use the commutative algebra $R_n$ (actually, $R_\ell$ for another $\ell \in \mathbb{N}$) as our ground ring.

**Remark 4.14.** Much of the theory to come could in fact be developed slightly more generally with $R_n$ replaced by the largest supercommutative quotient $\tilde{R}_n$ of $O\text{Sym}_n$, which is the quotient of $O\text{Sym}_n$ by the two-sided ideal generated by $(o^{(n)})^2$ and $[e^{(n)}, e_2^{(n)}]$. This superalgebra can also be identified at this point with the graded supercommutative superalgebra $R_n[O^{(n)}]$ obtained from $R_n$ by adjoining an odd generator $o^{(n)}$ of degree 2 if $n$ is odd, or with $R_n[o^{(n)}]/(e_n^{(n)} o^{(n)})$ if $n$ is even. We have taken the view that the additional complications involved in working over a ground ring that is merely supercommutative, not actually commutative, are not worth the extra effort, but we will discuss this possibility again in several remarks later on.

5. **Odd nil-Hecke algebras**

This section is largely an exposition of results from [EKL]. The *odd nil-Hecke algebra* is the graded superalgebra $O\text{NH}_n$ with generators $x_i (i = 1, \ldots, n)$ and $\tau_j (j = 1, \ldots, n - 1)$ which are odd of degrees 2 and $-2$, respectively, subject to the following relations:

$$x_i x_j = -x_j x_i \quad (i \neq j) \quad (5.1)$$

$$\tau_i \tau_j = -\tau_j \tau_i \quad ([i - j] > 1) \quad (5.2)$$

$$x_i \tau_j = -\tau_j x_i \quad (i \neq j, j + 1) \quad (5.3)$$

$$\tau_j^2 = 0 \quad (5.4)$$

$$\tau_j \tau_{j+1} \tau_j = -\tau_{j+1} \tau_j \tau_{j+1} \quad (5.5)$$

$$x_i \tau_i - \tau_i x_i = 1 = \tau_i x_i - x_i \tau_i. \quad (5.6)$$

We warn the reader that the above is *not* the standard form of the presentation for this algebra which appears in all of the existing literature. The difference is in the relations (5.5) and (5.6), in which our minus signs become plus signs in the standard presentation. To obtain the above presentation from the standard one, note that our generators $x_i$ and $\tau_j$ are equal to the elements denoted $(-1)^{i-1} x_i$ and $(-1)^{j-1} \tau_j$ elsewhere in the literature. This change certainly impacts many other formulae below, but it is usually straightforward to make the appropriate adaptation. One advantage of our modified sign convention can already be seen in the definitions (4.33) and (4.34) above—the corresponding formulae in [EKL] involve some additional signs.

Let $S_n$ act on the left on $O\text{Pol}_n$ by graded superalgebra automorphisms so that $w x_i = (-1)^{\ell(w)+\ell(i)-\ell(w(i))} x_{w(i)}$ for $w \in S_n$, $1 \leq i \leq n$. In particular:

$$s_i^j x_i = \begin{cases} x_{j+1} & \text{if } i = j \\ x_j & \text{if } i = j + 1 \\ -x_i & \text{otherwise.} \end{cases} \quad (5.7)$$
The odd Demazure operator \( \partial_j : OPol_n \to OPol_n \) is the linear map defined on \( f \in OPol_n \) by
\[
\partial_j(f) = \frac{(x_j + x_{j+1})f - (t^i f)(x_j + x_{j+1})}{x_j^2 - x_{j+1}^2},
\] (5.8)
which makes sense because the denominator is central. This formula which first appeared in [KKO1, (4.10)] remembering, of course, our modified choice of signs. We actually never use this form of the definition of \( \partial_j \), preferring the following recursive definition: \( \partial_j \) is the unique odd linear map of degree \(-2\) such that
\[
\partial_j(x_i) = \delta_{i,j} - \delta_{i,j+1}, \quad \partial_j(fg) = \partial_j(f)g + ({}^t f) \partial_j(g)
\] (5.9)
for \( f, g \in OPol_n \). Now we make the graded vector superspace \( OPol_n \) into a left \( ONH_n \)-supermodule so that \( x_i \in ONH_n \) acts on \( f \in OPol_n \) by \( x_i \cdot f := x_i f \), and \( \tau_j \in ONH_n \) acts by \( \tau_j \cdot f := \partial_j(f) \). A relation check is required at this point to show that this definition makes sense.

In is straightforward to show by induction on \( r \) that
\[
\tau_i \cdot x_i^{r+1} = \sum_{s=0}^{r} x_i^{r-s} x_i^{r-s}, \quad \tau_i \cdot x_i^{r+1} = - \sum_{s=0}^{r} x_i^{r-s} x_i^{r-s}.
\] (5.10)
The power series \((t-x)^{-1} = t^{-1} + t^{-2}x + t^{-3}x^2 + \cdots \in \mathbb{F}[x][[t^{-1}]]\) appears often. For example, using it, one can rewrite (5.10) as the generating function identities:
\[
\tau_i \cdot (t-x_i)^{-1} = (t-x_{i+1})^{-1}(t-x_i)^{-1}, \quad \tau_i \cdot (t-x_{i+1})^{-1} = -(t-x_i)^{-1}(t-x_{i+1})^{-1}.
\] (5.11)
From the former, we get that
\[
\tau_{n-1} \cdots \tau_1 \cdot (t-x_1)^{-1} = (t-x_n)^{-1} \cdots (t-x_1)^{-1}.
\] (5.12)
Hence, recalling (4.44), we get that
\[
\tau_{n-1} \cdots \tau_1 \cdot x_1^{n+r-1} = h_r^{(n)}
\] (5.13)
on computing \( t^{n-r} \)-coefficients.

For each \( w \in S_n \), we fix a choice of reduced expression \( w = s_{j_1} \cdots s_{j_l} \) then set \( \tau_w := \tau_{j_1} \cdots \tau_{j_l} \).

Although \( \tau_w \) depends on the choice of reduced expression for \( w \), the dependency is only up to a sign. They appear often, so we introduce the shorthands\(^2\):
\[
\omega_n := \tau_{w_n}, \quad \chi_n = x_{n-1} \cdots x_2 x_1^{n-1}.
\] (5.14)
A key calculation at this point shows that \( \omega_n \cdot \chi_n = \pm 1 \). This is done in [EKL, Lem. 2.10] using a particular choice of reduced expression for \( w_n \) in order to be able to determine the sign precisely. If one is prepared to ignore the sign this is just the same easy inductive calculation as in the ordinary even setting. To eliminate subsequent issues with this sign, we assume henceforth that our fixed reduced expression for \( w_n \) has been chosen in such a way that
\[
\omega_n \cdot \chi_n = 1.
\] (5.15)
To see that this is possible, \( \omega_1 = 1, \omega_2 = \tau_1 \) and \( \omega_3 = \tau_2 \tau_1 \tau_2 \) have the property (5.15) by direct calculation. After that, pick two reduced expressions for \( w_n \) which differ by a single braid relation of the form \( s_i s_j = s_j s_i \). Since \( \tau_i \tau_j = -\tau_j \tau_i \), one of these reduced expressions produces an admissible choice of \( \omega_n \) satisfying (5.15). With our conventions, we in fact have that
\[
\omega_n = (\tau_{n-1} \tau_{n-2} \cdots \tau_1)(\tau_{n-1} \cdots \tau_2) \cdots (\tau_{n-1} \tau_{n-2})(\tau_{n-1}),
\] (5.16)
\(^2\)One reason we prefer this particular ordering for the monomial \( \chi_n \) can be seen in the proof of Lemma 8.8—we want \( x_1 \) to appear on the right so that the right hand column of the companion matrix computed there does not depend on \( y \).
although this will never be needed explicitly. It can be checked in a similar way to the proof of [EKL, Lem. 2.10].

An important role will be played by the odd Schubert polynomials

\[ p_w^{(n)} := \tau_{w^{-1}w_n} \cdot \chi_n \in \text{OPol}_n. \]  (5.17)

For example, we have that \( p_1^{(3)} = 1, p_{s_1}^{(3)} = -x_1, p_{s_2}^{(3)} = x_1 + x_2, p_{s_2s_1}^{(3)} = x_1^2, p_{s_1s_2}^{(3)} = -x_2x_1 \) and \( p_{s_2s_1s_2}^{(3)} = x_2x_1^2. \) In general, \( p_w^{(n)} \) depends up to sign on the choice of reduced expression for \( w^{-1}w_n, \) but we always have that \( p_w^{(n)} = \chi_n \) and \( p_1^{(n)} = 1 \) thanks to (5.15). Note also that \( \deg(p_w^{(n)}) = 2\ell(w) \) and \( \text{Par}(p_w^{(n)}) \equiv \ell(w) \pmod{2}. \)

**Theorem 5.1** ([EKL, Prop. 2.11]). *The elements \([x^j\tau_w] \mid w \in S_n, k \in \mathbb{Z}^n\) give a basis for ONH.* Moreover, \( \text{OPol}_n \) is a faithful ONH-module.

**Proof.** First one shows using (5.2), (5.4) and (5.5) that any word in \( \tau_j (j = 1, \ldots, n - 1) \) can be reduced to 0 or \( \pm \tau_n \) for some \( w \in S_n. \) It follows that the set in Theorem 5.1 spans \( \text{ONH}_n. \) Then to establish the linear independence, suppose that we have some non-trivial linear relation

\[ \sum_{w \in S_n, x \in \mathbb{Z}^n} c_{w,x} x^j \tau_w = 0 \]

between the elements of this set. Pick \( w \) of minimal length such that \( c_{w,x} \neq 0 \) for some \( x. \) Then we act on \( p_w^{(n)} \) for \( \ell(w') \not\equiv \ell(w) \) that \( \tau_{w'} \cdot p_w^{(n)} = 0 \) by the relations (5.2), (5.4) and (5.5), and \( \tau_w \cdot p_1^{(n)} = \pm 1 \) by (5.15). So we deduce that \( \sum_{x \in \mathbb{Z}^n} c_{w,x} x^j = 0, \) which is a contradiction. This also shows \( \text{OPol}_n \) is faithful. \( \square \)

**Corollary 5.2.** \( \dim_{q,\pi} \text{ONH}_n = \dim_{q,\pi} \text{OSym}_n \times q^{\binom{n}{2}}[n]_{q,\pi}^{\dagger} \times q^{-\binom{n}{2}}[n]_{q,\pi}^{\dagger}, \)

**Proof.** The theorem gives that

\[ \dim_{q,\pi} \text{ONH}_n = \dim_{q,\pi} \text{OPol}_n \times \sum_{w \in S_n} (\pi q^2)^{-\ell(w)}. \]

Now replace \( \dim_{q,\pi} \text{OPol}_n \) by the first formula for it from Corollary 4.8, and replace the summation by a product using (3.3). \( \square \)

The basis theorem implies that the obvious homomorphism \( \text{OPol}_n \rightarrow \text{ONH}_n \) is injective. Henceforth, we identify \( \text{OPol}_n, \) hence, also \( \text{OSym}_n, \) with a subalgebra of \( \text{ONH}_n \) via this map. By the relations (5.1) to (5.6), \( \text{ONH}_n \) admits an algebra involution \( \varepsilon_n \) and a superalgebra anti-involution \(*\) defined by

\[ \varepsilon_n : \text{ONH}_n \rightarrow \text{ONH}_n, \quad x_i \mapsto x_{n+1-i}, \quad \tau_j \mapsto -\tau_{n-j}, \]  (5.18)

\[ * : \text{ONH}_n \rightarrow \text{ONH}_n, \quad x_i \mapsto x_i, \quad \tau_j \mapsto -\tau_j. \]  (5.19)

These commute with each other. Moreover, the restrictions of \( \varepsilon_n \) and \(*\) to \( \text{OPol}_n \) coincide with the symmetries \( \varepsilon_n \) and \(*\) from Theorem 4.4. In terms of diagrams, \( \varepsilon_n \) reflects a diagram in the horizontal axis multiplying by \((-1)^X\) where \( X \) is the number of crossings, and \(*\) reflects a diagram in the vertical axis multiplying by \((-1)^{X+\binom{Y}{2}}\) where \( X \) is the number of crossings and \( Y \) is the number of dots. We note that

\[ \varepsilon_n(a) \cdot \varepsilon_n(f) = \varepsilon_n(a \cdot f) \]  (5.20)

for any \( a \in \text{ONH}_n \) and \( f \in \text{OPol}_n. \) It is also clear that

\[ \varepsilon_n(\omega_n) = \xi_n \omega_n \]  (5.21)
for some $\xi_n \in \{\pm 1\}$. One can verify explicitly via (5.16) that $\xi_n = (-1)^{(n+1)/2}$; the calculation is similar to the proof of [EKL, Lem. 3.2]. The only place this sign is used is in the proof of Theorem 6.8, and in that place we actually do not need to know its actual value.

Let $ONH_n^{\text{fin}}$ be the subalgebra of $ONH_n$ with basis $\{\tau_w \mid w \in S_n\}$. As an algebra, $ONH_n^{\text{fin}}$ is generated by the elements $\tau_j (j = 1, \ldots, n-1)$ subject just to (5.2), (5.4) and (5.5). There is a unique way to make the ground field $\mathbb{F}$ into a purely even graded left $ONH_n^{\text{fin}}$-supermodule concentrated in degree zero; each $\tau_j$ acts as zero. There is then a canonical isomorphism of graded $ONH_n$-supermodules

$$ONH_n \otimes_{ONH_n^{\text{fin}}} \mathbb{F} \sim \text{OPol}_n, \quad x^i \otimes 1 \mapsto x^i. \quad (5.22)$$

This isomorphism explains the origin of the polynomial representation of $ONH_n$.

Another application of the basis theorem shows that there is an injective homomorphism $ONH_n \hookrightarrow ONH_{n+1}$ taking $x_i$ to $x_i$ and $\tau_j$ to $\tau_j$. Thus, we have a tower of graded superalgebras

$$ONH_0 \subset ONH_1 \subset ONH_2 \subset \cdots. \quad (5.23)$$

Mirroring the notation for symmetric groups from General conventions, we also have the embedding

$$\sigma_n : ONH_{n'} \hookrightarrow ONH_{n+n'}, \quad x_i \mapsto x_{i+n}, \tau_j \mapsto \tau_{j+n}. \quad (5.24)$$

For $\alpha = \Lambda(k, n)$, we let $ONH_\alpha$ be the subalgebra $\{x^i \tau_w \mid w \in S_n, k \in \mathbb{N}_0\}$ of $ONH_n$. The restrictions of natural and unnatural embeddings of $ONH_{n'}$ into $ONH_{n+n'}$ just introduced give analogous embeddings with $ONH$ replaced by $\text{OPol}$ or $ONH^{\text{fin}}$.

Now recall the subalgebra $OSym_n$ of $\text{OPol}_n$ which was defined just after Theorem 4.4—it is the subalgebra of $\text{OPol}_n$ generated by the odd symmetric polynomials $e^{(n)}_r$ from (4.33). A different formulation of the definition of $OSym_n$ was adopted in [EKL], where $OSym_n$ was defined from the outset to be $\bigcap_{i=1}^{n-1} \ker \partial_i$, which is a subalgebra of $\text{OPol}_n$. We will deduce the equality of $OSym_n$ with this subalgebra in Corollary 5.4, but one containment is obvious: we have that

$$OSym_n \subseteq \bigcap_{i=1}^{n-1} \ker \partial_i. \quad (5.25)$$

To see this, it suffices to check that $\partial_i(e^{(n)}_r) = 0$ for all $i$ and $r = 1, \ldots, n$, which follows from the definitions since $\partial_i(x_i + x_{i+1}) = \partial_i(x_i x_{i+1}) = 0$.

**Theorem 5.3 ([EKL, Prop. 2.13, Cor. 2.14]).** The graded right $OSym_n$-supermodule $\text{OPol}_n$ is free of graded rank $q^{(2)}(n)_q^{(w)}$ with basis $\{p^{(n)}_w \mid w \in S_n\}$ given by the odd Schubert polynomials from (5.17). So we have that

$$\text{OPol}_n = \bigoplus_{w \in S_n} p^{(n)}_w OSym_n \quad \text{with} \quad p^{(n)}_w OSym_n \simeq (\Pi Q^{2})^{(w)} OSym_n \quad (5.26)$$

as graded right $OSym_n$-supermodules. Moreover, the action of $ONH_n$ on $\text{OPol}_n$ induces a graded superalgebra isomorphism

$$ONH_n \sim \text{End}_{OSym_n}(\text{OPol}_n). \quad (5.27)$$

**Proof.** We claim that the polynomials $p^{(n)}_w (w \in S_n)$ are linearly independent over $OSym_n$. To see this, take a non-trivial linear relation

$$\sum_{w \in S_n} p^{(n)}_w b_w = 0$$

for $b_w \in OSym_n$. Choose $w$ of maximal length such that $b_w \neq 0$. Then we act with $\tau_w$. We have that $\tau_w \cdot p^{(n)}_w b_{w'} = 0$ for $w' \neq w$ by the relations (5.2), (5.4) and (5.5), and $\tau_w \cdot p^{(n)}_w b_w = \pm b_w$, so we deduce that $b_w = 0$, a contradiction. The claim implies that the $OSym_n$-submodule of $\text{OPol}_n$ generated by $p^{(n)}_w (w \in S_n)$ is of graded superdimension $\dim_{q, \pi} OSym_n \times \sum_{w \in S_n} (\pi q^{2})^{(w)}$, which is equal to $\dim_{q, \pi} \text{OPol}_n$. 

by Corollary 4.8. Hence, the $p_{w}^{(n)} (w \in S_n)$ also span $OPol_n$ as an $OSym_n$-module, and we have proved (5.26).

To establish (5.27), we first note by (5.26) that

$$\dim_{q,e} \text{End}_{OSym_n}(OPol_n) = \sum_{x,y \in S_n} (\pi q^2)^{\ell(x) - \ell(y)} \left( \sum_{x \in S_n} (\pi q^2)^{\ell(x)} \bigg| \sum_{y \in S_n} (\pi q^2)^{-\ell(y)} \right) = q^{(2)}[n]_{q,e} - q^{(2)}[n]_{q,e},$$

aplying (3.3). The homomorphism $\rho : ONH_n \to \text{End}_{OSym_n}(OPol_n)$ is injective by Theorem 5.1. Therefore it is an isomorphism because the graded superdimensions are the same thanks to Corollary 5.2.

**Corollary 5.4.** We have that $OSym_n = \bigcap_{i=1}^{n-1} \ker \partial_i = \bigcap_{i=1}^{n-1} \text{im} \partial_i$.

**Proof.** It is easy to see that $\ker \partial_i = \text{im} \partial_i$ for each $i$, hence, the second equality holds. For the first one, we have already noted in (5.25) that $OSym_n \subseteq \bigcap_{i=1}^{n-1} \ker \partial_i$. Conversely, take $f \in \cap_{i=1}^{n-1} \ker \partial_i$ and write it as $f = \sum_{w \in S_n} p_{w}^{(n)} b_w$ for $b_w \in OSym_n$. We need to show that $b_w = 0$ except when $w = 1$. Suppose for a contradiction that this is not the case, and pick $w$ of maximal length such that $b_w \neq 0$. Then we act on $f$ with $\tau_w$ to see that $b_w = 0$, contradiction. $\square$

**Remark 5.5.** As well as the basis $F := \{ p_{w}^{(n)} \mid w \in S_n \}$ of odd Schubert polynomials from Theorem 5.3, the monomials $G := \{ x^\kappa \mid \kappa \in \mathbb{N}^n \}$ with $0 \leq \kappa_i \leq n - i$ form a basis for $OPol_n$ as a free right $OSym_n$-module. To see this, it suffices to show that $F \subseteq G$. The elements of $F$ are linearly independent over $\mathbb{F}$ by Theorem 5.3, so $\dim F = n!$. Also $\dim G = n!$ obviously. So we are reduced to checking that $F \subseteq G$. To see this, we note first that $G$ is invariant under the action of each $\tau_i$, as may be seen directly using (5.10) plus $\tau_i \cdot x_{i+1}^r x_j^l = 0$. Since $\chi_n \in F$, it follows that $p_{w}^{(n)} = \tau_w^{-1} w_n \cdot \chi_n \in G$ for each $w \in S_n$ as claimed.

Let $M_{q^{(2)}[n]_{q,e}}^{\ell(x)}(OSym_m)$ denote the usual algebra of matrices $A = (a_{w,w'})_{w,w' \in S_n}$ with entries in $OSym_n$ viewed as a graded superalgebra so that the matrix with $a \in (OSym_n)_{i,p}$ in its $(w,w')$-entry and zeros elsewhere is of degree $i + 2\ell(w) - 2\ell(w')$ and parity $p + \ell(w) - \ell(w') \mod 2$. This graded superalgebra may be identified with $\text{End}_{OSym_n}(OPol_n)$ so that the matrix $A$ just described corresponds to the unique right $OSym_n$-supermodule endomorphism of $OPol_n$ taking $p_{w'}^{(n)}$ to $\sum_{w \in S_n} p_{w}^{(n)} a_{w,w'}$ for every $w' \in S_n$. Thus, Theorem 5.3 shows that $ONH_n \cong M_{q^{(2)}[n]_{q,e}}^{\ell(x)}(OSym_m)$. It follows that the graded superfunctors

$$- \otimes_{ONH_n} OPol_n : \text{gsMod-ONH}_n \to \text{gsMod-OSym}_n, \quad (5.28)$$

$$\text{Hom}_{ONH_n}(OPol_n, -) : \text{ONH}_n \text{-gsMod} \to \text{OSym}_n \text{-gsMod} \quad (5.29)$$

are equivalences of graded $(\mathcal{Q}, \Pi)$-supercategories.

**Theorem 5.6 ([EKL, Prop. 2.15]).** The even center $Z(ONH_n)_0$ of $ONH_n$ is the graded algebra consisting of symmetric polynomials in $x_1^2, \ldots, x_n^2$. This coincides with the even center $Z(OSym_n)_0$ of $OSym_n$ embedded into $ONH_n$ in the natural way.

**Proof.** This is proved in [EKL] but we give a slightly different argument since there are some minor issues in the first paragraph of the original proof, which does not restrict attention to the even center. Take $z \in Z(ONH_n)$. Using Theorem 5.1, we have that

$$z = \sum_{w \in S_n} f_w \tau_w$$
for unique \( f_w \in \text{OPol}_n \). The first step is to show that \( f_w = 0 \) unless \( w = 1 \). To see this, suppose for a contradiction that it is not the case. Let \( w \) be of maximal length such that \( f_w \neq 0 \). Pick \( i \in \{1, \ldots, n\} \) such that \( j := w(i) \neq i \). We have that
\[
x_i z = \sum_{w \in S_n} x_i f_w \tau_w.
\]
Now we use the relations to express \( x_i z \) as a linear combination \( \sum_{v \in S_n} g_v \tau_v \) for \( g_v \in \text{OPol}_n \). Since \( \tau_w x_i = \pm x_i \tau_w \) plus a linear combination of \( \tau_{w'} \) for \( w' \) with \( \ell(w') < \ell(w) \), we see that \( g_w = \pm f_w x_j \). Thus, we must have that \( x_i f_w = \pm f_w x_j \). Since \( i \neq j \), it is easy to see that this implies that \( f_w = 0 \). So now we have proved that \( z \in \text{OPol}_n \). Next, assuming also that \( z \) is even, we show that \( z \) is in fact in \( \mathbb{F}[x_1^2, \ldots, x_n^2] \) essentially following the idea from the proof in [EKL]. Take any \( 1 \leq i \leq n \) and suppose that \( z = \sum_{k \geq 0} f_k x_i^k \) for \( f_k \) belonging to the subalgebra of \( \text{OPol}_n \) generated by \( x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \). Since \( x_i z = x_i z_i \), we get that each \( f_k \) must be even. Since \( z \) is even too it follows that \( f_k = 0 \) unless \( k \) is even. This shows that \( z \) only involves even powers of \( x_i \). This is true for each \( i \), so \( z \in \mathbb{F}[x_1^2, \ldots, x_n^2] \) as claimed. To complete the proof that \( z \) is actually a symmetric polynomial in \( x_1, \ldots, x_n \), and to show that any such polynomial is central, we can now refer the reader to the argument given in the second two paragraphs of the proof of [EKL, Prop. 2.15].

Finally we explain how to see that \( Z(\text{ONH}_n)_0 \) coincides with \( Z(\text{OSym}_n)_0 \). The center of the matrix algebra \( M_{q^{(n)}[n]}_{q,a} (\text{OSym}_n) \) is isomorphic to \( Z(\text{OSym}_n) \) via the map taking \( z \in Z(\text{OSym}_n) \) to the matrix \( \text{diag}(z, \ldots, z) \). It follows that the even centers are isomorphic too. Given \( z \) in the even center of \( \text{ONH}_n \), we have just shown that it is a polynomial in \( x_1^2, \ldots, x_n^2 \), so we have that \( z p_w^{(n)} = p_w^{(n)} z \) for all \( w \in S_n \). It follows that \( z \) acts on \( \text{OPol}_n \) in the same way as the matrix \( \text{diag}(z, \ldots, z) \) under the identification of \( \text{ONH}_n \) with matrices described above. This shows that the natural embedding of \( \text{OSym}_n \) into \( \text{ONH}_n \) restricts to give an isomorphism between the even centers of \( \text{OSym}_n \) and \( \text{ONH}_n \).

The idempotents in \( \text{ONH}_n \) corresponding to the diagonal matrix units \( e_{w,w} \in M_{q^{(n)}[n]}_{q,a} (\text{OSym}_n) \), that is, the elements which act on \( \text{OPol}_n \) as the projections onto the indecomposable summands in (5.26), give a complete set of primitive idempotents in \( \text{ONH}_n \). The element \( \omega_n \chi_n \omega_n \) is a multiple of \( \omega_n \) by degree considerations, and both \( \omega_n \chi_n \omega_n \) and \( \omega_n \) map \( \chi_n \omega_n \) to 1 by (5.15), hence, we have that
\[
\omega_n \chi_n \omega_n = \omega_n.
\]

(5.30)

It follows that the following are both idempotents:
\[
(\chi \omega)_n := \chi_n \omega_n, \quad (\omega \chi)_n := \omega_n \chi_n.
\]

(5.31)

The first of these, \((\chi \omega)_n\), is exactly the matrix unit \( e_{w,w} \) which projects \( \text{OPol}_n \) onto the top degree component \( p_w^{(n)} \text{OSym}_n \). This follows almost immediately since (5.15) shows that \( \chi_n \omega_n \cdot p_w^{(n)} = p_w^{(n)} \) and \( \chi_n \omega_n \cdot p_w^{(n)} = 0 \) for all other \( w \in S_n \) as \( \omega_n \cdot p_w^{(n)} = 0 \) by degree considerations. In particular, this shows that \((\chi \omega)_n\) is a primitive idempotent. The second one, \((\omega \chi)_n\), is also primitive since we have that
\[
(\omega \chi)_n = (\chi \omega)_n^*.
\]

(5.32)

To see this, it is clear from the definitions that \((\chi \omega)_n^* = \pm (\omega \chi)_n\), and the sign must be plus since \((\omega \chi)_n\) is an idempotent. Note also that \((\omega \chi)_n \text{OPol}_n = \text{OSym}_n \). To see this, every \( \partial_i \) annihilates \( (\omega \chi)_n \cdot \text{OPol}_n \), so \((\omega \chi)_n \cdot \text{OPol}_n \subseteq \text{OSym}_n \) thanks to Corollary 5.4, and it is easy to see directly that \((\omega \chi)_n \cdot f = f \) for any \( f \in \text{OSym}_n \) giving the other containment. Thus we have shown that
\[
(\chi \omega)_n \cdot \text{OPol}_n = \chi_n \text{OSym}_n, \quad (\omega \chi)_n \cdot \text{OPol}_n = \text{OSym}_n.
\]

(5.33)
For \( n \geq 2 \), \((\omega \chi)_n\) is not the idempotent corresponding to the matrix unit \( e_{1,1} \) in the matrix algebra \( M_{d^2}[n]_{k,\ast} \) \((\text{OSym}_n)\), i.e., it is a projection of \( \text{OPol}_n \) onto \( \text{OSym}_n \), but along a different direct sum decomposition to (5.26). It is convenient to work with since left multiplication by \( \omega_n \) defines a homogeneous isomorphism \((\chi \omega)_n \cdot \text{OPol}_n \rightarrow (\omega \chi)_n \cdot \text{OPol}_n \), with inverse defined by left multiplication by \( \chi_n \).

**Lemma 5.7.** We have that \( \text{ONH}_n \cong \bigoplus_{w \in \mathbb{S}_n} (\Pi Q^2)^{\ell(w)} (\omega \chi)_n \text{ONH}_n \cong \bigoplus_{w \in \mathbb{S}_n} (\Pi Q^2)^{\ell(w)} (\chi \omega)_n \text{ONH}_n \) as a graded right \( \text{ONH}_n \)-supermodule.

**Proof.** Left multiplication by \( \omega_n \) defines an isomorphism \((\chi \omega)_n \cdot \text{ONH}_n \cong (\Pi Q^2)^{\ell(w)} (\omega \chi)_n \text{ONH}_n \) with inverse given by left multiplication by \( \chi_n \). Therefore it suffices to prove the first isomorphism. Since (5.28) is a graded superequivalence, we can apply it to reduce the problem to proving that \( \text{OPol}_n \cong \bigoplus_{w \in \mathbb{S}_n} (\Pi Q^2)^{\ell(w)} \text{OSym}_n \)

as graded right \( \text{OSym}_n \)-supermodules, where we have used that \((\omega \chi)_n \cdot \text{OPol}_n = \text{OSym}_n \) by (5.33). This follows from (5.26).

**Lemma 5.8.** The map \( \iota : \text{OPol}_n \rightarrow \text{ONH}_n(\omega \chi)_n, f \mapsto f(\omega \chi)_n \) is an even degree zero isomorphism of graded left \( \text{ONH}_n \)-supermodules. The map \( j : \text{OSym}_n \rightarrow (\omega \chi)_n \text{ONH}_n(\omega \chi)_n \) defined by the composition of the natural inclusion of \( \text{OSym}_n \) into \( \text{ONH}_n \) followed by the projection \( a \mapsto (\omega \chi)_n a(\omega \chi)_n \) is a graded superalgebra isomorphism. Moreover, we have that \( \iota(f a) = \iota(f) j(a) \) for all \( f \in \text{OPol}_n \) and \( a \in \text{OSym}_n \).

**Proof.** Since \( \tau(\omega \chi)_n = 0 \) by degree considerations, there is a unique graded left \( \text{ONH}_n \)-supermodule homomorphism \( \text{OPol}_n \rightarrow \text{ONH}_n(\omega \chi)_n \) taking \( 1 \) to \((\omega \chi)_n \) thanks to (5.22). This is \( \iota \). Also \((\omega \chi)_n \cdot 1 = 1 \), so there is a homomorphism \( \text{ONH}_n(\omega \chi)_n \rightarrow \text{OPol}_n, a \mapsto a \cdot 1 \). These two maps are mutual inverses, hence, \( \iota \) is an isomorphism. The restriction of \( \iota \) gives an isomorphism \( (\omega \chi)_n \cdot \text{OPol}_n \rightarrow \text{OPol}_n, a \mapsto a \cdot 1 \). Recalling that \((\omega \chi)_n \cdot \text{OPol}_n = \text{OSym}_n \) by (5.33), this restriction is the map \( j \) as defined in the statement of the lemma. Since \( j \) is an algebra homomorphism, this shows that \( j \) is an isomorphism. The final assertion follows using \((\omega \chi)_n a(\omega \chi)_n = a(\omega \chi)_n \) for \( a \in \text{OSym}_n \).

**Corollary 5.9.** Using \( j \) to identify \( \text{OSym}_n \) with \((\omega \chi)_n \text{ONH}_n(\omega \chi)_n \), the superfunctors \(- \otimes_{\text{ONH}_n} \text{OPol}_n \) and \( \text{Hom}_{\text{ONH}_n}(\text{OPol}_n, -) \) from (5.28) and (5.29) are isomorphic to the idempotent truncation functors defined by right and left multiplication by the idempotent \((\omega \chi)_n \), respectively.

We note finally that there is also a right action of \( \text{ONH}_n \) on \( \text{OPol}_n \), and everything in this section could be reformulated in terms of this viewed as an \((\text{OSym}_n, \text{ONH}_n)\)-superbimodule. This right action may be defined succinctly from

\[
f \cdot a := (-1)^{\text{par}(a)} \text{par}(f)(a^* \cdot f^* )^* \tag{5.34}
\]

for \( f \in \text{OPol}_n, a \in \text{ONH}_n \). The following more explicit description similar to (5.9) can easily be derived from this:

\[
x_i \cdot \tau_j = \delta_{i,j} - \delta_{i,j+1}, \quad (f g) \cdot \tau_j = f(g \cdot \tau_j) + (f \cdot \tau_j)(\delta^i g). \tag{5.35}
\]

for \( f, g \in \text{OPol}_n \). The right action of \( \text{ONH}_n \) on \( \text{OPol}_n \) obviously commutes with the natural action of \( \text{OSym}_n \) by left multiplication. Theorem 5.3 and (5.32) imply that

\[
\text{OPol}_n = (\Pi Q^2)^{\ell(w)} \text{OSym}_n \tag{5.36}
\]

as a graded left \( \text{OSym}_n \)-supermodule, with the “bottom” summand that is \( \text{OSym}_n \) itself being the image of the idempotent \((\chi \omega)_n \) acting on the right. We stress that the action (5.34) is different from the right action defined via \( f \cdot a := (-1)^{\text{par}(a)} \text{par}(f)a^* \circ f \); the latter action does not commute with the left action of
Lemma 5.10. \((t + x_2)^{-1}x_1' \cdot \tau_1 = (t + x_2)^{-1}x_2'(t + (-1)^r x_1)^{-1} + \sum_{q=0}^{r-1} (-1)^q (t + x_2)^{-1}x_2^q x_1'^{-q-1}\).

**Proof.** Similarly to (5.10) and (5.11), one shows that \((\cdot)\) and \((\cdot)\) one shows that \((t + x_2)^{-1} \cdot \tau_1 = (t + x_2)^{-1}(t + x_1)^{-1}\) and \(x_1' \cdot \tau_1 = \sum_{q=0}^{r-1} x_1^q x_2^{q-1}\). These combine using (5.35) to give the final formula. One also needs to commute all \(x_2\) to the left of all \(x_1\) producing some additional signs. \(\square\)

6. Schur polynomials and "odd cohomology" of Grassmannians

Another important basis of \(O\text{Sym}^n\) is introduced in [EK, Sec. 3.3]: the basis of *odd Schur functions* \(\{s_\lambda \mid \lambda \in \Lambda^+\}\). As explained after [EK, Cor. 3.9], this is the basis of \(O\text{Sym}^n\) characterized uniquely by the properties that \((s_\lambda, h_\mu)^- = 0\) if \(\mu >_{\text{lex}} \lambda\) and \(s_\lambda = h_{\lambda + 1}(a \mathbb{Z}\)-linear combination of other \(h_\mu\), for \(\mu >_{\text{lex}} \lambda\). Some examples can be found in the appendix of [EK]. The key property of odd Schur functions is that they are signed-orthonormal:

**Theorem 6.1** ([EK, Cor. 3.9]). For \(\lambda, \mu \in \Lambda^+\), we have that \((s_\lambda, s_\mu)^- = (-1)^{\delta_{\lambda, \mu}}\).

The odd Kostka matrix \((K_{\lambda, \mu})_{\lambda, \mu \in \Lambda^+}\) is the transition matrix defined from

\[
h_\mu = \sum_{\lambda \in \Lambda^+} K_{\lambda, \mu}s_\lambda. \quad (6.1)
\]

There is an explicit formula for the entries of this matrix derived in [EK, (3.7)], as follows. For a \(\lambda\)-tableau \(T\) \((=\text{a function from the Young diagram of } \lambda \text{ to } \mathbb{Z})\), we let \(N(T)\) be the number of pairs of boxes \((A, B)\) such that \(B\) is north of \(A\) and also \(T(B) < T(A)\). For example, if \(T\) is the unique semistandard \(\lambda\)-tableau of content \(\lambda\) (so all entries on row \(i\) are equal to \(i\)) then \(N(T) = 0\). Then

\[
K_{\lambda, \mu} = \sum_{T} (-1)^{N(T)} \quad (6.2)
\]

summing over semistandard \(\lambda\)-tableaux \(T\) of content \(\mu\). Note from this description that \(K_{\lambda, \mu} = 0\) unless \(\lambda \geq \mu\) in the dominance order.

Since the involution \(\psi_1 \psi_2\) in [EK] is our \(\psi\) by Remark 4.3, [EK, Lem. 3.11] shows that

\[
\psi(s_\lambda) = (-1)^{\delta_{\lambda, \mu}} s_\lambda. \quad (6.3)
\]

for any \(\lambda \in \Lambda^+\). Hence, also applying Theorem 6.1 and (4.29), we have that

\[
(s_\lambda, s_\mu)^+ = (-1)^{d_{W(\lambda)}^{\delta_{\lambda, \mu}}}. \quad (6.4)
\]

Using (6.3) and the first identity from (4.29), it follows that \(s_\lambda\) can also be characterized as the unique element of \(O\text{Sym}^n\) such that \((s_\lambda, e_\mu)^+ = 0\) for \(\mu >_{\text{lex}} \lambda^T\) and \(s_\lambda = (-1)^{\delta_{\lambda, \mu}} e_\mu\). This characterization plus Lemma 4.2 and the second identity from (4.29) imply that

\[
e(s_\lambda)^+ = (-1)^{d_{W(\lambda)}^{\delta_{\lambda, \mu}}} e_{\lambda} s_\lambda. \quad (6.5)
\]

It also follows easily now that

\[
h_r = s_{(r)}, \quad e_r = s_{(1^r)}. \quad (6.6)
\]

Let \(s_\lambda^{(n)}\) denote the image of \(s_\lambda\) under the quotient map \(\pi_n : O\text{Sym}^n \to O\text{Sym}_n\). The *dual odd Schur polynomial* associated to \(\lambda\) is

\[
(s_\lambda^{(n)})^+ = (-1)^{d_{W(\lambda)}} e_n(s_\lambda^{(n)}). \quad (6.7)
\]
Up to a sign, these coincide with the polynomials introduced in [EKL, Def. 4.10]. They will play an important role in Theorem 6.8 below.

**Theorem 6.2.** The set \( \{ s^{(n)}_\lambda \mid \lambda \in \Lambda^+ \text{ with } \text{ht}(\lambda) \leq n \} \) is a basis for OSym\(_n\). Moreover, for any \( \lambda \in \Lambda^+ \), we have that

\[
s^{(n)}_\lambda = \begin{cases} x^1 + (a \mathbb{Z}-\text{linear combination of } x^k \text{ for } k \in \mathbb{N} \text{ with } k < \lambda) & \text{if } \text{ht}(\lambda) \leq n \\ 0 & \text{if } \text{ht}(\lambda) > n. \end{cases}
\]  

(6.8)

**Proof.** We have already explained enough to see that \( s^{(n)}_\lambda = (-1)^{\Lambda_E(\lambda)} \epsilon^{(n)}_\lambda + (a \mathbb{Z}-\text{linear combination of other } \epsilon^{(n)}_\mu \text{ for } \mu < \lambda). \) The result follows from this using Theorem 4.5 and (4.36).

 The next result was originally formulated as a conjecture in [EKL, Conj. 5.3], and the conjecture was proved in [E, Th. 3.8]. However, we also need to reformulate it using our sign conventions, and for this we need a preliminary lemma.

**Lemma 6.3.** For \( f \in \text{OSym}_{n-1} \), \( m \geq 0 \) and \( k = 1, \ldots, n \), we have that

\[ \tau_{k-1} \cdots \tau_1 x^{m+k-1}_1 \cdot \sigma_1(f) = \sum_{i=0}^{k-1} \sigma_i(h^{(k-i)}_{m+i}) \tau_i \cdots \tau_1 \cdot \sigma_1(f), \]

equality in the ONH\(_n\)-supermodule OPol\(_n\).

**Proof.** We prove this by induction on \( k \), the case \( k = 1 \) being trivial. For the induction step, we have by induction that \( \tau_{k-1} \cdots \tau_1 x^{m+k}_1 \cdot \sigma_1(f) = \sum_{i=0}^{k-1} \sigma_i(h^{(k-i)}_{m+i+1}) \tau_i \cdots \tau_1 \cdot \sigma_1(f) \).Applying \( \tau_k \) to both sides, we deduce that

\[ \tau_k \cdots \tau_1 x^{m+k}_1 \cdot \sigma_1(f) = \sum_{i=0}^{k-1} \left( \tau_k \cdot \sigma_i(h^{(k-i)}_{m+i}) \right) \tau_i \cdots \tau_1 \cdot \sigma_1(f) + \sum_{i=0}^{k-1} s_i \sigma_i(h^{(k-i)}_{m+i+1}) \tau_k \tau_i \cdots \tau_1 \cdot \sigma_1(f). \]

By (5.10), we have that \( \tau_{k-i} \cdot h^{(k-i)}_{m+i+1} = h^{(k-i)}_{m+i} \), so the \( i \)-th term in the first summation becomes

\[ \sigma_i(\tau_{k-i} \cdot h^{(k-i)}_{m+i+1}) \tau_i \cdots \tau_1 \cdot \sigma_1(f) = \sigma_i(h^{(k-i)}_{m+i}) \tau_i \cdots \tau_1 \cdot \sigma_1(f). \]

The second summation gives zero except when \( i = k-1 \), when it gives \( \sigma_k(h^{(1)}_{m+k}) \tau_k \cdots \tau_1 \cdot \sigma_1(f) \). In total, we obtain the desired \( \sum_{i=0}^{k} \sigma_i(h^{(1)}_{m+i+1}) \tau_i \cdots \tau_1 \cdot \sigma_1(f). \)

**Theorem 6.4 ([E, Th. 3.8]).** For \( \lambda \in \Lambda^+ \), we have that \( s^{(n)}_\lambda = (\omega \chi)_n \cdot x^1 \).

**Proof.** The original formula from [EKL] took the form

\[ s_\lambda = \left(-1\right)^{\lambda} \left[ \partial_{w_0} \left( x_1^{\lambda_1} \cdots x_n^{\lambda_n} x_1^{n-1} \cdots x_{n-2} x_{n-1} \right) \right]^{(n)}, \]

(6.9)

using their notation everywhere. The result was proved in [E] with exactly this in place of our \((\omega \chi)_n \cdot x^1\). In (6.9), the conjugation by \( w_0 \) corresponds up to a sign to an application of our involution \( e_n \), which commutes with the action of \( \partial_{w_0} \) again up to a sign, due to (5.20) and (5.21). This shows that the right hand side of (6.9) is equal to \((\omega \chi)_n \cdot x^1\) up to a sign. Hence, \((\omega \chi)_n \cdot x^1 = \pm s^{(n)}_\lambda\). Presumably, one could see that the sign is actually a plus by carefully keeping track of all of the sign changes in this translation. However, this is rather prone to error, so we give an alternative approach. It suffices by (6.8) to check that the \( x^1 \)-coefficient of \((\omega \chi)_n \cdot x^1\) is 1. We have that \( \omega_n = \tau_{n-1} \cdots \tau_1 \sigma_1(\omega_{n-1}), \chi_n = \sigma_1(\chi_{n-1}) x_1^{n-1} \) and \( x^1 = x_1^1 \sigma_1(x^d) \) where \( \mu = (\lambda_2, \ldots, \lambda_n) \). Using these and induction on \( n \), we get that

\[ (\omega \chi)_n \cdot x^1 = \tau_{n-1} \cdots \tau_1 x_1^{\lambda_1+n-1} \cdot \sigma_1(\omega_{n-1} \chi_{n-1} \cdot x^d) \]
Similarly to (5.31), it follows that the elements

\[ \tau_{n-1} \cdots \tau_1 \Lambda^1 + \cdots + \sigma_1 (\lambda_{\mu}^{(n-1)}) = \sum_{i=0}^{n-1} \sigma_i (h_{\Lambda^1+\cdots+\lambda_{\mu}}) \tau_i \cdots \tau_1 \cdot \sigma_1 (\lambda_{\mu}^{(n-1)}), \]

the last equality being an application of Lemma 6.3. Now we express this in terms of the monomial basis for \( OPol_\mu \). The only place a monomial whose \( x_1 \)-exponent is \( \geq \Lambda_1 \) can arise is from the \( i = 0 \) term, which is \( h_{\Lambda_1}^0 \sigma_1 (\lambda_{\mu}^{(n-1)}) \). This has leading term exactly \( x_1 \), as required. \( \Box \)

Now we are going to discuss a graded superalgebra which may be interpreted as the odd analog of the equivariant cohomology algebra of the Grassmannian. We set things up initially in greater generality. Switching our default choice of variable from \( n \) to \( \ell \) for reasons that will become clear shortly, suppose that \( \alpha \in \Lambda(k, \ell) \). This represents the “shape” of a partial flag variety, Grassmannians being the special case that \( k = 2 \). Let

\[
OSym_\alpha := \bigcap_{\{i \in \{1, \ldots, \ell\} \mid i \notin \{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\cdots+\alpha_k\}\}} \ker \partial_i = \bigcup_{\{i \in \{1, \ldots, \ell\} \mid i \notin \{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\cdots+\alpha_k\}\}} \text{im} \partial_i, \tag{6.10}
\]

which is a subalgebra of \( OPol_\ell \) containing \( OSym_\ell \). We think of \( OSym_\alpha \) as being the odd analog of the ring of “partial” invariants \( \mathbb{F}[x_1, \ldots, x_\ell]^{w_\alpha} \). For example, we have that \( OSym_\alpha = OSym_\ell \) if \( \alpha = (\ell) \), and \( OSym_\alpha = OPol_\ell \) if \( \alpha = (1^\ell) \). Note also that the superalgebra anti-involution \( * \) of \( OPol_\ell \) leaves \( OSym_\alpha \) invariant, whereas the involution \( \epsilon \) takes \( OSym_\alpha \) to \( OSym_{w_\alpha} \) where \( w_\alpha (\alpha) = (\alpha_k, \ldots, \alpha_1) \) is the reversed composition.

Consider the following diagram:

\[
\begin{array}{ccc}
OSym & \xrightarrow{\Delta_k^\ast} & OSym \otimes \cdots \otimes OSym \\
\downarrow \pi \downarrow & & \downarrow \pi_{\alpha_1} \otimes \cdots \otimes \pi_{\alpha_k} \\
OSym_\ell & \xhookrightarrow{\subseteq} & OSym_\alpha \\
\downarrow & & \downarrow \\
OPol_\ell & \xrightarrow{\subseteq} & OPol_{\alpha_1} \otimes \cdots \otimes OPol_{\alpha_k} \\
\end{array}
\tag{6.11}
\]

The top horizontal map \( \Delta_k^\ast \) is the \( k \)th iteration of the comultiplication \( \Delta^\ast : OSym \to OSym \otimes OSym \). The bottom equality is the canonical identification explained just after (2.4), and the outside square commutes thanks to (4.35). In view of Corollary 5.4, the subalgebra \( OSym_\alpha \) of \( OPol_\ell \) is identified with the subalgebra \( OSym_{\alpha_1} \otimes \cdots \otimes OSym_{\alpha_k} \) of \( OPol_{\alpha_1} \otimes \cdots \otimes OPol_{\alpha_k} \). This shows that the natural inclusion of \( OSym_\ell \) into \( OSym_\alpha \) is induced by the comultiplication \( \Delta^\ast \).

Recall that \( w_\alpha \) is the longest element of \( S_\alpha \) and \( w^\alpha \) is the longest element of \( [S_\ell/S_\alpha]_{\text{min}} \), so that \( w_\ell = w^\alpha w_\alpha \). Noting that \( w_\alpha = w_\alpha \sigma_1 (w_\beta) \) where \( \beta := (\alpha_2, \ldots, \alpha_k) \), we recursively define

\[
\omega_\alpha := \omega_{\alpha_1} \sigma_1 (\omega_\beta) \quad (= \pm \tau_{w_\beta}), \quad \chi_\alpha := \sigma_1 (\chi_\beta) \chi_{\alpha_1}. \tag{6.12}
\]

We get from (5.15) and induction on \( k \) that

\[
\omega_\alpha \cdot \chi_\alpha = 1 \tag{6.13}
\]

for any \( \alpha \). The following identity is proved in the same way as (5.30):

\[
\omega_\alpha \chi_\alpha = \omega_\alpha. \tag{6.14}
\]

Similarly to (5.31), it follows that the elements

\[
(\chi \omega)_\alpha := \chi_\alpha \omega_\alpha, \quad (\omega \chi)_\alpha := \omega_\alpha \chi_\alpha \tag{6.15}
\]
are primitive idempotents in $ONH_\alpha$ such that
\[(\chi_\alpha)_\alpha \cdot OPol_\ell = \chi_\alpha OSym_\alpha, \quad (\omega_\chi)_\alpha \cdot OPol_\ell = OSym_\alpha. \quad (6.16)\]
Also let $\omega^\alpha := \pm \tau_w^\alpha$ for the particular sign chosen so that
\[\omega_\ell = \omega^\alpha \omega_\alpha \quad (6.17)\]
and let
\[\chi^\alpha := \omega_\alpha \cdot \chi_\ell \in OSym_\alpha. \quad (6.18)\]

We have that
\[\omega^\alpha \cdot \chi^\alpha = 1, \quad \chi^\alpha = \chi_\alpha \chi^\alpha. \quad (6.19)\]
The first of these equalities follows because $\omega^\alpha \cdot \chi^\alpha = \omega^\alpha \omega_\alpha \cdot \chi_\ell = \omega_\ell \cdot \chi_\ell = 1$ by (5.15) and (6.17). The second follows because we clearly have that $\chi_\alpha \chi^\alpha = \pm \chi_\ell$ for some choice of sign, and the sign is plus because $\omega_\ell \cdot \chi_\alpha \chi^\alpha = \omega^\alpha \omega_\alpha \cdot \chi_\ell = \omega^\alpha \chi^\alpha = 1 = \omega_\ell \cdot \chi_\ell$. Finally, we have that
\[\omega_\alpha \chi_\alpha \omega_\ell = \omega_\ell = \omega_\ell \chi_\alpha \omega_\ell. \quad (6.20)\]

This follows because all three expressions act in the same way on $\chi_\ell \in OPol_\ell$ due to (5.15), (6.13), (6.18) and (6.19).

**Theorem 6.5.** For $\alpha \in \Lambda(k, \ell)$, the graded superalgebra $OSym_\alpha$ is free as a right $OSym_\ell$-supermodule with basis $\{p_w^{(\ell)} \mid w \in [S_\ell/S_\alpha]_{\text{min}}\}$. Each $p_w^{(\ell)}$ in this basis belongs to the subalgebra $OSym_\alpha \cap OPol_{\ell-a_1}$.

**Proof.** By Corollary 4.8 and (3.8), we have that
\[\dim_{q,\pi} OSym_\alpha / \dim_{q,\pi} OSym_\ell = \frac{q^{(\ell)} [\alpha]_{q,\pi}}{\prod_{i=1}^k q^{(\ell)} [\alpha_i]_{q,\pi}} = q^{N[\alpha]} \left[\ell\right]_{q,\pi} = \sum_{w \in [S_\ell/S_\alpha]_{\text{min}}} (\pi q^2)^{\ell(w)}. \quad (6.21)\]

This is the graded rank of a free graded right $OSym_\ell$-supermodule with basis $\{p_w^{(\ell)} \mid w \in [S_\ell/S_\alpha]_{\text{min}}\}$. So, to prove the theorem, it just remains to show that the elements $p_w^{(\ell)}$ ($w \in [S_\ell/S_\alpha]_{\text{min}}$) belong to $OSym_\alpha \cap OPol_{\ell-a_1}$ and are linearly independent over $OSym_\ell$. The linear independence is immediate from Theorem 5.3.

To show that $p_w^{(\ell)} \in OSym_\alpha$, we need to show that $\partial_i(p_w^{(\ell)}) = 0$ for all $i$ such that $s_i \in S_\alpha$. We have that $w^{-1}w_\ell = w_\alpha w'$ for some $w' \in [S_\alpha/S_\ell]_{\text{min}}$. So $p_w^{(\ell)} = \tau_{w^{-1}w_\ell} \cdot \chi_\ell = \pm \omega_\alpha \tau_{w'} \cdot \chi_\ell$. Since $\ell(w_\alpha) < \ell(w)$ when $s_i \in S_\alpha$, the relations in $ONH_\ell$ now imply that $\tau_i \cdot p_w^{(\ell)} = 0$.

To show that $p_w^{(\ell)} \in OPol_{\ell-a_1}$, we again use $w^{-1}w_\ell = w_\alpha w'$ to deduce that $\tau_{w^{-1}w_\ell} = \pm \sigma_{\ell-a_1}(\omega_\alpha) \tau_{w''}$ for some $w'' \in S_\ell$. By the argument explained in the last sentence of Remark 5.5, it follows that $p_w^{(\ell)}$ is a linear combination of terms of the form $\sigma_{\ell-a_1}(\omega_\alpha_i) \cdot x_\ell^k \cdot x_{\ell-a_1}^{k+1}$ is a scalar by (5.15) and degree considerations.

**Corollary 6.6.** Suppose that $\alpha \in \Lambda(k, \ell)$ for $k \geq 1$.

1. The graded superalgebra $OSym_\alpha$ is free as a graded right $OSym_{(a_1, \ell-a_1)}$-supermodule with basis $\left\{\sigma_{a_1}(p_w^{(\ell-a_1)}) \mid w \in (S_{\ell-a_1}/S_{(a_2, \ldots, a_k)})_{\text{min}}\right\}$.
2. The graded superalgebra $OSym_\alpha$ is free as a graded right $OSym_{(\ell-a_1,a_1)}$-supermodule with basis $\left\{\epsilon_{\ell-a_1}(p_w^{(\ell-a_1)}) \mid w \in (S_{\ell-a_1}/S_{(a_2, \ldots, a_k)})_{\text{min}}\right\}$.

All vectors in the bases described in (1)–(2) belong to the subalgebra $\sigma_{a_1}(OSym_{(a_2, \ldots, a_k)})$. 

Proof. (1) This follows immediately from the theorem. (2) This follows by applying the involution $\epsilon_\ell$ to the result from (1) with $\alpha$ replaced by the reverse composition $\alpha^R$. \hfill $\Box$

Continuing with $\alpha \in \Lambda(k, \ell)$, we need a few more pieces of notation. For $i = 1, \ldots, k$, we define

$$h_r^{(\alpha,i)} := \sigma_{1+a_1+\cdots+a_{i-1}}(h_r^{(\alpha)}), \quad e_r^{(\alpha,i)} := \sigma_{1+a_1+\cdots+a_{i-1}}(e_r^{(\alpha)}).$$

(6.22)

Under the identification of $\text{OSym}_\alpha$ with $\text{OSym}_{\alpha_r} \otimes \cdots \otimes \text{OSym}_{\alpha_k}$ from (6.11), these are $1^{\otimes(i-1)} \otimes h_r^{(\alpha)} \otimes 1^{\otimes(k-i)}$ and $1^{\otimes(i-1)} \otimes e_r^{(\alpha)} \otimes 1^{\otimes(k-i)}$, respectively. We use similar notation for other elements of $\text{OSym}_\alpha$ such as $e_A^{(\alpha,i)}$, $h_A^{(\alpha,i)}$ and $s_A^{(\alpha,i)}$ for $\lambda \in \Lambda^+$. From (4.33) and (4.34), we get that

$$e_r^{(\ell)} = \sum_{r_1, \ldots, r_k \geq 0 \atop r_1 + \cdots + r_k = r} e_{r_1}^{(\alpha,1)} \cdots e_{r_k}^{(\alpha,k)}, \quad h_r^{(\ell)} = \sum_{r_1, \ldots, r_k \geq 0 \atop r_1 + \cdots + r_k = r} h_{r_1}^{(\alpha,1)} \cdots h_{r_k}^{(\alpha,k)}.$$  

(6.23)

These are really more convenient when written in terms of the generating functions

$$e^{(\alpha,i)}(t) := \sum_{r \geq 0} e_r^{(\alpha,i)} t^{-r}, \quad h^{(\alpha,i)}(t) := \sum_{r \geq 0} h_r^{(\alpha,i)} t^{-r}.$$  

(6.24)

Now the identities (6.23) become

$$e^{(\ell)}(t) := e^{(\alpha,1)}(t) e^{(\alpha,2)}(t) \cdots e^{(\alpha,k)}(t), \quad h^{(\ell)}(t) := h^{(\alpha,1)}(t) h^{(\alpha,2)}(t) h^{(\alpha,1)}(t).$$  

(6.25)

These identities, which generalize (4.44), together with (4.46) are useful when moving between different families of generators. The following lemma is the first example of this principle.

Lemma 6.7. In $\text{OSym}_\ell$, we have that $\sum_{s=0}^r (-1)^s x_1^{r-s} e_s^{(\ell)} = 0$ for all $r \geq \ell$.

Proof. The identity (6.25) when $\mu = (1, \ell - 1)$ plus the infinite Grassmannian relation (4.46) gives that $\sigma_1(e_{\ell-1}(-t)) = -h^{(1)}(t) e^{(\ell)}(-t)$. The left hand side is a polynomial, so the $t^{\ell-1-r}$ coefficients on the right hand side are zero for $r \geq \ell$. The lemma follows by computing these coefficients explicitly, using that $h^{(1)}_{\mu_{r-s}} = x_1^{r-s}$. \hfill $\Box$

Now we focus on the most important case $k = 2$, so $\alpha = (n, n') \in \Lambda(2, \ell)$ for some $n, n' \geq 0$. Then $\chi_\alpha = \chi_{(n,n')} = \sigma_n(\chi_{n'}) \chi_n$.

Theorem 6.8. Suppose that $\ell = n + n'$ for $n, n' \geq 0$. Then $\text{OSym}_{(n,n')}$ has the following two bases as a free graded right $\text{OSym}_{\ell}$-supermodule:

(1) $\{ e^{(n)}_\lambda \mid \lambda \in \Lambda^+_n \}$;
(2) $\{ \sigma_n(s^{(n')}_\mu) \mid \mu \in \Lambda^+_{n'} \}$.

Also let $\text{tr} : \text{OSym}_{(n,n')} \rightarrow \text{OSym}_\ell$ be the linear map $a \mapsto \omega_\ell \chi_{(n,n')} \cdot a$. This map is a homogeneous homomorphism of graded right $\text{OSym}_{\ell}$-supermodules of degree $-2nn'$ and parity $nn' \pmod{2}$, and the bases (1)–(2) satisfy

$$\text{tr} \left( s^{(n)}_\lambda \sigma_n(s^{(n')}_\mu) \right) = \begin{cases} \text{sgn}(\mu) & \text{if } \mu_{n+1-i} = n' - \lambda_{n+1-i} \text{ for } i = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

(6.26)

for $\text{sgn}(\mu) \in \{\pm 1\}$ with $\text{sgn}(\emptyset) = 1$; see Corollary 6.14 below for a formula for $\text{sgn}(\mu)$ for general $\mu$. 


Proof. The main work here is to prove (6.26). This turns out to be significantly harder than the analogous formula in the ordinary even theory; see [EKL, Rem. 4.12] for an illuminating example. Fortunately, the details are already worked out in [EKL, Prop. 4.11] up to an undetermined sign since our conventions are different. To keep track of this sign, we repeat the first few steps of the proof in [EKL] in our set up. By Theorem 6.4 and (5.20), (5.21), (6.5) and (6.20), we have that

\[ tr\left(s_{\mu}^{(n)}(s_{\nu}^{(n')})^{*}\right) = (-1)^{dN(\mu)+dW(\mu)} \omega_{\ell}(\chi(\mu,\nu')) \cdot s_{\mu}^{(n)}(s_{\nu}^{(n')}) \]

\[ = (-1)^{dN(\mu)+dW(\mu)} \omega_{\ell}(\chi(\mu,\nu')) \cdot (\omega_{\nu}(\chi_{\mu} \cdot x^{\ell})) \]

\[ = \xi_{\mu}(\omega_{\nu}(\chi_{\mu} \cdot x^{\ell}))(\chi_{\mu} \cdot x^{\ell}) s_{\mu}(\nu) \]

\[ = \xi_{\mu}(\omega_{\nu}(\chi_{\mu} \cdot x^{\ell}))(\chi_{\mu} \cdot x^{\ell}) s_{\mu}(\nu). \quad (6.27) \]

Up to another sign, the monomial appearing after the \( \cdot \) in (6.27) is as considered in [EKL, Lem. 4.9], so applying that lemma gives that \( tr\left(s_{\mu}^{(n)}(s_{\nu}^{(n')})^{*}\right) \) is \( \pm 1 \) if \( \mu_{i}^{n'} = n' - \lambda_{n+1} \) for \( i = 1, \ldots, n \), and it is zero otherwise. It remains to check that (6.27) equals \( +1 \) in the special case that \( \lambda = (n^{n'}) \) and \( \mu = \emptyset \). To see this, one first checks that \( \chi(x_{1}^{n'} \cdots x_{n}^{n'}) = x_{n}^{n'} x_{n-1}^{n'} \cdots x_{1}^{n'+1} \). Hence, letting \( \omega_{\ell} = \pi\sigma_{\mu}(\omega_{\nu}) \) for \( \tau \in ONH_{\ell} \), (6.27) simplifies in this case to give

\[ tr\left(s_{\mu}^{(n)}(s_{\nu}^{(n')})^{*}\right) = \xi_{\mu}(\omega_{\nu})(\chi_{\mu} \cdot x_{n}^{n'} x_{n-1}^{n'} \cdots x_{1}^{n'+1} \chi_{\mu} \cdot x_{n}^{n'} x_{n-1}^{n'} \cdots x_{1}^{n'+1} = \omega_{\ell} \cdot \chi_{\ell} = 1. \]

Now (6.26) is proved.

It is clear from the definition that \( tr \) is a homogeneous homomorphism of graded right \( OSym_{\ell} \)-supermodules of degree \(-2nm' \) and parity \( nm' \) (mod 2). It remains to show that the elements (1) and (2) are bases. To see that the elements (1) are linearly independent over \( OSym_{\ell} \), take a linear relation \( \sum_{A} s_{A}^{(n)}(a_{A}) = 0 \). Then we have using (6.26) that

\[ 0 = tr\left(s_{A}(s_{\mu}^{(n')})^{*}\right) s_{A}^{(n)}(a_{A}) = \sum_{A'} (-1)^{|A'|} \cdot dW(\mu') \cdot (\chi_{\mu} \cdot x_{n}^{n'} x_{n-1}^{n'} \cdots x_{1}^{n'+1} \cdot \chi_{\mu} \cdot x_{n}^{n'} x_{n-1}^{n'} \cdots x_{1}^{n'+1} = \omega_{\ell} \cdot \chi_{\ell} = 1. \]

This establishes the linear independence. As \( s_{A}^{(n)}(a_{A}) \) is of degree \( 2|A'| \) and parity \( |A'| \) (mod 2), we deduce from Corollary 3.2 that the elements (1) generate a free graded right \( OSym_{\ell} \)-supermodule of graded rank \( (\mu q)^{|A'|}_{n} \). In view of Theorem 6.5, it follows that this submodule is all of \( OSym_{(n,n')} \). This proves that (1) is a basis. A similar argument gives that (2) is a basis too. \( \square \)

**Corollary 6.9.** Suppose that \( \ell = n + d + n' \) for \( n, d, n' \geq 0 \). Then \( OSym_{(a,d,n')} \) is a free right \( OSym_{(n,n')} \)-supermodule with basis \( \{ s_{A}(s_{\mu}^{(n'})^{*} \mid A \in A_{d,n,n'} \} \) and \( OSym_{(n,n')} \) is a free right \( OSym_{(a,d,n')} \)-supermodule with basis \( \{ s_{A}(s_{\mu}^{(n'})^{*} \mid A \in A_{d,n,n'} \} \). \( \square \)

**Proof.** This follows from Theorem 6.8(1)–(2) in the same way that Corollary 6.6(1) was deduced from Theorem 6.5. \( \square \)

Continue with \( \ell = n + n' \). When all of the algebras involved are commutative, the analog of the map tr from Theorem 6.8 is actually a graded bimodule homomorphism, so that it gives a trace making \( Sym_{(n,n')} \) into a graded Frobenius algebra over \( Sym_{\ell} \). However, in the odd case, \( OSym_{(n,n')} \) is usually not a Frobenius extension of \( OSym_{\ell} \), e.g., it is already false in the case \( n = n' = 1 \) since one can check directly that \( OSym_{2} < OPol_{2} \) has no complement as a graded \( OSym_{2}, OSym_{2} \)-superbimodule. This is a significant obstruction to the development of the odd theory. At this point in
[EKL, Sec. 5], the obstruction is avoided by passing to the finite-dimensional graded superalgebra

\[
\overline{OH}_n := \text{OSym}_n / \langle h^{(n)}_r | r > n' \rangle \quad (= \text{OSym} / \langle h, e_s | r > n', s > n \rangle).
\]

This is called the odd Grassmannian cohomology algebra since it is an odd analog of the cohomology algebra \( H^*(X; \mathbb{F}) \) of the Grassmannian \( X \) of \( n \)-dimensional subspaces of \( \mathbb{C}^f \).

We denote the image of \( a \in \text{OSym}_n \) in \( \overline{OH}_n \) by \( \bar{a} \). The first part of following theorem is [EKL, Prop. 5.4], but we give a different argument which gives extra information.

**Theorem 6.10.** Suppose that \( \ell = n + n' \). The odd Schur polynomials \( s^{(n)}_\lambda \) for \( \lambda \in \Lambda^+_{n,n'} \) give a linear basis for \( \overline{OH}_n \), and all other \( s^{(n)}_\lambda \) are zero. Moreover, viewing \( \mathbb{F} \) as a graded \( \text{OSym}_\ell \)-supermodule in the obvious way, there is a commuting diagram

\[
\begin{array}{ccc}
\overline{OH}_n & \xrightarrow{\bar{a} \mapsto a \otimes 1} & \text{OSym}_{(n,n')} \otimes \text{OSym}_\ell \mathbb{F} \\
\mathbb{F} & \xrightarrow{\psi_n} & \mathbb{F} \otimes \text{OSym}_\ell \text{OSym}_{(n,n')} \\
\overline{OH}_n & \xrightarrow{\bar{a} \mapsto (-1)^{\text{par}(a)} 1 \otimes \sigma_n(a)} & \mathbb{F} \otimes \text{OSym}_\ell \text{OSym}_{(n,n')} \\
\end{array}
\]

(6.29)

of isomorphisms in which

1. the top map is an even degree 0 isomorphism of graded left \( \text{OSym}_{n'} \)-supermodules;
2. the bottom map is an even degree 0 isomorphism of graded right \( \text{OSym}_{n'} \)-supermodules for the action on \( \mathbb{F} \otimes \text{OSym}_\ell \text{OSym}_{(n,n')} \) defined by restriction along \( \sigma_n \circ \text{par} : \text{OSym}_{n'} \rightarrow \text{OSym}_{(n,n')} \);
3. the right hand map is an even linear isomorphism of degree 0;
4. the left hand map \( \psi_n \) is the graded superalgebra isomorphism induced by the involution \( \psi = \text{par} \) of \( \text{OSym} \).

**Remark 6.11.** The inclusion of the parity involution \( \text{par} \) in the definition of the left and bottom maps in (6.29) is hard to justify at this point—it could simply be omitted in both places and the simplified result is also true. However, the signs we have incorporated here persist through many subsequent definitions and, eventually, they prove to be essential in the construction of the adjunction in Theorem 9.11 below; see especially (9.42).

**Proof of Theorem 6.10.** (1) To construct the top map so that it is a homomorphism of graded left \( \text{OSym}_{n'} \)-supermodules, we must show that \( \text{OSym}_{(n,n')} \otimes \text{OSym}_\ell \mathbb{F} \) can be made into a graded left \( \overline{OH}_n \)-supermodule so that \( \bar{a} \cdot (b \otimes 1) = ab \otimes 1 \) for all \( a \in \text{OSym}_n, b \in \text{OSym}_{(n,n')} \). Since it is already a graded left \( \text{OSym}_{n'} \)-supermodule, and \( \overline{OH}_n \) is the quotient of \( \text{OSym}_n \) by the relations \( h^{(n)}_r = 0 \) for \( r > n' \), it suffices to check that \( h^{(n)}_r \) acts as zero on \( \text{OSym}_{(n,n')} \otimes \text{OSym}_\ell \mathbb{F} \) for all \( r > n' \). By Theorem 6.8(2), any homogeneous element of \( \text{OSym}_{(n,n')} \otimes \text{OSym}_\ell \mathbb{F} \) can be written as \( \sigma_n(b) \otimes 1 \) for \( b \in \text{OSym}_{n'} \). Now we must show that \( h^{(n)}_r \sigma_n(b) \otimes 1 = 0 \) for \( r > n' \). This follows from the calculation

\[
h^{(n)}_r \sigma_n(b) \otimes 1 = (-1)^{\text{par}(b)} \sigma_n(b) h^{(n)}_r \otimes 1 = (-1)^{\text{par}(b)} \sigma_n(b) \sum_{s=0}^r (-1)^s h^{(n)}_{r-s} e^{(f)}_s \otimes 1
\]

\[
= (-1)^{\text{par}(b)} \sigma_n(b) \sigma_n(-1)^s e^{(f)}_s \otimes 1 = 0.
\]

The second equality is just the observation that \( e^{(f)}_s \otimes 1 = 1 \otimes e^{(f)}_s = 0 \) for \( s > 0 \). For the penultimate equality, we used (6.25) and the infinite Grassmannian relation to see that \( h^{(n)}_r e^{(f)}(-t) = (-1)^s \sigma_n(e^{(f)}(-t)) \).
hence, equating coefficients of $t^{n'-r}$, we have that
\[
\sum_{s=0}^{r} (-1)^s \tilde{h}_{r-s}^{(n)} e_s^{(f)} = (-1)^r \sigma_n(e_r^{(n)}')
\] (6.30)
in $O\text{Sym}_{(r, n')}$ This is zero for $r > n$.

Now consider $\tilde{s}_\lambda^{(n)} \in \overline{O\text{H}}^f_n$. If $ht(\lambda) > n$, we already know this is zero by Theorem 6.2. We have by (6.1) that $\tilde{s}_\lambda^{(n)} = \tilde{h}_{A}^{(n)} + (\text{a linear combination of } \tilde{h}_{\mu}^{(n)} \text{ for } \mu > \lambda)$. We deduce that $\tilde{s}_\lambda^{(n)} = 0$ if $\lambda_1 > n'$ since $\tilde{h}_A^{(n)}$ and all of the $\tilde{h}_\mu^{(n)}$ appearing in this expansion are zero by the defining relations of $\overline{O\text{H}}^f_n$. This shows that $\overline{O\text{H}}^f_n$ is spanned by the elements $\tilde{s}_\lambda^{(n)}$ ($\lambda \in \Lambda^+_{n,n'}$). To see that these elements are linearly independent, hence, a basis for $\overline{O\text{H}}^f_n$, we act on the vector $1 \otimes 1 \in O\text{Sym}_{(n,n')} \otimes O\text{Sym}_{n'} \mathbb{F}$ to obtain the vectors $\tilde{s}_\lambda^{(n)} \otimes 1$ ($\lambda \in \Lambda^+_{n,n'}$) which constitute a basis for $O\text{Sym}_{(n,n')} \otimes O\text{Sym}_{n'} \mathbb{F}$ by Theorem 6.8(1). This argument also shows that the map (1) is an isomorphism.

(2) Similarly, to construct the bottom map, we must make $\mathbb{F} \otimes O\text{Sym}_{(n,n')} \otimes O\text{Sym}_{n'} \mathbb{F}$ into a graded right $\overline{O\text{H}}^f_{\mu'}$-supermodule so that $(1 \otimes b) \cdot \tilde{a} = (-1)^{par(a)\text{par}(b)} \otimes b\sigma_n(a)$ for $b \in O\text{Sym}_{(n,n')}$ and $a \in O\text{Sym}_{n'}$. To do this, one first applies $\ast$ to Theorem 6.8 to deduce that $\mathbb{F} \otimes O\text{Sym}_{(n,n')} \otimes O\text{Sym}_{n'} \mathbb{F}$ is spanned by vectors of the form $1 \otimes a$ for $a \in O\text{Sym}_n$. This plus the identity
\[
\sum_{s=0}^{r} (-1)^s \tilde{e}_s^{(f)} \gamma_n(h_{r-s}^{(n)}) = (-1)^r \tilde{e}_r^{(n)}
\] (6.31)
are then used to establish the well-definedness of the action. The fact that the bottom map is an isomorphism could be deduced using Theorem 6.8(2) like in the previous paragraph, but it also follows once we have checked the commutativity of the diagram using that the other three maps (1), (3) and (4) are all isomorphisms.

(3) To obtain the map (3), we start with the isomorphism $O\text{Sym}_{(n,n')} \sim O\text{Sym}_{(n,n')}$, $a \mapsto a^\ast$ where $\ast$ here is the restriction of the superalgebra anti-involution $\ast: O\text{Pol}_{\ell} \to O\text{Pol}_{\ell}$. Since we have that $(ab)^\ast = (-1)^{\text{par}(a)\text{par}(b)} b^\ast a^\ast$ for any $b \in O\text{Sym}_{(n,n')}$ and $a \in O\text{Sym}_{(n',\ell)}$ again, this induces the desired isomorphism $\mathbb{F} \otimes O\text{Sym}_{(n,n')} \sim O\text{Sym}_{(n,n')} \otimes O\text{Sym}_{n'} \mathbb{F}$.

(4) By definition, $\overline{O\text{H}}^f_n$ is the quotient of $O\text{Sym}$ by the two-sided ideal generated by $\{e_r | r > n \} \cup \{h_r | r > n' \}$ and $\overline{O\text{H}}^f_{\mu'}$ is the quotient of $O\text{Sym}$ by the two-sided ideal generated by $\{h_r | r > n \} \cup \{e_r | r > n' \}$. The involution $\psi \circ P_{n}$ interchanges these ideals so it factors through the quotients to induce an isomorphism $\overline{\psi}_{\mu} : \overline{O\text{H}}^f_n \sim \overline{O\text{H}}^f_{\mu'}$. This gives the graded superalgebra isomorphism (4).

To complete the proof, it just remains to show that the diagram commutes. Consider $\tilde{h}_1^{(n)} \cdots \tilde{h}_k^{(n)} \in \overline{O\text{H}}^f_n$ for $r_1, \ldots, r_k > 0$ and $k \geq 0$. The map (1) takes it to $h_1^{(n)} \cdots h_k^{(n)} \otimes 1$. As in the opening paragraph of the proof, we have that
\[
f h_r^{(n)} \otimes 1 = (-1)^f f \sigma_n(e_r^{(n)}') \otimes 1 = (-1)^{r+\text{par}(f)} \sigma_n(e_r^{(n)}') f \otimes 1
\]
for any $r$ and $f \in O\text{Sym}_n$. By induction, it follows that
\[
h_1^{(n)} \cdots h_k^{(n)} \otimes 1 = (-1)^{r_1+\cdots+r_k+\sum_{i<j} r_ir_j} \sigma_n(e_{r_1}^{(n)} \cdots e_{r_k}^{(n)}') \otimes 1 = (-1)^{r_1+\cdots+r_k+\sum_{i<j} r_ir_j} \sigma_n(e_{r_1}^{(n)} \cdots e_{r_k}^{(n)}') \otimes 1.
\]
This is the same as the image of $\tilde{h}_1^{(n)} \cdots \tilde{h}_k^{(n)}$ going around the other three sides of the square. \hfill \Box

**Corollary 6.12.** In $O\text{Sym}_{(n,n')} \otimes O\text{Sym}_{n'} \mathbb{F}$, we have that $\sigma_n(s_{\mu}^{(n')})^\ast \otimes 1 = (-1)^{\text{VE}(\mu)+|\mu|} s_{\mu}^{(n')} \otimes 1$ for every $\mu \in \Lambda^+$. 

Proof. Note that $(-1)^{NE(\mu)+|\mu|}s^{(n)}_{\mu}$ is the image of $(-1)^{NE(\mu)+|\mu|}\tilde{s}^{(n)}_{\mu}$ under the top map in the commuting square (6.29). Now we compute the image of $(-1)^{NE(\mu)+|\mu|}\tilde{s}^{(n)}_{\mu}$ around the other three edges of this square. It maps first to $(-1)^{|\mu|}\tilde{s}^{(m')}_{\mu}$ thanks to (6.3), then to $1 \otimes \sigma_n(s^{(n')}_{\mu})$, then to $\sigma_n(s^{(n')}_{\mu})^* \otimes 1$. □

**Corollary 6.13.** There is a unique (up to scalars) trace map $\tilde{\text{tr}} : \overline{OH}_n^\ell \to \mathbb{F}$ making $\overline{OH}_n^\ell$ into a graded Frobenius superalgebra over $\mathbb{F}$ of degree $2mn'$ and parity $nn'$ (mod 2). Moreover, normalizing $\tilde{\text{tr}}$ so that $\tilde{\text{tr}}(\tilde{s}^{(n)}_{0,0}) = 1$ and recalling the definition of $\text{tr}$ from Theorem 6.8, we have that $\text{tr}(\lambda) \otimes 1 = 1 \otimes \tilde{\text{tr}}(\lambda)$ in $O\text{Sym}_{(n,m')} \otimes O\text{Sym}_n \mathbb{F}$ for any $a \in O\text{Sym}_n$.

Proof. If it exists, the trace map is unique up to a non-zero scalar. Now we define $\tilde{\text{tr}} : \overline{OH}_n^\ell \to \mathbb{F}$ so that $\text{tr}(\lambda) \otimes 1 = 1 \otimes \tilde{\text{tr}}(\lambda)$ and check that is a trace map sending $\tilde{s}^{(n)}_{0,0}$ to 1. The latter statement follows because we know in (6.26) that $\text{sgn}(\varnothing) = 1$. To show that $\tilde{\text{tr}}$ is a trace map, we need to show that there exist elements $b_1, \ldots, b_m, b_1', \ldots, b_m' \in \overline{OH}_n^\ell$ satisfying (2.10) and (2.11). We take $b_1, \ldots, b_m$ and $b_1', \ldots, b_m'$ to be the elements $\tilde{s}^{(n)}_{\lambda}$ and $(-1)^{NE(\mu)+|\mu|} \text{sgn}(\mu)\tilde{s}^{(n)}_{\mu}$ for $\lambda \in \Lambda^+_{nn'}$ and $\mu \in \Lambda^+_{n',nn}$ enumerated so that $b_\nu = \tilde{s}^{(n)}_{\nu}$ and $b'_\nu = (-1)^{NE(\mu)+|\mu|} \text{sgn}(\mu)\tilde{s}^{(n)}_{\mu}$ if and only if $\mu|^n = n' - \lambda_{n+1-i}$ for each $i = 1, \ldots, n$. The properties (2.10) obviously hold, and (2.11) follows from Corollary 6.12 and (6.26). □

**Corollary 6.14.** Let $\lambda, \mu \in \Lambda^+$ be related as in the first case of (6.26). Then $\text{sgn}(\nu) = (-1)^{NE(\mu)+|\mu|} LR^\nu_{\lambda,\mu}$, where $\nu := (n'-n)$ and $LR^\nu_{\lambda,\mu}$ denotes the odd Littlewood-Richardson coefficient, that is, the coefficient of $s_\nu$ when $s_\lambda s_\mu^*$ is expanded in terms of odd Schur functions.

Proof. By the previous two corollaries and the definition (6.26), we have that

$$\text{sgn}(\nu) = (-1)^{NE(\mu)+|\mu|}LR^\nu_{\lambda,\mu}. \quad (6.32)$$

Some very special odd Littlewood-Richardson coefficients arise in the odd analog of the Pieri formula proved in [EKL, (72.7)]:

$$s_\lambda h_r = \sum_\mu (-1)^{NE(\lambda)+NE(\mu)+S(\lambda,\mu)} s_\mu. \quad (6.33)$$

The sum here is over all partitions $\mu$ whose Young diagram is obtained by adding one box to the bottom of $r$ different columns of the Young diagram of $\lambda$, and $S(\lambda,\mu) := \sum_{1 \leq j \leq r} \sum_{k=j+1}^{i_j} 1$ assuming these columns are indexed by $i_1 < \cdots < i_r$. The ghastly signs appearing in (6.32) and in the next lemma fortunately play no significant role.

**Lemma 6.15.** The inclusion $O\text{Sym}_n \hookrightarrow O\text{Sym}_{(n-1,1)}$ maps

$$s^{(n)}_{\mu} \mapsto \sum_{r,\lambda} (-1)^{NE(\lambda)+NE(\mu)+dW(\lambda)+dW(\mu)+S(\lambda,\mu)+(\lambda)_{(n-1)}} s^{(n)}_{\lambda} x^{(\lambda)}_n$$

where the sum is over all $\lambda \geq 0$ and partitions $\lambda$ whose Young diagram is obtained by removing one box from the bottom of $r$ different columns of the Young diagram of $\mu$, including all boxes from its $n$th row since $s^{(n-1)}_{\lambda} = 0$ if $\lambda_n > 0$.

Proof. From (6.11), it follows that the coefficient of $s^{(n-1)}_{\lambda} x^{(\lambda)}_n$ when $s^{(n)}_{\mu}$ is expanded in terms of the Schur basis for $S_{(n-1,1)}$ is equal to the $s_\lambda \otimes h_r$-coefficient of $\Delta^+(s_\mu)$. Using (4.29), (4.31) and (6.4), this is

$$(-1)^{dW(\lambda)+c(\lambda)}(s_\lambda \otimes h_r, \Delta^+(s_\mu))^+ = (-1)^{dW(\lambda)+c(\lambda)}(s_\lambda h_r, s_\mu)^+.$$ 

Now use (6.32) plus (6.4) again. □

\[ \]
We note finally, that a general formula for odd Littlewood-Richardson coefficients is derived in [E, Th. 4.8], showing that they can be computed by counting the same set of semi-standard skew tableaux that appear in the ordinary Littlewood-Richardson rule, but counting each one with a sign ±. A useful consequence of this is that if an ordinary even Littlewood-Richardson coefficient is zero then so is the corresponding odd Littlewood-Richardson coefficient. This, together with the odd Pieri rule, is all that we actually use below.

7. Odd equivariant Grassmannian cohomology algebras

Recall from Corollary 4.13 that the largest commutative quotient of $OSym_\ell$ is identified with the purely even graded superalgebra $R_\ell$ of symmetric polynomials of rank $\lfloor \ell/2 \rfloor$. We use the notation $\tilde{c}$ to denote the image of $c \in OSym_\ell$ in $R_\ell$. In particular, we have the elements $e^{(s)}_r, h^{(s)}_r \in R_\ell$, which are both zero if $r$ is odd. We are now going to work over this as our base ring.

**Definition 7.1.** For $\ell = n + n'$, the odd equivariant Grassmannian cohomology algebra\(^3\) is the graded $R_\ell$-superalgebra

$$OH^{\ell}_n := OSym_n \otimes R_\ell \left( \sum_{r > n'} (-1)^r h^{(r)}_{\sum_{s \neq r} c^{(s)}_s} \right).$$

(7.1)

This may be viewed as an odd analog of the $GL_\ell(\mathbb{C})$-equivariant cohomology algebra of the Grassmannian of $n$-dimensional subspaces of $\mathbb{C}^{\ell}$. The odd Grassmannian cohomology algebra $OH^{\ell}_n$ from (6.28) is naturally identified with the specialization $OH^{\ell}_n \otimes_{R_\ell} \mathbb{F}$.

For $a \in OSym_n$ and $c \in OSym_\ell$, we denote the canonical image of $a \otimes \tilde{c} \in OSym_n \otimes R_\ell$ in the quotient $OH^{\ell}_n$ by $a \otimes \tilde{c}$.

**Example 7.2.** We have that $OH^{\ell}_1 \cong \mathbb{F}[x]$ via the isomorphism $h^{(1)}_r \otimes 1 \mapsto x^r$ and $1 \otimes e^{(2)}_s \mapsto -x^2$.

**Lemma 7.3.** For $\ell = n + n'$, there is a surjective graded superalgebra homomorphism $\alpha^{(s)}_r : OH^{\ell}_n \twoheadrightarrow R_n'$ taking $a \otimes 1$ to zero for $a \in OSym_n$ of positive degree and $1 \otimes \tilde{c}^{\ell}$ to $\tilde{c}^{(s)}$ for $c \in OSym$.

**Proof.** This is clear from the nature of the defining relations (7.1). \hfill \Box

**Theorem 7.4.** For $\ell = n + n'$, $OH^{\ell}_n$ is free as an $R_\ell$-supermodule with basis given by the odd Schur polynomials $s^{(n)}_\lambda \otimes 1$ for $\lambda \in A^+_{n,n'}$. Moreover, there is a commuting diagram

$$\begin{array}{ccc}
OH^{\ell}_n & \xrightarrow{\varphi^{(s)}_r} & OSym_{(n,n')} \otimes_{OSym_\ell} R_\ell \\
\otimes \ \\
OH^{\ell}_n & \xrightarrow{a \otimes \tilde{c} \mapsto (-1)^{n \text{par}(a)} \tilde{c} \otimes \sigma(a)} & R_\ell \otimes_{OSym_\ell} OSym_{(n,n')} \\
\end{array}$$

(7.2)

of isomorphisms in which

1. the top map is an even degree 0 isomorphism of graded $(OSym_n, R_\ell)$-superbimodules;
2. the bottom map is an even degree 0 isomorphism of graded $(R_\ell, OSym_n)$-superbimodules for the right action of $OSym_n$ defined by restriction along $\sigma_n \circ p^n : OSym_n' \rightarrow OSym_{(n,n')}$;
3. the right hand map is an even degree 0 graded $R_\ell$-supermodule isomorphism;

\(^3\)Since $e^{(s)}_r = 0$ if $s$ is odd, the relations in (7.1) are written in an unnecessarily complicated form! One could just omit the sign $(-1)^r$ entirely, or replace $\sum_{s \neq r} (-1)^r h^{(s)}_r \otimes e^{(s)}_s$ with $\sum_{s \neq r} h^{(s)}_r \otimes e^{(s)}_s$. We use this form to be consistent with the slightly more general version of this superalgebra defined in Remark 7.9.
\( (4) \) the left hand map \( \psi_n' \) is a graded \( R_\ell \)-superalgebra isomorphism such that

\[
\psi_n'(a \widehat{\otimes} 1) = \sum_{i=1}^{p} a_i \otimes \tilde{c}_i
\]

for \( a \in O\text{Sym}_n \) with \( a = \sum_{i=1}^{p} (-1)^{\mu(a)} \sigma_n(a)^* c_i \) for \( a_i \in O\text{Sym}_n', c_i \in O\text{Sym}_\ell \).

**Proof.** (1) We must show that \( O\text{Sym}_{(n,n')} \otimes O\text{Sym}_n \ell \) can be made into a graded left \( OH_n^\ell \)-supermodule so that \( a \widehat{\otimes} \tilde{c}(b \otimes 1) = ab \otimes \tilde{c} \) for \( a \in O\text{Sym}_n, b \in O\text{Sym}_{(n,n')} \) and \( c \in O\text{Sym}_\ell \). To see this is well defined, we know by Theorem 6.8(2) that \( O\text{Sym}_{(n,n')} \otimes O\text{Sym}_n \ell \) is spanned by vectors of the form \( \sigma_n(b) \otimes 1 \) for \( b \in O\text{Sym}_n' \), so it suffices to check that

\[
\sum_{s=0}^{r} (-1)^{s} h_{r-s}^{(n)} \otimes e_s^{(f)} \cdot (\sigma_n(b) \otimes 1) = (-1)^{r+\text{par}(b)} \sigma_n(b) \sum_{s=0}^{r} (-1)^{s} h_{r-s}^{(n)} \otimes e_s^{(f)} = 0
\]

for \( b \in O\text{Sym}_n' \) and \( r > n' \). This follows from (6.30). Thus, we have defined the top map.

Next, we show that the elements \( \{s_{\lambda}^{(n)} \otimes 1 \mid \lambda \in \Lambda^+_{n,n'} \} \) generate \( OH_n^\ell \) as a graded \( R_\ell \)-superalgebra. This is not quite as easy as in the proof of Theorem 6.10 since it is no longer the case that \( s_{\lambda}^{(n)} \otimes 1 = 0 \) in \( OH_n^\ell \) when \( \lambda_1 > n \). Instead, mimicking the earlier argument, one shows by induction on \( |\lambda| \) that any \( s_{\lambda}^{(n)} \otimes 1 \) with \( \lambda_1 > n \) can be written as an \( R_\ell \)-linear combination of other \( s_{\mu}^{(n)} \otimes 1 \) for \( \mu \) with \( \mu_1 \leq n \) and \( |\mu| < |\lambda| \).

Now the proof of the first part of the theorem can be completed. The spanning set for \( OH_n^\ell \) just constructed is also linearly independent because it becomes the basis for \( O\text{Sym}_{(n,n')} \otimes O\text{Sym}_n \ell \) arising from Theorem 6.8(1) when we act on the vector \( 1 \otimes 1 \). This also shows that the top map (1) is an isomorphism.

(2) We need to make \( R_\ell \otimes O\text{Sym}_n \otimes O\text{Sym}_{(n,n')} \) into a graded right \( OH_n^\ell \)-supermodule so that \( (1 \otimes b) \cdot a \widehat{\otimes} \tilde{c} = (-1)^{\text{par}(\tilde{c})} a \otimes b \sigma_n(a) \) for \( a \in O\text{Sym}_n', b \in O\text{Sym}_{(n,n')} \) and \( c \in O\text{Sym}_\ell \). To check that this action is well defined, it suffices to check that \( \sum_{s=0}^{r} (-1)^{s} h_{r-s}^{(n')} \otimes e_s^{(f)} \cdot \sigma_n(b) \) acts as zero on \( 1 \otimes b \) for \( b \in O\text{Sym}_n \) and \( r > n \). This follows using (6.31) and also that \( e_s^{(f)} = 0 \) if \( s \) is odd:

\[
(1 \otimes b) \cdot \sum_{s=0}^{r} (-1)^{s} h_{r-s}^{(n')} \otimes e_s^{(f)} = (-1)^{r+\text{par}(b)} \sum_{s=0}^{r} (-1)^{s+n(r-s)} e_s^{(f)} \otimes \sigma_n(h_{r-s}^{(n')}) b
\]

\[
= (-1)^{r+\text{par}(b)+n} \sum_{s=0}^{r} (-1)^s 1 \otimes e_s^{(f)} \sigma_n(h_{r-s}^{(n')}) b = (-1)^{r+\text{par}(b)+n} 1 \otimes c_s^{(n')} b = 0.
\]

Applying \( * \) to the basis from Theorem 6.8(2), we deduce that \( R_\ell \otimes O\text{Sym}_n \otimes O\text{Sym}_{(n,n')} \) is a free graded \( R_\ell \)-supermodule with basis \( \{1 \otimes \sigma_n(s_{\mu}^{(n')}) \mid \mu \in \Lambda^+_{n,n'} \} \). Also the elements \( \{s_{\mu}^{(n')} \otimes 1 \mid \mu \in \Lambda^+_{n,n'} \} \) span \( OH_n^\ell \) as in the previous paragraph. Acting on \( 1 \otimes 1 \) shows finally that these elements form a basis for \( OH_n^\ell \) and that the bottom map is an isomorphism.

(3) To construct the right hand map, start with the map

\[
R_\ell \otimes O\text{Sym}_{(n,n')} \rightarrow O\text{Sym}_{(n,n')} \otimes O\text{Sym}_n \ell, \quad \tilde{c} \otimes aa^* \otimes \tilde{c}
\]

for \( c \in O\text{Sym}_n, a \in O\text{Sym}_{(n,n')} \). Since \( R_\ell \) is purely even and commutative, this is a homomorphism of graded \( R_\ell \)-supermodules. To check that the map is balanced, we need to show that the images of \( \tilde{c}_1 \tilde{c}_2 \otimes a \) and \( \tilde{c}_1 \otimes c_2 a \) are the same for \( a \in O\text{Sym}_{(n,n')} \) and \( c_1, c_2 \in O\text{Sym}_n \). Thus, we must show that \( a^* \otimes \tilde{c}_1 \tilde{c}_2 = (-1)^{\text{par}(a)\text{par}(c_1)c_1^*} \tilde{c}_1 \tilde{c}_2 \otimes \tilde{c}_1 \) are equal. Note that the image of \( c_2^* \) in \( R_\ell \) is the same as the image of \( c_2 \). This follows because \( R_\ell \) is a purely even commutative algebra and \( * \) fixes the generators \( e_i^{(f)} \) of
$O\text{Sym}_t$. So we have that $a^*c_2^* \otimes \hat{c}_1 = a^* \otimes \hat{c}_2\hat{c}_1$. Thus, we are reduced to checking that $a^* \otimes \hat{c}_1\hat{c}_2 = (-1)^{\text{par}(a)\text{par}(c)}a^* \otimes \hat{c}_2\hat{c}_1$, which follows because both sides are zero if $c_2$ is odd, and $R_t$ is commutative. So this map induces the required graded $R_t$-supermodule isomorphism.

(4) We define $\psi_n^{'(a)}$ to be the unique graded $R_t$-supermodule isomorphism making the diagram commute. If $a = \sum_{i=1}^{p}(-1)^{\text{par}(a)}\sigma_n(a_i)^*\hat{c}_i$ for $a_i \in O\text{Sym}_m$, $c_i \in O\text{Sym}_t$ then the image of $a \otimes \hat{1}$ under the top map is equal to $\sum_{i=1}^{p}(-1)^{\text{par}(a)}\sigma_n(a_i)^* \otimes \hat{c}_i$. It follows that $\psi_n^{'(a \otimes \hat{1})} = \sum_{i=1}^{p} a_i \otimes \hat{c}_i$ because the latter expression also maps to $\sum_{i=1}^{p}(-1)^{\text{par}(a)}\sigma_n(a_i)^* \otimes \hat{c}_i$ when the bottom map followed by the right hand map is applied.

We still need to show that $\psi_n^{'(a)}$ is actually a graded $R_t$-superalgebra isomorphism. For this, we take $a, b \in O\text{Sym}_m$ such that $a = \sum_{i=1}^{p}(-1)^{\text{par}(a)}\sigma_n(a_i)^*\hat{c}_i$ and $b = \sum_{j=1}^{q}(-1)^{\text{par}(b)}\sigma_n(b_j)^*\hat{d}_j$ for $a_i, b_j \in O\text{Sym}_m$ and $c_i, d_j \in O\text{Sym}_t$. Then we have that

$$\psi_n^{'(a)}(\psi_n^{'(b)}(b)) = \left(\sum_{i=1}^{p} a_i \otimes \hat{c}_i\right)\left(\sum_{j=1}^{q} b_j \otimes \hat{d}_j\right) = \sum_{i=1}^{p} \sum_{j=1}^{q} a_i b_j \otimes \hat{c}_i \hat{d}_j.$$

To see that this equals $\psi_n^{'(ab)}$, we have that

$$ab = \sum_{j=1}^{q}(-1)^{\text{par}(b_j)}a\sigma_n(b_j)^*d_j = (-1)^{\text{par}(a)\text{par}(b)}\sum_{j=1}^{q} \sigma_n(b_j)^*ad_j$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{q}(-1)^{\text{par}(a_i)\text{par}(b_j)}\sigma_n(a_i)^*\sigma_n(b_j)^*c_i d_j$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{q}(-1)^{\text{par}(a_i)\text{par}(b)_j}\sigma_n(a_i b_j)^*c_i d_j.$$

So

$$\psi_n^{'(ab)} = \sum_{i=1}^{p} \sum_{j=1}^{q}(-1)^{\text{par}(c_i)\text{par}(b_j)}a_i b_j \otimes \hat{c}_i \hat{d}_j = \sum_{i=1}^{p} \sum_{j=1}^{q} a_i b_j \otimes \hat{c}_i \hat{d}_j$$

as required to complete this check.

\begin{corollary}
There is a unique (up to scalars) trace map $\tilde{\text{tr}} : O\hat{H}_n \rightarrow R_t$ making $O\hat{H}_n$ into a graded Frobenius superalgebra over $R_t$ of degree $2m'$ and parity $m'$ (mod 2). Moreover, normalizing $\tilde{\text{tr}}$ so that $\tilde{\text{tr}}(s_{(n)}^{(2)}) = 1$ and recalling the definition of tr from Theorem 6.8, we have that $\text{tr}(a)\otimes 1 = 1 \otimes \text{tr}(a \otimes 1)$ in $O\text{Sym}_{(n,n')} \otimes O\text{Sym}_t R_t$ for any $a \in O\text{Sym}_n$.
\end{corollary}

\begin{proof}
Define $\tilde{\text{tr}} : O\hat{H}_n \rightarrow R_t$ to be the unique graded $R_t$-supermodule homomorphism such that $\text{tr}(a) \otimes 1 = 1 \otimes \tilde{\text{tr}}(a \otimes 1)$ for $a \in O\text{Sym}_n$. This is a homogeneous linear map of degree $-2m'$ and parity $m'$ (mod 2) such that $\tilde{\text{tr}}(s_{(n)}^{(2n)}) = 1$. It remains to check (2.10) and (2.11). We define $b_1, \ldots, b_m, b_1^-, \ldots, b_m^- \in O\hat{H}_n$ so that the $b_i$ are the elements $s_{(n)}^{(2)} \otimes 1$ and the images of the $b_i^-$ under $\psi_n^{'(a)}$ are the elements $\text{sgn}(\lambda)\lambda^{(2-n)} s_{(n)}^{(2)} \otimes 1 \in O\hat{H}_n$ for $\lambda \in \Lambda^n$ with $\text{ht}(\lambda) \leq n$ and $\lambda_1 \leq n'$. The commutativity of the diagram (7.2) shows that this $b_i$ maps to $s_{(n)}^{(2)} \otimes 1$ and this $b_i^-$ maps to $\text{sgn}(\lambda)\lambda^{(2-n)} s_{(n)}^{(2)} \otimes 1$. Then (2.11) follows using (6.26).
\end{proof}

\begin{corollary}
$O\hat{H}_n$ is a graded $R_t$-supermodule of graded rank $q^{n(-n)}[\ell]_{a,q,x}^t$.
\end{corollary}

\begin{proof}
This follows from the basis theorem that is the first assertion of Theorem 7.4 plus Corollary 3.2.
\end{proof}
The next lemma investigates the isomorphism $\psi_n^\ell : OH_n^\ell \wedge OH_n^\ell$ constructed in Theorem 7.4(4).

**Lemma 7.7.** For $\ell = n + n'$, the isomorphism $\psi_n^\ell : OH_n^\ell \wedge OH_n^\ell \rightarrow OH_n^\ell$ maps

$$h_r^{(n)} \otimes 1 \mapsto (-1)^{(n+1)r} \sum_{s=0}^r (-1)^s e_{r-s}^{(n)} \otimes h_s^{(f)}, \quad e_r^{(n)} \otimes 1 \mapsto (-1)^{(n+1)r} \sum_{s=0}^r (-1)^s h_{r-s}^{(n)} \otimes e_s^{(f)} \quad (7.4)$$

for $r \geq 0$.

**Proof.** We first observe that

$$h_r^{(n)}(t) = (-1)^r \sigma_n(e^{(n)}(-t)) h^{(f)}(t), \quad e_r^{(n)}(t) = (-1)^r \sigma_n(e^{(n)}(-t)) e^{(f)}(t)$$

in $OSym(n,n')$. These follow from (4.46) using the identities $h^{(f)}(t) = \sigma_n(h^{(n)}(t)) h^{(n)}(t)$ and $e^{(f)}(t) = \sigma_n(e^{(n)}(t)) e^{(n)}(t)$. Hence, using (7.3), we deduce that $\psi_n^\ell$ maps

$$h_r^{(n)}(t) \otimes 1 \mapsto (-1)^{(n+1)r} e^{(n)}((-1)^{(n+1)r}) \otimes h^{(f)}(t), \quad e_r^{(n)}(t) \otimes 1 \mapsto (-1)^{(n+1)r} e^{(n)}((-1)^{(n+1)r}) \otimes e^{(f)}(t).$$

The first formula in (7.4) follows from the first of these identities on equating $r^{n+n'}$-coefficients in the first of these equations, remembering that $e_r^{(n)}$ is fixed by $\ast$ and $e_{r-s} = 0$ if $r \not\equiv s \pmod{2}$. Similarly, equating $r^{n+n'}$-coefficients in the second identity gives that

$$\psi_n^\ell(e_r^{(n)} \otimes 1) = (-1)^{(n+1)r} \sum_{s=0}^r (-1)^s \sum_{t=0}^{r-s} (-1)^t h_{r-t}^{(n)} \otimes e_s^{(f)}.$$

Replacing $e_r^{(n)}$ with $(-1)^{(r)} e_r^{(n)}$, $(e^{(n)} e^{(n)})^\ast$ with $(-1)^{(r)} h_{r-t}^{(n)}$ and $e_s^{(f)} = (-1)^{(r)} e_s^{(f)}$ gives the second formula in (7.4). \hfill $\square$

**Corollary 7.8.** $\psi_n^\ell \circ \psi_n^\ell = p^\ell$.

**Proof.** We simply use (7.4) to compute $(\psi_n^\ell \circ \psi_n^\ell)(h_r^{(n)} \otimes 1)$, to see that it equals

$$(-1)^{(n+1)r} \sum_{s=0}^r (-1)^s h_{r-s}^{(n)} \otimes e_s^{(f)}.$$

The expression in the brackets here is $\delta_{r,s}$ thanks to (4.55), so this is $(-1)^r h_r^{(n)} \otimes 1$, which is the image of $\hat{h}_r^{(n)} \otimes 1$ under $p^\ell$. \hfill $\square$

**Remark 7.9.** We mentioned already in Remark 4.14 the possibility of working not over the commutative ground ring $R_\ell$ but over the ground ring $\widetilde{R}_\ell$ that is the largest supercommutative quotient of $OSym_\ell$. In that case, instead of (7.1), we consider

$$\widehat{OH}_n^\ell := OSym_n \otimes \widetilde{R}_\ell \left/ \left\{ \sum_{s=0}^r (-1)^s h_{r-s}^{(n)} \otimes e_s^{(f)} \bigg| r > n' \right\} \right.$$  \hspace{-3.5em} (7.5)

where $\hat{a}$ denotes the image of $a \in OSym_\ell$ in $\widetilde{R}_\ell$. Note that $OH_n^\ell = \overline{OH}_n^\ell \otimes_{\overline{R}_\ell} R_\ell$. With this extended algebra, it is still possible to prove a version of Theorem 7.4. The diagram (7.2) becomes

$$\begin{array}{ccc}
OH_n^\ell & \xrightarrow{a \otimes \iota \mapsto \hat{a} \otimes \hat{c}} & OSym_{(n,n')} \otimes_{OSym_\ell} \widetilde{R}_\ell \\
\downarrow \psi_n^\ell & & \uparrow e^{\otimes \mp \left((-1)^{\text{par}(n)\text{par}(n')} e \otimes \sigma(a)\right)} \\
OH_n^\ell & \xrightarrow{a \otimes \iota \mapsto (-1)^{\text{par}(n)\text{par}(n')} e \otimes \sigma(a)} & \widetilde{R}_\ell \otimes_{OSym_\ell} OSym_{(n,n')} 
\end{array}$$

(7.6)
The only difference compared to (7.2) is that there are some extra signs since “scalars” can now be odd. However, the proof is more delicate: to show that the bottom map is well defined, one needs to observe that the two-sided ideal defining $\overline{OH}_n^\ell$ as a quotient of $OSym_n \otimes R_\ell$ contains the elements $\sum_{r=0}^{r'} h_{r'} \otimes e_r^{(f)}$ for all $r > n' + 1$, as may be checked using (4.15). In this extended setting, Corollary 8.7 becomes more complicated: one finds that

$$\psi_n^f(\overline{\psi}_n(h_{n}^{(n)} \otimes 1)) = \begin{cases} (-1)^{r} h_{r}^{(n)} \otimes 1 + (-1)^{r(r-1)} 2 h_{r-1}^{(n)} \otimes e_r^{(f)} & \text{if } r > 0 \text{ and } r \equiv n \pmod{2} \\ (-1)^{r} h_{r}^{(n)} \otimes 1 & \text{otherwise.} \end{cases}$$

(7.7)

Many of the results below can be upgraded from $OH_n^\ell$ to $\overline{OH}_n^\ell$, as we will note in several subsequent remarks, but the second adjunction constructed in Corollary 9.13 is more complicated in this more general setup due to the additional term appearing in (7.7). This is the main reason we have chosen to work simply over the purely even ground ring $R_\ell$.

8. Odd deformed cyclotomic nil-Hecke algebras

The odd cyclotomic nil-Hecke algebra $\overline{ONH}_n^\ell$ is the quotient of $ONH_n^\ell$ by the two-sided ideal generated by the element $x_1^\ell$. This algebra was introduced originally in [EKL, Sec. 5]. In particular, in [EKL, Prop. 5.2], it is shown that $\overline{ONH}_n^\ell$ is zero unless $0 \leq n \leq \ell$, in which case it is isomorphic to the graded matrix superalgebra $M_{q^{\ell}}(\overline{OH}_n^\ell)$, notation as explained after Remark 5.5.

**Definition 8.1.** The odd deformed cyclotomic nil-Hecke algebra is the quotient algebra

$$ONH_n^\ell := ONH_n \otimes R_\ell \left\{ \sum_{r=0}^{\ell} (-1)^{r} x_1^{\ell-r} \otimes e_r^{(f)} \right\}$$

(8.1)

for $n > 0$. We also let $ONH_0^\ell := \mathbb{F} \otimes R_\ell$ so that $ONH_0^\ell$ makes sense for all $n \geq 0$.

We denote the image of $a \otimes \hat{c}$ in $ONH_n^\ell$ by $\hat{a} \otimes \hat{c}$. It is a graded $R_\ell$-superalgebra which is the odd analog of the algebras defined for $sl_2$ in [R1, Sec. 5.2.1] and for other Cartan types in [R2, Sec. 4.4.1]; our definition (8.1) looks more like the latter formulation.

**Theorem 8.2.** The odd deformed cyclotomic nil-Hecke algebra $ONH_n^\ell$ is zero unless $0 \leq n \leq \ell$. Assuming $0 \leq n \leq \ell$, the natural left action of $ONH_n^\ell \otimes R_\ell$ on $OPol_n \otimes OSym_n OH_n^\ell$ factors through the quotient $ONH_n^\ell$ to make $OPol_n \otimes OSym_n OH_n^\ell$ into a graded $(ONH_n^\ell, OH_n^\ell)$-superbimodule. The associated representation

$$\rho : ONH_n^\ell \to \text{End}_{OH_n^\ell}(OPol_n \otimes OSym_n OH_n^\ell)$$

is an isomorphism of graded superalgebras. Moreover, $OPol_n \otimes OSym_n OH_n^\ell$ is free as a graded right $OH_n^\ell$-supermodule with basis $\{ x^k \otimes 1 \mid k \in K \}$ where $K := \{ k \in \mathbb{N}^n \mid 0 \leq k_i \leq n - i \text{ for } i = 1, \ldots, n \}$. Hence, $ONH_n^\ell$ is isomorphic to the graded matrix superalgebra $M_{q^{\ell}}(\overline{OH}_n^\ell)$.

We will prove Theorem 8.2 shortly, ultimately deducing it from Theorem 5.3, but first we state some corollaries. The first is analogous to Corollary 5.9.

**Corollary 8.3.** The maps $i$ and $j$ from Lemma 5.8 induce an isomorphism $OH_n^\ell \equiv (\omega \chi)_n ONH_n^\ell(\omega \chi)_n$ of graded superalgebras and an isomorphism $OPol_n \otimes OSym_n OH_n^\ell \equiv ONH_n^\ell(\omega \chi)_n$ of graded $(ONH_n^\ell, OH_n^\ell)$-superbimodule. Making these identifications, the idempotent truncation functor

$$(\omega \chi)_n : ONH_n^\ell \text{-gsMod} \to OH_n^\ell \text{-gsMod}$$

is an equivalence of graded $(Q, \Pi)$-supercategories.
Corollary 8.4. For $0 \leq n \leq \ell$, $\text{ONH}_n^\ell$ is a free graded $R_\ell$-supermodule of graded rank $q^{n(\ell-n)}[n]_{q,\bar{x}}[\ell]_{q,\bar{x}}^1/([\ell-n]_{q,\bar{x}})$.  

Proof. This follows from the final part of the theorem and Corollary 7.6.

Corollary 8.5. For $0 \leq n \leq \ell$, the monomials 
$$\{x^\ell\tau_w \otimes 1 \mid w \in S_n, \kappa \in \mathbb{N}^n \text{ with } 0 \leq \kappa_i \leq \ell - i \text{ for } i = 1, \ldots, n\}$$

form a basis for $\text{ONH}_n^\ell$ as a free $R_\ell$-supermodule.

Proof. The free graded $R_\ell$-supermodule with basis given by elements of the same degrees and parities as these monomials is graded rank $q^{n(\ell-n)}[n]_{q,\bar{x}}[\ell]_{q,\bar{x}}^1/([\ell-n]_{q,\bar{x}})$, which is the same as the graded rank of $\text{ONH}_n^\ell$ according to Corollary 8.4. Therefore it suffices to show that the monomials $\{x^\ell\tau_w \otimes 1 \mid w \in S_n, \kappa \in \mathbb{N}^n \text{ with } 0 \leq \kappa_i \leq \ell - i \text{ for } i = 1, \ldots, n\}$ are linearly independent over $R_\ell$.

We first prove this linear independence in the special case that $n = \ell$, in which case we have simply that $\text{OH}_\ell^\ell = R_\ell$. Suppose we have a linear relation $\sum_{w, \kappa} x^\ell\tau_w \otimes c_{w, \kappa} = 0$ in $\text{ONH}_\ell^\ell$ for $c_{w, \kappa} \in R_\ell$ that are not all zero, summing over $w \in S_\ell, \kappa \in \mathbb{N}^\ell$ with $0 \leq \kappa_i \leq \ell - i$ for all $i = 1, \ldots, \ell$. Pick $w$ of minimal length such that $c_{w, \kappa} \neq 0$ for some $\kappa$. Then we act on the vector $p_{\ell, \kappa}^{(\ell)} \otimes 1 \in \text{OPol}_\ell \otimes_{\text{OSym}_\ell} R_\ell$ to deduce as in the proof of Theorem 5.1 that $\sum_{\kappa} (-1)^{\text{par}(c_{w, \kappa})} x^\ell \otimes c_{w, \kappa} = 0$. The elements $x^\ell \otimes 1$ for $\kappa \in \mathbb{N}^\ell$ with $0 \leq \kappa_i \leq \ell - i$ for all $i = 1, \ldots, \ell$ are linearly independent over $R_\ell$ thanks to Remark 5.5. So this implies that $c_{w, \kappa} = 0$ for all $\kappa$, which is a contradiction.

Now we treat the general case. The inclusion $\text{ONH}_n^\ell \otimes R_\ell \hookrightarrow \text{ONH}_n^\ell$ induces an $R_\ell$-superalgebra homomorphism $i : \text{ONH}_n^\ell \to \text{ONH}_n^\ell$. The monomials in $\text{ONH}_n^\ell$ which we are trying to show are linearly independent map to a subset of the monomials shown to be linearly independent in the previous paragraph. This completes the proof (and also shows that $i$ is injective).

Corollary 8.6. The graded superalgebra $\text{ONH}_n^\ell$ is a graded Frobenius superalgebra of degree $n(2\ell - n - 1)$ and parity $n\ell - \binom{n}{2}$ (mod 2) over $R_\ell$ with its unique (up to scalars) trace map taking an element to its $x_1^{\ell-1}x_2^{\ell-2}\cdots x_n^{\ell-n} \otimes 1$-coefficient when expanded in terms of the monomial basis from Corollary 8.5.

Proof. It is a graded Frobenius superalgebra of the asserted degree and parity by theorem and Corollary 7.5. The trace has to project onto the top graded component spanned by $x_1^{\ell-1}x_2^{\ell-2}\cdots x_n^{\ell-n} \otimes 1$ in the decomposition of $\text{ONH}_n^\ell$ as a free graded $R_\ell$-supermodule.

Remark 8.7. Since $\text{ONH}_n^\ell \cong \text{ONH}_n^\ell \otimes R_\ell \mathbb{F}$ and $\mathbb{O}^\ell_n \cong \text{OH}_n^\ell \otimes R_\ell \mathbb{F}$, Corollary 8.4 implies that $\overline{\text{ONH}}_n^\ell$ is isomorphic to the graded matrix superalgebra $M_{q^{\ell}]\mathbb{N}^n\mathbb{F}q\ell\mathbb{F}} ([\text{OH}_n^\ell]$), recovering [EKL, Prop. 5.2]. Also Corollary 8.5 implies that $\overline{\text{ONH}}_n^\ell$ has basis given by the canonical images of the monomials 
$$\{x^\ell\tau_w \mid w \in S_n, \kappa \in \mathbb{N}^n \text{ with } 0 \leq \kappa_i \leq \ell - i \text{ for } i = 1, \ldots, n\}$$

recovering [HS, Th. 4.10]. The proof that these monomials span given in [HS] gives an explicit algorithm to “straighten” arbitrary monomials into this form.

In the remainder of the section, we prove Theorem 8.2. The approach is based on the proof of [EKL, Prop. 5.2] (the result which we are generalizing). First we record some preliminary lemmas.

Lemma 8.8. Suppose that $n \geq 1$. Let $y$ be a non-zero homogeneous element of $\sigma_1(\text{OPol}_{n-1})$ and consider the (free) right $\text{OSym}_n$-submodule of $\text{OPol}_n$ with basis $v_1, \ldots, v_n$ defined from $v_i := yx_1^{i-1}$. The matrix of the endomorphism of this subspace defined by the left action of $(-1)^{\text{par}(y)}x_1$ is equal to the
(non-commutative) companion matrix

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & (-1)^{n-1} e_n^{(n)} \\
1 & 0 & \cdots & 0 & (-1)^{n-2} e_{n-1}^{(n)} \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0 & -e_2^{(n)} \\
0 & \cdots & 0 & 1 & e_1^{(n)}
\end{pmatrix}
\]

(8.2)

of the polynomial \( (t - x_1) \cdots (t - x_n) \in \text{OPol}_n[t] \).

**Proof.** We have that \((-1)^{\text{par}(y)} xy^{i-1} = xy^i\). This gives all but the last column of the matrix already. For the last column, use Lemma 6.7. □

**Lemma 8.9.** Let \( C \) be the \( n \times n \) companion matrix from (8.2). For any \( 1 \leq i, j \leq n \) and \( k \geq 0 \), the \((i, j)\)-entry of \( C^k \) is equal to

\[
c_{i,j;k} := \sum_{t=0}^{\min(k+j-i,n-i)} (-1)^t e_t^{(n)} h_{k+j-i-1}^{(n)},
\]

(8.3)

which is zero if \( k < i - j \) and \( 1 \) if \( k = i - j \).

**Proof.** This goes by induction on \( k = 0, 1, \ldots \). When \( k = 0 \), we have that

\[
c_{i,j;k} = \sum_{t=0}^{j-i} (-1)^t e_t^{(n)} h_{j-i-1}^{(n)}
\]

which is zero if \( i > j \) as it is the empty sum, and it is \( \delta_{i,j} \) if \( i \leq j \) by the infinite Grassmannian relation. This checks the induction base. Then we take \( k \geq 1 \) and consider the \((i, j)\)-entry of \( C^{k+1} = CC^k \). Since \( C \) has at most two non-zero entries in its \( i \)th row, namely, its \((i, i-1)\)-entry \( 1 \) if \( i > 1 \) and its \((i, n)\)-entry \((-1)^{n-i} e_{n+1-i}^{(n)}\), we get by induction that the \((i, j)\)-entry of \( C^{k+1} \) is equal to \( c_{i-1,j;k} + (-1)^{n-i} e_{n+1-i}^{(n)} h_{n,j,i}^{(n)} \) where the first term should be omitted in the case that \( i = 1 \). This is equal to

\[
\sum_{t=0}^{\min(k+1+j-i,n+1-i)} (-1)^t e_t^{(n)} h_{k+1+j-i-t}^{(n)} + (-1)^{n-i} e_{n+1-i}^{(n)} h_{k,j-n}^{(n)},
\]

interpreting \( h_{k,j-n}^{(n)} \) as zero if \( k < n - j \) and noting in the case that \( i = 1 \) that the first term here is zero by the infinite Grassmannian relation (so there is no need to omit it). To complete the proof, we need to show that

\[
\sum_{t=0}^{\min(k+1+j-i,n+1-i)} (-1)^t e_t^{(n)} h_{k+1+j-i-t}^{(n)} = \sum_{t=0}^{\min(k+1+j-i,n+1-i)} (-1)^t e_t^{(n)} h_{k+1+j-i-t}^{(n)} + (-1)^{n-i} e_{n+1-i}^{(n)} h_{k,j-n}^{(n)}.
\]

If \( k + j - i \leq n - i \) both sums are over \( 0 \leq t \leq k + 1 + j - i \) and the second term on the right hand side is zero by convention since \( k + j - n < 0 \), so the equality is true. If \( k + j - i > n - i \) then the sum on the right hand side has one extra term when \( t = n + 1 - i \) compared to the sum on the left hand side. But this extra term is \((-1)^{n+1-i} e_{n+1-i}^{(n)} h_{k,j-n}^{(n)} \) which cancels with the final term on the right hand side. □

**Proof of Theorem 8.2.** Recall from the statement of the theorem that

\[
K = \{ k \in \mathbb{N}^n \mid 0 \leq k_i \leq n - i \text{ for } i = 1, \ldots, n \}.
\]

Let \( A := \text{ONH}_n \otimes R_t, B := \text{OSym}_n \otimes R_t \) and \( V := \text{OPol}_n \otimes R_t \), which is a graded \((A, B)\)-superbimodule. By Theorem 5.3, \( V \) is free as a graded right \( B \)-supermodule with basis \( \{ x^k \otimes 1 \mid k \in K \} \). Also \( A \cong \text{End}_g(V) \) so that \( A \) can be identified with the graded matrix superalgebra consisting of matrices \((a_{k,s})_{k,s \in K}\), for
$a_{\kappa \kappa'} \in B$, this matrix representing the endomorphism $x^\kappa \otimes 1 \mapsto \sum_{\kappa \in K} x^\kappa \otimes 1 a_{\kappa \kappa'}$. In this situation, Morita theory implies that there are bijections between the sets of graded superideals of $A$, graded sub-

superbimodules of $V$ and graded superideals of $B$ so that $I \leftrightarrow IV \leftrightarrow J$. For $I \subseteq A$ corresponding to $J \subseteq B$ in this way, a set of generators for $J$ is given by the matrix entries of a set of generators of $I$, and we have that $A/I \cong \text{End}_B(\mathcal{V}/IV)$.

Thus, to prove the theorem, we start from the two-sided ideal $I$ of $A$ from (8.1), which may be described equivalently as the two-sided ideal generated by the elements $\sum_{r=0}^n (-1)^r x_1^{-r} \otimes e_s^{(f)}$ ($r \geq \ell$). We must show that the two-sided ideal $J$ of $B$ generated by the matrix entries of the generators of $I$ is equal to $B$ if $n > \ell$ and it is equal to the two-sided ideal of $B$ from (7.1) if $n \leq \ell$. Consider the matrix associated to the generator $\sum_{r=0}^n (-1)^r x_1^{-r} \otimes e_s^{(f)}$ of $I$ for $r \geq \ell$. Lemma 8.8 implies that it is a block diagonal matrix with $n \times n$ blocks parametrized by all $\kappa \in K$ with $\kappa_1 = 0$, the $(i, j)$-entry of this block being the $x^\kappa x_1^{-i} \otimes 1$-coefficient of $\sum_{s=0}^r (-1)^r x_1^{-s} \otimes e_s^{(f)} \cdot (x_\kappa x_1^{-1} \otimes 1)$ for $1 \leq i, j \leq n$. Denote this matrix entry by $f_{i,j;\kappa}$. Using Lemmas 8.8 and 8.9, we have that

$$f_{i,j;\kappa} = (-1)^{r(i)} \sum_{s=0}^{r-\min(n+j-i,s-n-i)} (-1)^{sj} \sum_{t=0}^{\min(n+j-i,-s)} (-1)^t e_s^{(f)} h_{r+j-i-t,s} \otimes e_s^{(f)}. \quad (8.4)$$

Apart from the leading sign which is irrelevant for the problem in hand, this does not depend on $\kappa$, so we may as well assume from now on that $\kappa = 0 = (0, \ldots, 0)$. If $n > \ell$, we take $r = \ell$, $j = 1$ and $i = \ell + 1$, in which case the summations collapse and we deduce that $\sum_{\kappa \in K} f_{i,j;\kappa} = 1 \in J$. So $J = B$ as required in this case. Now assume that $n \leq \ell$ and set $n' := \ell - n$. It remains to show that the elements

$$F := \left\{ f_{i,j;r}^{(0)} \bigg| 1 \leq i \leq j \leq n, r \geq \ell \right\}, \quad G := \left\{ \sum_{r=0}^n (-1)^r h_{r-n,s}^{(n)} \otimes e_s^{(f)} \bigg| r > n' \right\}$$

generate the same two-sided ideal of $B$. We switch the summations in (8.4) to deduce that

$$f_{i,j;r}^{(0)} = \sum_{t=0}^{\min(r+j-i,s-n-i)} (-1)^t e_s^{(f)} \left( \sum_{s=0}^{\min(r+j-i,t)} (-1)^s h_{r+j-i-t,s}^{(n)} \otimes e_s^{(f)} \right).$$

We have that $r + j - i \geq \ell + 1 - i > n - i$ so the first summation is over $0 \leq t \leq n - i$. Since $e_s^{(f)} = 0$ for $s > \ell$, the second summation is over $0 \leq s \leq r + j - i - t$. Taking $i = n$ and $j = 1$ gives us the elements $\sum_{s=0}^{r-n+1} (-1)^s h_{r-n+1,s}^{(n)} \otimes e_s^{(f)}$ for all $r \geq \ell$. Since we have all $r$ so that $r - n + 1 > n'$, these already give us all of the elements of $G$, demonstrating one containment. It remains to show that all $f_{i,j;r}^{(0)}$ for $1 \leq i, j \leq n$ and $r \geq \ell$ also lie in $\langle G \rangle$. In fact, given that $t \leq n - i$, we have that

$$\sum_{s=0}^{r+j-i-t} (-1)^s h_{r+j-i-t,s}^{(n)} \otimes e_s^{(f)} \in \langle G \rangle$$

because $r + j - i - t \geq n' + j$ and $e_s^{(f)} = 0$ if $s$ is odd.

**Remark 8.10.** Continuing from Remarks 4.14 and 7.9, there is also an extended version of $\text{ONH}^L_n$ with the ground ring $\tilde{R}_t$ upgraded to $\tilde{R}_t$, namely,

$$\tilde{\text{ONH}}^L_n := \text{ONH}_n \otimes \tilde{R}_t \left/ \left( \sum_{r=0}^\ell (-1)^r x_1^{-r} \otimes e_r^{(f)} \right) \right. \quad (8.5)$$

for $n > 0$. Theorem 8.2 can be extended to this setup, showing that $\tilde{\text{ONH}}^L_n$ is isomorphic to the graded matrix superalgebra $M_q(\tilde{\mathcal{H}}^L_n)$ for $\tilde{\mathcal{H}}^L_n$ as in Remark 7.9.
9. Odd Grassmannian bimodules

Throughout the section, \( \ell \) is fixed. We will work always over the ground ring \( R_\ell \), but note that everything in this section also makes sense on base change from \( R_\ell \) to any supercommutative \( R_\ell \)-superalgebra, including the most important case when the ground ring is the field \( \mathbb{F} \). We denote the element of the graded \( R_\ell \)-superalgebra \( OH_{\alpha}^\ell \) formerly denoted by \( a \otimes 1 \) simply by \( a \) \((a \in OSym_n)\). When \( R_\ell \) is specialized to \( \mathbb{F} \), this is consistent with our earlier notation for elements of \( \overline{OH}_{\alpha}^\ell \). Suppose that we are given \( n, d, n' \geq 0 \) with \( n + d + n' = \ell \) and \( \alpha \in \Lambda(k, d) \). The cases \( \alpha = (d) \) and \( \alpha = (1^d) \) will be particularly important. Let

\[
\overline{V}_{\ell, \alpha} := \text{OSym}_{(n, a_1, \ldots, a_k, n')}(OSym_{n} \otimes R_\ell), \quad U_{\ell, \alpha} := R_\ell \otimes_{OSym_n} \text{OSym}_{(n', a_1, \ldots, a_k, n')}. \tag{9.1}
\]

These are graded \( R_\ell \)-supermodules. We refer to \( \overline{V}_{\ell, \alpha} \) and \( U_{\ell, \alpha} \) as odd Grassmannian bimodules. According to the following lemma, they are graded superbimodules over odd equivariant Grassmannian cohomology algebras.

**Lemma 9.1.** Let \( \ell = n + d + n' \) and \( \alpha \in \Lambda(k, d) \) be fixed as above.

1. There is a unique way to make \( \overline{V}_{\ell, \alpha} \) into a graded \((OH_{\alpha}^\ell, OH_{\alpha}^\ell)\)-superbimodule so that the left action of \( \bar{a} \) \((a \in OSym_n)\) is defined by \( \bar{a}(b \otimes 1) := ab \otimes 1 \) for \( b \in OSym_{(n, a_1, \ldots, a_k, n')} \), and the right action of \( \bar{a} \) \((a \in OSym_{n + d})\) is defined by \( (b \otimes 1) \bar{a} := ba \otimes 1 \) for \( b \in OSym_{(n, a_1, \ldots, a_k, n')} \). Moreover:
   
   a. \( \overline{V}_{\ell, \alpha} \) has basis \( \sigma_n(p_{w}^{(n+d)}) \otimes 1 \in \{ w \in [S_{n + d}/S_{(a_2, ..., a_k, n')}]|_{\text{min}} \} \) as a right \( OH_{\alpha}^{n + d} \)-supermodule;
   
   b. \( \overline{V}_{\ell, \alpha} \) has basis \( \sigma_n(p_{w}^{(n+d)}) \otimes 1 \in \{ w \in [S_{n + d}/S_{(a_2, ..., a_k, n')}]|_{\text{min}} \} \) as a left \( OH_{\alpha}^{n + d} \)-supermodule.

2. There is a unique way to make \( U_{\ell, \alpha} \) into a graded \((OH_{\alpha}^{n + d}, OH_{\alpha}^{n + d})\)-superbimodule so that the right action of \( \bar{a} \) \((a \in OSym_n)\) is defined by \( (1 \otimes b) \bar{a} := (-1)^{\text{par}(a)}b \otimes \sigma_{n + d}(a) \) for \( b \in OSym_{(n', a_1, \ldots, a_k, n')} \), and the left action of \( \bar{a} \) \((a \in OSym_{n + d})\) is defined by \( (1 \otimes \sigma_{n + d}(b)) := (-1)^{\text{par}(a)}b \otimes \sigma_{n + d}(a) \) for \( b \in OSym_{(n, a_1, \ldots, a_k, n')} \). Moreover:
   
   a. \( U_{\ell, \alpha} \) has basis \( 1 \otimes \epsilon_{n + d}(p_{w}^{(n+d)}) \in \{ w \in [S_{n + d}/S_{(a_2, ..., a_k, n')}]|_{\text{min}} \} \) as a right \( OH_{\alpha}^{n + d} \)-supermodule;
   
   b. \( U_{\ell, \alpha} \) has basis \( 1 \otimes \epsilon_{n + d}(p_{w}^{(n+d)}) \in \{ w \in [S_{n + d}/S_{(a_2, ..., a_k, n')}]|_{\text{min}} \} \) as a left \( OH_{\alpha}^{n + d} \)-supermodule.

**Proof.**

1. By Theorem 6.5, \( OSym_{(n, a_1, \ldots, a_k, n')} \) is generated by the elements of \( OSym_{(n, a_1, \ldots, a_k)} \) as a right \( OSym_{n} \)-supermodule. Hence, \( \overline{V}_{\ell, \alpha} \) is generated as an \( R_\ell \)-supermodule by elements of the form \( b \otimes 1 \) for \( b \in OSym_{(n, a_1, \ldots, a_k)} \). In view of this, provided that it is well defined, there is a unique way to make \( \overline{V}_{\ell, \alpha} \) into a graded right \( OH_{\alpha}^{n + d} \)-supermodule such that \( (b \otimes 1) \bar{a} = ba \otimes 1 \) for all \( b \in OSym_{(n, a_1, \ldots, a_k)} \) and \( a \in OSym_{n + d} \). It is also clear that the right action of \( OH_{\alpha}^{n + d} \) defined in this way and the left action of \( OH_{\alpha}^{n + d} \) from the statement of the lemma commute with each other, again assuming that both actions are well defined.

To see that the right action is well defined, we have that

\[
\overline{V}_{\ell, \alpha} \simeq OSym_{(n, a_1, \ldots, a_k, n')} \otimes OSym_{(n', a_1, \ldots, a_k, n')} \otimes OSym_{R_\ell}.
\]

By Corollary 6.6(2), we deduce that

\[
\overline{V}_{\ell, \alpha} \simeq \bigoplus_{w \in \{ S_{n + d}/S_{(a_2, ..., a_k, n')} \}_{\text{min}}} \epsilon_{n + d}(p_{w}^{(n+d)}) \otimes (OSym_{(n', a_1, \ldots, a_k, n')} \otimes OSym_{R_\ell})
\]

with each summand being a copy of \( OSym_{(n', a_1, \ldots, a_k, n')} \) shifted in degree and parity. Hence, each of these subspaces is isomorphic to \( OH_{\alpha}^{n + d} \), via the isomorphism from Theorem 7.4(1). The right action of \( OH_{\alpha}^{n + d} \) we are defining is just the natural right action of \( OH_{\alpha}^{n + d} \) on itself transported through these isomorphisms. So it is well defined. We have also proved (1a).
For the left action, we have that
\[ \overline{V}^\ell_{n,\alpha} \cong \text{OSym}_{(n,\alpha_1,\ldots,\alpha_k,\alpha')} \otimes \text{OSym}_{(n',\alpha')} \otimes \text{OSym}_{n',d} \otimes \text{OSym}_n \mathfrak{R}_\ell. \]

By Corollary 6.6(1), we deduce that
\[ \overline{V}^\ell_{n,\alpha} = \bigoplus_{w \in [S_{n'+d}/S_{(n,\alpha_1,\ldots,\alpha_k,\alpha')}]} \sigma_n(p^w_{n'+d}) \otimes (\text{OSym}_{n',d}) \otimes \text{OSym}_n \mathfrak{R}_\ell \]

with each summand being a graded left $\text{OSym}_n \otimes \mathfrak{R}_\ell$-submodule isomorphic to $\text{OSym}_{(n',d)} \otimes \text{OSym}_n \mathfrak{R}_\ell$ (shifted in parity and degree). It remains to apply Theorem 7.4(1) to see that the left action of $\mathfrak{O}H^\ell_n$ is well defined. This also establishes (1b).

(2) This is similar to the proof of (1), using the isomorphism from Theorem 7.4(2) in place of the one from Theorem 7.4(1), and the left supermodule analogs of Theorem 6.5 and Corollary 6.6 obtained by applying $\ast$ to those assertions.

We proceed to develop the properties of odd Grassmannian bimodules in a systematic way. Take $\ell = n + d + n'$ and $\alpha \in \Lambda(k, d)$. There are even degree 0 isomorphisms of graded $\mathfrak{R}_\ell$-supermodules
\[ \ast : \overline{V}^\ell_{n,\alpha} \rightarrow U_{n',\alpha'}, \quad a \otimes \overline{\psi} \mapsto \overline{\psi} \otimes a^*, \quad (9.2) \]

\[ \ast : U_{n,\alpha} \rightarrow \overline{V}^\ell_{n',\alpha}, \quad \overline{\psi} \otimes a \mapsto a^* \otimes \overline{\psi}. \quad (9.3) \]

The first of these with the roles of $n$ and $n'$ switched is the two-sided inverse of the second one.

**Lemma 9.2.** Continue with $\ell = n + d + n'$ and $\alpha \in \Lambda(k, d)$. The isomorphism from (9.2) satisfies
\[ (\bar{a}_1 \bar{v} \bar{a}_2)^\ast = \psi_n(\bar{a}_1)v^\ast \psi_{n+d}(\bar{a}_2) \]

for $a_1 \in \text{OSym}_n$, $a_2 \in \text{OSym}_{n+d}$ and $v \in \overline{V}^\ell_{n,\alpha}$.

**Proof.** We first consider right actions. Take $a \in \text{OSym}_{n+d}$. According to (7.3), we have that \[ \psi_{n+d}(\bar{a}) = \sum_{i=1}^p \bar{a}_i \bar{c}_i, \]
where $a = \sum_{i=1}^p (1)_{n+d+\text{par}(a)} c_{n+d}(a_i) c_i$. We saw in the proof of Lemma 9.1(1) that $\overline{V}^\ell_{n,\alpha}$ is spanned as an $\mathfrak{R}_\ell$-supermodule by vectors of the form $b \otimes 1$ for $b \in \text{OSym}_{(n,\alpha_1,\ldots,\alpha_k,\alpha')}$. So we may assume that $v = b \otimes 1$ for such a $b$. We have that
\[(\bar{v} \bar{a})^\ast = (ba \otimes 1)^\ast = \left( \sum_{i=1}^p (1)_{n+d+\text{par}(a)} b \sigma_{n+d}(a_i) c_i \otimes 1 \right)^\ast = \left( \sum_{i=1}^p (1)_{n+d+\text{par}(b)} \text{par}(a_i) \sigma_{n+d}(a_i^\ast) b \otimes \bar{c}_i \right)^\ast \]
\[= \sum_{i=1}^p (1)_{n+d+\text{par}(a_i)} c_i \otimes b^\ast \sigma_{n+d}(a_i) = (1 \otimes b^\ast) \sum_{i=1}^p \bar{a}_i \bar{c}_i = v^\ast \psi_{n+d}(\bar{a}). \]

For left actions, take $a \in \text{OSym}_n$ with $a = \sum_{i=1}^p (1)_{n+d+\text{par}(a)} \sigma_{n}(a_i) c_i$, so that \[ \psi_n(\bar{a}) = \sum_{i=1}^p \bar{a}_i \bar{c}_i. \] We may assume that $v = \sigma_{n}(b) \otimes 1$ for $b \in \text{OSym}_{(n,\alpha_1,\ldots,\alpha_k,\alpha')}$. Then we have that
\[(\bar{v} \bar{a})^\ast = \sigma_{n}(b) \otimes 1)^\ast = (1)_{n+d+\text{par}(b)} \sigma_{n}(ba \otimes 1)^\ast = \left( \sum_{i=1}^p (1)_{n+d+\text{par}(a_i) + n \text{par}(a)} \sigma_{n}(ba_i^\ast) \otimes \bar{c}_i \right)^\ast \]
\[= \sum_{i=1}^p (1)_{n+d+\text{par}(a_i)} c_i \otimes \sigma_{n}(a_i b^\ast) = \left( \sum_{i=1}^p a_i \bar{c}_i \right) \cdot (1 \otimes \sigma_{n}(b^\ast)) = \psi_n(\bar{a})v^\ast. \]

□
Assuming still that \( \ell = n + d + n' \) and \( \alpha \in \Lambda(k, d) \), we next introduce some special notation for elements of \( \overline{V}_{n, \alpha}^{\ell} \) and \( U_{\alpha, n}^{\ell} \). Recall that \( \alpha^s \) denotes the reversed composition \((\alpha_k, \ldots, \alpha_1)\), and note that the involution \( \epsilon_n : O\text{Sym}_d \to O\text{Sym}_d \) interchanges the subalgebras \( O\text{Sym}_\alpha \) and \( O\text{Sym}_{\alpha^s} \). For \( f \in O\text{Sym}_\alpha \) and \( g \in O\text{Sym}_{\alpha^s} \), we let

\[
\overline{v}_{n, \alpha}(f) := \sigma_n(f) \otimes 1 \in \overline{V}_{n, \alpha}^{\ell}, \quad u_{\alpha, n}(g) := (-1)^{n^\text{par}(g)} \otimes \sigma_n(\epsilon_n(g)) \in U_{\alpha, n}^{\ell}.
\]  

(9.4) The isomorphisms \( * \) from (9.2) and (9.3) satisfy

\[
\overline{v}_{n, \alpha}(f)^* = (-1)^{n^\text{par}(f)} u_{\alpha, n}(\epsilon_n(g)^*), \quad u_{\alpha, n}(g)^* = (-1)^{n^\text{par}(g)} \overline{v}_{n, \alpha}(\epsilon_n(g)^*). \quad (9.5)
\]

**Lemma 9.3.** Suppose that \( \ell = n + d + n' \) and \( \alpha \in \Lambda(k, d) \). The supermodule \( \overline{V}_{n, \alpha}^{\ell} \) is generated either as a left \( OH_n^{\ell} \)-supermodule or as a right \( OH_n^{\ell+\alpha^s} \)-supermodule by vectors of the form \( \overline{v}_{n, \alpha}(f) \) \( (f \in O\text{Sym}_\alpha) \). Similarly, the supermodule \( U_{\alpha, n}^{\ell} \) is generated either as a right \( OH_n^{\ell^s} \)-supermodule or as a left \( OH_n^{\ell^s+\alpha} \)-supermodule by vectors of the form \( u_{\alpha, n}(g) \) \( (g \in O\text{Sym}_{\alpha^s}) \).

**Proof.** The fact that \( \overline{V}_{n, \alpha}^{\ell} \) is generated by the vectors \( \overline{v}_{n, \alpha}(f) \) either as a right \( OH_n^{\ell+\alpha^s} \)-supermodule or as a left \( OH_n^{\ell^s} \)-supermodule follows from Lemma 9.1(1a)–(1b), since the basis vectors there are of this form; cf. Corollary 6.6. Similarly, \( U_{\alpha, n}^{\ell} \) is generated by the vectors \( u_{\alpha, n}(g) \) as a left \( OH_n^{\ell^s} \)-supermodule or right \( OH_n^{\ell^s+\alpha} \)-supermodule by Lemma 9.1(2a)–(2b). \( \square \)

The following lemma gives “Schur bases” for \( \overline{V}_{n, \alpha}^{\ell} \) and \( U_{\alpha, n}^{\ell} \) when \( \alpha \) has just one part.

**Lemma 9.4.** Suppose that \( \ell = n + d + n' \).

1. The supermodule \( \overline{V}_{n,d}^{\ell} \) has basis \( \{ \overline{v}_{n,d}(\sigma_n(\lambda)) \mid \lambda \in \Lambda_{\text{dual}}^+ \} \) as a free left \( OH_n^{\ell} \)-supermodule, and basis \( \{ \overline{v}_{n,d}(\sigma_n(\lambda^s)) \mid \lambda \in \Lambda_{\text{dual}}^+ \} \) as a right \( OH_n^{\ell^s+\alpha} \)-supermodule.

2. The supermodule \( U_{d,n}^{\ell} \) has basis \( \{ u_{d,n}(s_{\lambda}) \mid \lambda \in \Lambda_{\text{dual}}^+ \} \) as a free left \( OH_n^{\ell^s} \)-supermodule, and basis \( \{ u_{d,n}(s_{\lambda}) \mid \lambda \in \Lambda_{\text{dual}}^+ \} \) as a right \( OH_n^{\ell^s} \)-supermodule.

**Proof.** The existence of the Schur bases for \( \overline{V}_{n,d}^{\ell} \) follow in the same way as the bases in Lemma 9.1(1) were constructed, using Corollary 6.9 in place of Corollary 6.6. Then, in view of Lemma 9.2, the existence of the Schur bases for \( U_{d,n}^{\ell} \) can be deduced by applying \( * \) to the ones for \( \overline{V}_{n,d}^{\ell} \), using also (9.5) and (6.5). \( \square \)

Later on, we will also need the following, which gives a little more information about the basis for \( \overline{V}_{n,d}^{\ell} \) as a right \( OH_n^{\ell^s+\alpha} \)-supermodule constructed in Lemma 9.4(1). Applying \( * \), one also obtains an analogous result about the basis for \( U_{d,n}^{\ell} \) as a left \( OH_n^{\ell^s+\alpha^s} \)-supermodule from Lemma 9.4(2).

**Lemma 9.5.** For \( \ell = n + d + n' \), \( \lambda \in \Lambda^+ \) and \( f \in O\text{Sym}_d \), we have that

\[
\overline{v}_{n,d}(f(s_{\lambda}^{(d)}))^{*} \otimes 1 = (-1)^{N(E(\lambda)+\lambda^s)\lambda^s} s_{\lambda}^{(d)} \overline{v}_{n,d}(f) \otimes 1
\]

in \( \overline{V}_{n,d}^{\ell} \otimes OH_{n,d}^{\ell^s} \). Hence, we have that \( \overline{v}_{n,d}(f(s_{\lambda}^{(d)}))^{*} \otimes 1 = 0 \) unless \( \lambda \in \Lambda^+_{\text{dual}} \). The remaining \( \overline{v}_{n,d}(f(s_{\lambda}^{(d)}))^{*} \otimes 1 \) for \( \lambda \in \Lambda^+_{\text{dual}} \) give a linear basis for \( \overline{V}_{n,d}^{\ell} \otimes OH_{n,d}^{\ell^s} \).

**Proof.** In \( O\text{Sym}_{n,d} \otimes O\text{Sym}_{n,d} \) we have that \( \sigma_n(f(s_{\lambda}^{(d)}))^{*} \otimes 1 = (-1)^{N(E(\lambda)+\lambda^s)\lambda^s} s_{\lambda}^{(d)} \otimes 1 \) thanks to Corollary 6.12. Multiplying this identity on the left by \( \sigma_n(f) \) for \( f \in O\text{Sym}_d \) gives that

\[
\sigma_n(f)^{*} \otimes 1 = (-1)^{N(E(\lambda)+\lambda^s)\lambda^s} s_{\lambda}^{(d)} \otimes 1
\]

This implies that \( \overline{v}_n(f(s_{\lambda}^{(d)}))^{*} \otimes 1 = (-1)^{N(E(\lambda)+\lambda^s)\lambda^s} \overline{v}_n(f) \otimes 1 \).
If \( \lambda \notin \Lambda_{d+n}^+ \) we either have that \( \lambda_1 > n \), in which case \( s_1^{(n)} = 0 \), or \( \text{ht}(\lambda) > d \), in which case \( s_1^{(d)} = 0 \). Hence, \( \bar{V}_{d,n}((s_1^{(d)})^+) \otimes 1 = 0 \). The final assertion that the remaining vectors form a linear basis is immediate from Lemma 9.4(1).

Next, we show that all \( \bar{V}^f_{n,\alpha} \) and \( U^f_{\alpha,n} \) can be obtained by taking tensor products of ones of the form \( \bar{V}^f_{n,\alpha} \) and \( U^f_{\alpha,n} \).

**Lemma 9.6.** Let \( \ell = n + d + d' + n' \), \( \alpha \in \Lambda(k, d) \) and \( \alpha' \in \Lambda(k', d') \).

1. There is a unique isomorphism of graded \((OH_n^\ell, OH_{n+d+d'}^f)\)-superbimodules such that
   \[
   \bar{c}_{\alpha,\alpha'} : \bar{V}^f_{n,\alpha} \otimes_{OH^f_{n+d}} \bar{V}^f_{n+d,\alpha'} \rightarrow (OH^f_{n+d+d'})_{\ell,\alpha',\alpha'}(f \sigma_d(f')) \rightarrow \bar{V}_{n,\alpha}(f) \otimes \bar{V}_{n+d,\alpha'}(f')
   \] (9.6)
   for \( f \in OSym_\alpha, f' \in OSym_{\alpha'} \).

2. Writing \( * \) for the appropriate isomorphisms from (9.2) and (9.3), the composition \((* \otimes *) \circ \bar{c}_{\alpha,\alpha'} \circ * \)
   is an isomorphism of graded \((OH^f_{n+d+d'}, OH^f_n)_\alpha \)-superbimodules
   \[
   c_{\alpha,\alpha'} : U^f_{\alpha,n} \otimes_{OH^f_{n+d}} U^f_{\alpha,n+d} \rightarrow \mathcal{U}_{\alpha,\alpha'}(f) \rightarrow (-1)^{\text{par}(f)} \mathcal{U}_{\alpha,\alpha'}(f) \otimes \mathcal{U}_{\alpha,\alpha'}(f')
   \] (9.7)
   for \( f \in OSym_\alpha, f' \in OSym_{\alpha'} \).

**Proof.** (1) Let \( \gamma := (n, a_1, \ldots, a_k, a'_1, \ldots, a'_{k'}, n') \). Consider the surjective \( \mathbb{F} \)-linear map

\[ OSym_\gamma : \bar{V}^f_{n,\alpha} \otimes_{OH^f_{n+d}} \bar{V}^f_{n+d,\alpha'} \rightarrow \mathcal{U}_{\alpha,\alpha'} \] for \( b_1 \in OSym(a_1, \ldots, a_k), b_2 \in OSym(a'_1, \ldots, a'_{k'}, n') \). It is easy to check that it is a right \( OSym_\gamma \)-supermodule homomorphism, so it induces an \( R_\ell \)-supermodule homomorphism \( \bar{c}_{\alpha,\alpha'} : OSym_\gamma \rightarrow \bar{V}^f_{n,\alpha} \otimes_{OH^f_{n+d}} \bar{V}^f_{n+d,\alpha'} \). This is the map from the statement of the lemma. The domain and range are free graded \( R_\ell \)-supermodules, so to see that \( \bar{c}_{\alpha,\alpha'} \) is an isomorphism it suffices to check that they have the same graded ranks. By Lemma 9.1(1a) and (3.8), \( \bar{V}^f_{n,\alpha} \) is a free graded right \( OH^f_{n+d} \)-supermodule of graded rank \( q^{N(\alpha) + nd}_{(n, a_1, \ldots, a_k)} \). By Lemma 9.1(1b) and (3.8), \( \bar{V}^f_{n+d,\alpha'} \) is a free graded left \( OH^f_{n+d} \)-supermodule of graded rank \( q^{N(\alpha') + n'd'}_{(a'_1, \ldots, a'_{k'}, n')} \). By Corollary 7.6, \( OH^f_{n+d} \) is a free graded \( R_\ell \)-supermodule of graded rank \( q^{(n+d)(n'+d')}_{n+d} \). Multiplying these together and using the identity

\[ N(\gamma) = N(\alpha) + N(\alpha') + nd + n'd' + (n + d)(n' + d') \]
gives that \( \bar{V}^f_{n,\alpha} \otimes_{OH^f_{n+d}} \bar{V}^f_{n+d,\alpha'} \) is a free graded \( R_\ell \)-supermodule of graded rank \( q^{N(\gamma)}_{(\gamma)} \). This is also the graded rank of \( \bar{V}^f_{n,\alpha} \otimes_{OH^f_{n+d}} \bar{V}^f_{n+d,\alpha'} \) as a free graded \( R_\ell \)-supermodule, as follows from Theorem 6.5 and (6.21).

We still need to show that \( \bar{c}_{\alpha,\alpha'} \) is a graded \((OH^f_n, OH^f_{n+d+d'})\)-superbimodule homomorphism. We just go through the details for the right action whose definition is slightly more complicated than the left action. We restrict to considering just to vectors \( b_1 \sigma_{n+d}(b_2) \otimes 1 \in \bar{V}^f_{n,\alpha,\lambda'} \), for \( b_1 \in OSym(a_1, \ldots, a_k) \) and \( b_2 \in OSym(a'_1, \ldots, a'_{k'}) \). We can do this because these vectors generate \( \bar{V}^f_{n,\alpha,\lambda'} \) as an \( R_\ell \)-supermodule. Then we take \( \alpha \in OSym_{n+d+d'} \), write it as \( \alpha = \sum_{i=1}^p a_i \sigma_{n+d}(a''_i) \) for \( a_i' \in OSym(a_1, \ldots, a_k) \) and \( a''_i \in OSym(a'_1, \ldots, a'_{k'}) \), and calculate:

\[
\bar{c}_{\alpha,\alpha'}((b_1 \sigma_{n+d}(b_2) \otimes 1) \bar{a}) = \sum_{i=1}^p \bar{c}_{\alpha,\alpha'}(b_1 \sigma_{n+d}(b_2) a'_i \sigma_{n+d}(a''_i) \otimes 1) = \sum_{i=1}^p (-1)^{\text{par}(a'_{i})} \bar{c}_{\alpha,\alpha'}(b_1 a'_i \sigma_{n+d}(b_2 a''_i) \otimes 1)
\]
\[ = \sum_{i=1}^{p} (-1)^{\text{par}(b_{2i}) \text{par}(a_i')} (b_{1i} a'_i \otimes 1) \otimes (\sigma_{n+d}(b_2) \sigma_{n+d}(a_i'') \otimes 1), \]

\[ \tilde{c}_{\alpha,\alpha'} (b_1 \sigma_{n+d}(b_2) \otimes 1) \tilde{a} = ((b_1 \otimes 1) \otimes (\sigma_{n+d}(b_2) \otimes 1)) \tilde{a} = \sum_{i=1}^{p} (b_{1i} \otimes 1) \otimes (\sigma_{n+d}(b_2) a'_i \sigma_{n+d}(a_i'') \otimes 1) \]

\[ = \sum_{i=1}^{p} (-1)^{\text{par}(b_{2i}) \text{par}(a_i')} (b_{1i} \otimes 1) \tilde{a}'_i (\sigma_{n+d}(b_2) \sigma_{n+d}(a_i'') \otimes 1) \]

\[ = \sum_{i=1}^{p} (-1)^{\text{par}(b_{2i}) \text{par}(a_i')} (b_{1i} \otimes 1) \tilde{a}_i' \otimes (\sigma_{n+d}(b_2) \sigma_{n+d}(a_i'') \otimes 1) \]

\[ = \sum_{i=1}^{p} (-1)^{\text{par}(b_{2i}) \text{par}(a_i')} (-1)^{\text{par}(b_2) \text{par}(a_i')} (b_{1i} a'_i \otimes 1) \otimes (\sigma_{n+d}(b_2) \sigma_{n+d}(a_i'') \otimes 1). \]

These are equal so \( \tilde{c}_{\alpha,\alpha'} \) is a right \( OH_n \)-supermodule homomorphism.

(2) This follows from (1) and Lemma 9.2. We just go through the explicit computation of the composition \( * \circ \tilde{c}_{\alpha,\alpha'} \circ (\ast \otimes * \ast) \). The appropriate diagram is

\[
\begin{array}{ccc}
U^\ell_{n':\ell,\alpha} & \xrightarrow{\tilde{c}_{\alpha,\alpha'}} & U^\ell_{n':\ell} \otimes OH^\ell_n \otimes U^\ell_{\alpha,\ell} \\
\downarrow * & & \uparrow * \otimes * \\
\tilde{V}^\ell_{n':\ell,\alpha} & \xrightarrow{\tilde{c}_{\alpha,\alpha'}} & \tilde{V}^\ell_{n':\ell} \otimes OH^\ell_n \otimes \tilde{V}^\ell_{\alpha,\ell} \\
\end{array}
\]

Take \( f \in OSym_{\alpha'} \), \( f' \in OSym_{\alpha'} \) so that \( \sigma_d(f') = ( -1)^{\text{par}(f') \text{par}(f')} \sigma_d(f') \in OSym_{\alpha'} \). We have that

\[
\tilde{c}_{\alpha',\alpha} \left( u_{\alpha':\ell,\alpha} (\sigma_d(f') f) \right) = ( -1)^{\text{par}(f') \text{par}(f')} \sigma_d(f') \left( \tilde{c}_{\alpha':\alpha} \left( u_{\alpha':\ell,\alpha} \left( \epsilon_d(f', f) \right) \right) \right) \]

which is the formula for \( \tilde{c}_{\alpha',\alpha} \left( u_{\alpha':\ell,\alpha} (\sigma_d(f') f) \right) \) from the statement of the lemma. \( \square \)

From this point onwards, we will denote \( \tilde{V}^\ell_{n(1)} \) and \( U^\ell_{(1)\alpha} \) simply by \( \tilde{V}^\ell_n \) and \( U^\ell_n \), respectively. Thus, \( U^\ell_n \) is a graded \( (OH_{n+1}, OH_n) \)-superbimodule, and \( \tilde{V}^\ell_n \) is a graded \( (OH_n, OH_{n+1}) \)-superbimodule. For further motivation for the significance of these, see Corollary 9.12 below. We denote the generators \( \tilde{v}_{n(1)}(f) \) and \( u_{n(1),\alpha}(g) \) of \( \tilde{V}^\ell_n \) and \( U^\ell_n \) from (9.4) simply by \( \tilde{v}_n(f) \) and \( u_n(g) \) for \( f, g \in OSym_1 \). It is also convenient to write simply \( x \) in place of \( x_1 \) when working in rank 1, i.e., we identify \( OSym_1 \) with the graded polynomial superalgebra \( \mathbb{F}[x] \) generated by the variable \( x \) that is odd of degree 2 so that \( x_1 \in OSym_1 \) is identified with \( x \in \mathbb{F}[x] \). So, for \( f(x) \in \mathbb{F}[x] \), we have that

\[
\tilde{v}_n(f(x)) = f(x_1) \otimes 1 \in \tilde{V}^\ell_n, \quad u_n(f(x)) = (-1)^{x \text{par}(f)} 1 \otimes f(x_1+1) \in U^\ell_n \quad (9.8)
\]

for \( n' \) defined from \( \ell = n + 1 + n' \). Applying a sequence of the isomorphisms from Lemma 9.6 in any way that makes sense, we obtain canonical even degree 0 isomorphisms

\[
\tilde{c}_{(1)\alpha'} : \tilde{V}^\ell_{n(1)1} \xrightarrow{\tilde{c}_{(1)\alpha'}} \tilde{V}^\ell_{n+1} \otimes OH^\ell_{n+2} \otimes OH^\ell_{n+3} \cdots \otimes OH^\ell_{n+d-1} \quad (9.9)
\]
Lemma 9.7. Suppose that the right action of \( \tilde{\nu}_n^{(d)_1} (x^{k_1} \cdots x^{k_d}) \mapsto \tilde{\nu}_n (x^{k_1}) \otimes \cdots \otimes \tilde{\nu}_{n+d-1} (x^{k_d}) \) of \((OH_n^{d}, OH^{d}_{n+d})\)-superbimodules, and

\[
\begin{align*}
& c'_1 : U^{d}_{(1),n} \to U^{d}_{n+d-1} \otimes OH^{d}_{n+d-1} \cdots \otimes OH^{d}_{n+2} U^{d}_{n+1} \otimes OH^{d}_{n+1} U^{d}_n \\
& u_{(1),n} (x^{k_1} \cdots x^{k_1}) \mapsto (-1)^{\sum_{i=1}^{d} (d-1) n_i} u_{n+d-1} (x^{k_d}) \otimes \cdots \otimes u_n (x^{k_1})
\end{align*}
\]

of \((OH^{d}_{n+d}, OH^{d}_n)\)-superbimodules.

\textbf{Lemma 9.7.} Suppose that \( \ell = n + d + n' \).

1. There is a left action of \( ONH_d \) on \( \overline{V}^{(d)}_{n(d_1)} \) making it into an \((OH_n^{d} \otimes ONH_d, OH^{d}_n)\)-superbimodule so that \( a \cdot \overline{V}^{(d)}_{n(d_1)} (f) = (-1)^{(n+d-1)\text{par}(a)} \overline{V}^{(d)}_{n(d_1)} (a \cdot f) \) for \( a \in ONH_d \) and \( f \in \text{OPol}_d \). Moreover, the inclusion \( OSym_d \hookrightarrow \text{OPol}_d \) induces an isomorphism \( \overline{V}^{(d)}_{n(d_1)} \to (\omega \chi)_d \cdot \overline{V}^{(d)}_{n(d_1)} \) of \((OH^{d}_n, OH^{d}_{n+d})\)-superbimodules taking \( \overline{V}^{(d)}_{n(d_1)} (f) \to \overline{V}^{(d)}_{n(d_1)} (f) \) for \( f \in OSym_d \). Hence, we have that

\[
\overline{V}^{(d)}_{n(d_1)} \simeq \bigoplus_{w \in S_d} (\Pi Q^2)^{(\ell)(w)} \overline{V}^{(d)}_{n(d_1)}.
\]

2. There is a right action of \( ONH_d \) on \( U^{d}_{(1),n} \) making it an \((OH^{d}_{n+d}, OH^{d}_n \otimes ONH_d)\)-superbimodule so that \( u_{(1),n} (f) \cdot a = (-1)^{(d-1)\text{par}(a)} u_{(1),n} (\cdot a ) \cdot f \) for \( a \in ONH_d \) and \( f \in \text{OPol}_d \); the right action of \( ONH_d \) on \( \text{OPol}_d \) being used here is the one from (5.34). Moreover, the inclusion \( OSym_d \hookrightarrow \text{OPol}_d \) induces an isomorphism \( U^{d}_{(d),n} \to U^{d}_{(1),n} \cdot (\chi \omega)_d \) taking \( u_{(d),n} (f) \to u_{(1),n} (f) \) for \( f \in OSym_n \). Hence, we have that

\[
U^{d}_{(1),n} \simeq \bigoplus_{w \in S_d} (\Pi Q^2)^{(\ell)(w)} U^{d}_{(d),n}.
\]

3. The actions in (1) and (2) are related under the isomorphism (9.2) by

\[
(a \cdot \overline{V}^{(d)}_{n(d_1)} (f))^* = (-1)^{\text{par}(a) \cdot \text{par}(f)} \overline{V}^{(d)}_{n(d_1)} (f)^* \cdot \varepsilon_d (a)^*.
\]

\textbf{Proof.} (1) Since we have defined \( \overline{V}^{(d)}_{n(d_1)} \) to be \( OSym_{n(d_1),n'} \otimes OSym_d R_d \) and \( OSym_{n(d_1),n'} = OSym_n \otimes \text{OPol}_d \otimes OSym_{d'} \), the left action of \( ONH_d \) on \( \text{OPol}_d \) from Section 5 composed with \( p^{d+d-1} : ONH_d \to ONH_d \) induces a left action on \( \overline{V}^{(d)}_{n(d_1)} \). This supercommutes with the left action of \( OH^{d}_n \) and commutes with the right action of \( OH^{d}_{n+d} \), so it makes \( \overline{V}^{(d)}_{n(d_1)} \) into an \((OH^{d}_n \otimes ONH_d, OH^{d}_{n+d})\)-superbimodule. Since \( OSym_d = (\omega \chi)_d \cdot \text{OPol}_d \), the inclusion \( OSym_d \hookrightarrow \text{OPol}_d \) induce an isomorphism \( \overline{V}^{(d)}_{n(d_1)} \to (\omega \chi)_d \cdot \overline{V}^{(d)}_{n(d_1)} \). The decomposition (9.11) follows from Theorem 5.3.

(2) This is similar, starting from the right action of \( ONH_d \) on \( \text{OPol}_d \) discussed at the end of Section 5 composed with \( p^{d-1} \). The last assertions in (2) follow because \( OSym_d = \text{OPol}_d \cdot (\chi \omega)_d \), and (9.12) follows from (5.36).

(3) This is easy to check directly using (9.5).

In view of Lemmas 9.6 and 9.7, all of the odd Grassmannian bimodules \( \overline{V}^{(d)}_{n,n} \) and \( U^{d}_{d,n} \) are isomorphic to 1-morphisms in the 2-supercategory \( OG\text{Bim}_d \) introduced in the next important definition.
Definition 9.8. The category $OGBim_\ell$ of odd Grassmannian bimodules is the full additive graded sub-
$(Q, \Pi)$-2-supercategory of the (weak) graded $(Q, \Pi)$-2-supercategory of graded superalgebras, graded
superbimodules and superbimodule homomorphisms consisting of objects that are the graded super-
algebras $OH_n^{-\ell}$ for $0 \leq n \leq \ell$ plus a distinguished object that is a trivial graded superalgebra, and
1-morphisms that generated by the odd Grassmannian bimodules $V_n^{\ell}$ and $U_n^{\ell}$ for $0 \leq n < \ell$.

Remark 9.9. Although not a strict 2-supercategory, we often work with $OGBim_\ell$ as though it was in
fact strict. More formally, when we do this, we are identifying $OGBim_\ell$ with a subcategory of the
strict graded 2-supercategory of graded superalgebras, graded superfunctors and graded supernatu-
ral transformations so that the graded superbimodule $OH_n^{\ell}$ is identified with the graded supercategory
$OH_n^{\ell}-\text{gsMod}$, a graded $(OH_n^{\ell}, OH_{n+1}^{\ell})$-superbimodule $M$ in $OGBim_\ell$ is identified with the superfunctor
$M \otimes_{OH_n} - : OH_n^{\ell}-\text{gsMod} \to OH_{n+1}^{\ell}-\text{gsMod}$, and a graded superbimodule endomorphism $f : M \to M'$ be-
tween two such superbimodules is identified with the supernatural transformation $f \otimes \text{id} : M \otimes_{OH_n} - \to M' \otimes_{OH_n} -$.

The next lemma gives an explicit presentation for the most important generating odd Grassmannian
bimodules $V_n^{\ell}$ and $U_n^{\ell}$. In formulating the result, we also incorporate an indeterminate $t$ into our notat-
ion, working in the $R_t^\ell(t^{-1})$-supermodules $V_n^\ell(t^{-1})$ and $U_n^\ell(t^{-1})$, this being a natural extension of the
generating function formalism developed already for odd symmetric functions. Recall also the generat-
ing functions (4.42), whose image in $OH_n^{\ell}$ are $\bar{\varepsilon}(n)(t)$ and $\bar{\gamma}(n)(t)$. The following lemma is the first place
that the significance of these signed versions of odd elementary and complete symmetric polynomials
becomes apparent.

Lemma 9.10. Suppose that $\ell = n + 1 + n'$.

1. Let $V := OH_n^{\ell} \otimes_{R_t^\ell} R_t[x] \otimes_{R_t^\ell} OH_{n+1}^{\ell}$, which is the free graded $(OH_n^{\ell}, OH_{n+1}^{\ell})$-superbimodule on
the graded $R_t^\ell$-supermodule $R_t[x]$. For $f(x) \in \mathbb{F}[x] \subseteq R_t[x]$, we denote $1 \otimes f(x) \otimes 1 \in V$ by
$\bar{v}(f(x))$. Let $S$ be the sub-bimodule of $V$ generated by the relations\(^5\)
$$\bar{v}(f(x))\bar{\varepsilon}(n+1)(t) = \begin{cases} \bar{v}(t + x)^{-1} f(x)\bar{\gamma}(n+1)(t) & \text{if } f \text{ is even} \\ \bar{v}(t + x)^{-1} f(x)\bar{\gamma}(n+1)(t) & \text{if } f \text{ is odd} \end{cases}$$
for $f(x) \in \mathbb{F}[x]$ that is either even or odd. Equivalently, $S$ is the sub-bimodule generated by the
relations
$$\bar{\varepsilon}(n)(t)\bar{v}(f(x)) = \begin{cases} \bar{v}(t + x)^{-1} f(x)\bar{\gamma}(n+1)(t) & \text{if } f \text{ is even} \\ \bar{v}(t + x)^{-1} f(x)\bar{\gamma}(n+1)(t) & \text{if } f \text{ is odd} \end{cases}$$
There is an isomorphism of graded $(OH_n^{\ell}, OH_{n+1}^{\ell})$-superbimodules
$$V / S \sim \bar{V}_n^{\ell}, \quad \bar{v}(f(x)) + S \mapsto \bar{v}_n(f(x)).$$
Moreover:
(a) $\bar{V}_n^{\ell}$ is free as a graded left $OH_{n+1}^{\ell}$-supermodule with basis $\{\bar{v}_n(x') | 0 \leq r \leq n'\}$;
(b) $\bar{V}_n^{\ell}$ is free as a graded right $OH_{n+1}^{\ell}$-supermodule with basis $\{\bar{v}_n(x') | 0 \leq r \leq n\}$;
(c) the vector $\bar{v}_n(1)$ generates $\bar{V}_n^{\ell}$ as a graded $(OH_n^{\ell}, OH_{n+1}^{\ell})$-superbimodule.

2. Let $U := OH_{n+1}^{\ell} \otimes_{R_t^\ell} R_t[x] \otimes_{R_t^\ell} OH_n^{\ell}$, which is the free graded $(OH_{n+1}^{\ell}, OH_n^{\ell})$-superbimodule on
the graded $R_t^\ell$-supermodule $R_t[x]$. For $f \in \mathbb{F}[x] \subseteq R_t[x]$, we denote $1 \otimes f \otimes 1 \in U$ by $u(f)$. Let $T$ be the sub-bimodule of $U$ generated by either of the following equivalent relations:
$$\bar{\varepsilon}(n+1)(t)u(f(x)) = \begin{cases} \bar{v}(t + x)^{-1} f(x)\bar{\gamma}(n+1)(t) & \text{if } f \text{ is even} \\ u(t + x)^{-1} f(x)\bar{\varepsilon}(n+1)(t) & \text{if } f \text{ is odd} \end{cases}$$
\(^5\)We mean the relations obtained by equating coefficients of powers of $t$ on both sides.
\[ u(f(x))\tilde{\varepsilon}^{(n)}(t) = \begin{cases} 
(\pm 1)^{\nu+1} \tilde{\varepsilon}^{(n+1)}(-t)u((t-x)^{-1}f(x)) & \text{if } f \text{ is even} \\
\tilde{\varepsilon}^{(n+1)}(t)u((t-x)^{-1}f(x)) & \text{if } f \text{ is odd} 
\end{cases} \tag{9.18} \]

There is an isomorphism of graded \((OH^t_{n+1}, OH^t_n)\)-superbimodules

\[ U / T \xrightarrow{\sim} U^t_n, \quad u(f(x)) + T \mapsto u_n(f(x)). \tag{9.19} \]

Moreover:
(a) \( U^t_n \) is free as a right \( OH^t_n \)-supermodule with basis \( \{ u_n(x^r) \mid 0 \leq r \leq n^t \} \);
(b) \( U^t_n \) is free as a graded left \( OH^t_{n+1} \)-supermodule with basis \( \{ u_n(x^r) \mid 0 \leq r \leq n \} \);
(c) the vector \( u_n(1) \) generates \( U^t_n \) as a graded \((OH^t_{n+1}, OH^t_n)\)-superbimodule.

**Proof.** Note to start with that (1a)–(1b) and (2a)–(2b) follow immediately from Lemma 9.4.

(1) We first check that the images of the relations derived both from (9.14) and (9.15) hold for the actions of \( OH^t_n \) and \( OH^t_{n+1} \) on \( \tilde{V}^t_n \). For (9.14), it suffices to check just that \( \tilde{v}_n(1)\tilde{\varepsilon}^{(n+1)}(t) = \tilde{\varepsilon}^{(n)}(t)\tilde{v}_n(t + x) \), for then we can act on the left with \( f(x) \in ONH_1 \) using Lemma 9.7 to deduce the more general formulæ.

By the definition of the actions and (4.44), both sides of this are equal to \( (t + x_1) \cdots (t + x_n)(t + x_{n+1}) \otimes 1 \in OSym_{(n,1,n')} \otimes OSym_1 R_t \), so this is clear. Similarly, for (9.15), it suffices to check just that \( \tilde{\varepsilon}^{(n)}(t)\tilde{v}_n(1) = \tilde{v}_n(t + x)^{-1}\tilde{\varepsilon}^{(n+1)}(t) \), and this follows because both sides equal \( (t + x_1) \cdots (t + x_1) \otimes 1 \) thanks to (4.45).

In this paragraph,\(^6\) we explain the equivalence of the relations (9.14) and (9.15). This is just a formal manipulation with power series. We found it to be convenient to go via the following intermediate set of relations:

\[ \tilde{\varepsilon}^{(n)}(t)\tilde{v}(f(x)) = \begin{cases} 
\tilde{v}(t(2 + x)^{-1}f(x))\tilde{\varepsilon}^{(n+1)}(t) + (-1)^n\tilde{v}(x(2 + x)^{-1}f(x))\tilde{\varepsilon}^{(n+1)}(-t) & \text{if } f \text{ is even} \\
(-1)^{n+1}\tilde{v}(t(2 + x)^{-1}f(x))\tilde{\varepsilon}^{(n+1)}(t) + \tilde{v}(x(2 + x)^{-1}f(x))\tilde{\varepsilon}^{(n+1)}(-t) & \text{if } f \text{ is odd}. 
\end{cases} \tag{9.20} \]

To go from (9.20) to (9.14), one looks at \( \tilde{\varepsilon}^{(n)}(t)\tilde{v}_n(f(x) + xf(x)) \) if \( f \) is even or \( -1)^n\tilde{\varepsilon}^{(n)}(-t)\tilde{v}_n(f(x) + xf(x)) \) if \( f \) is odd. After using (9.20) to commute the \( \tilde{\varepsilon}^{(n)}(t) \) to the right hand side, terms cancel leaving the desired \( \tilde{v}_n(f(x))\tilde{\varepsilon}^{(n+1)}(t) \). The reverse implication is similar, starting from the right hand side of (9.20) relations and simplifying using (9.14) to get the left hand side. To establish the equivalence of (9.20) and (9.15), we use the identities

\[ \varepsilon^{(n)}(t) = \frac{1 - it^{n+1} \varepsilon^{(n)}(it)}{1 - it}, \quad \varepsilon^{(n)}(-t) = \frac{1 - it^{n+1} \varepsilon^{(n)}(-it)}{1 + it} \tag{9.21} \]

where \( i \in \mathbb{F} \) denotes a square root of \(-1\). These may be verified by equating coefficients on each side. Then to calculate \( \varepsilon^{(n)}(t)\tilde{v}(f(x)) \), we replace \( \varepsilon^{(n)}(t) \) with this linear combination of \( \varepsilon^{(n)}(\pm it) \), then use (9.20) with \( t \) replaced by \( \pm it \) to commute \( \varepsilon^{(n)}(\pm it) \) to the right. At the end, a linear factor in the denominator \( t^2 - x^2 = (t - x)(t + x) \) cancels, and after that one converts back to \( \varepsilon^{(n)}(\pm t) \) using (9.21) again (with \( n \) replaced by \( n + 1 \)). This calculation is quite lengthy but elementary. In this way, we obtain (9.15) from (9.20). The argument can be reversed to obtain (9.20) from (9.15), hence these two sets of relations are equivalent.

The relations check made in the first paragraph of the proof implies that there is a well-defined graded superbimodule homomorphism \( V / S \to \tilde{V}^t_n \) taking \( \tilde{v}(f(x)) + S \) to \( \tilde{v}_n(f(x)) \) for all \( f(x) \in \mathbb{F}[x] \). Moreover, this is obviously surjective. To show that it is an isomorphism, it suffices by the first paragraph of the proof to show that \( V / S \) is generated as a right \( OH^t_{n+1} \)-module by the vectors \( \tilde{v}(1) + S, \tilde{v}(x) + S, \ldots, \tilde{v}(x^n) + S \). It is generated as a bimodule by all \( \tilde{v}(x^r) + S \) \( r \geq 0 \). The relation (9.15) shows that any \( \tilde{v}(g(x)) \) \( (a \in OSym_n, f(x) \in \mathbb{F}[x]) \) can be expanded as a linear combination of vectors of the form \( \tilde{v}(g(x)) \otimes b \) \( (b \in OSym_{n+1}, g(x) \in \mathbb{F}[x]) \). Hence, \( V \) is generated just as a right \( OH^t_{n+1} \)-module by the vectors

\(^6\)This is more of a curiosity and is in fact never needed subsequently!
\( \tilde{v}(x^r) + S \) \((r \geq 0)\). Now we show by induction on \( r = 0, 1, 2, \ldots \) that \( \tilde{v}(x^{n+r}) + S \) lies in the right \( OH^\ell_{n+1} \)-submodule \( V' \) of \( V \) generated by the vectors \( \tilde{v}(1) + S, \tilde{v}(x) + S, \ldots, \tilde{v}(x^r) + S \). The base \( r = 0 \) is vacuous. For the induction step, take \( r > 0 \). Consider the relation arising from the \( t^{-1}\)-coefficients in (9.15) taking \( f(x) := x^r \). The left hand side is a polynomial, so this coefficient is zero on the left hand side. Hence, it must also be zero on the right hand side. Working out this coefficient explicitly reveals that it equals 

\[-\tilde{v}(x^{n+r}) + \text{ terms where } \text{v} \text{ is } 0, 2, \ldots, n + r \text{ and } \alpha \in O\text{Sym}_{n+1} \text{ of positive degree. All of these "lower terms" are in } V' \text{ by induction, hence, } \tilde{v}(x^{n+r}) + S.

Finally, to establish (1c), looking at the \( t^\ell\)-coefficients of (9.14) when \( f(x) = x^r \) shows that \( \tilde{v}(x^r) + S \) lies in the sub-bimodule generated by \( \tilde{v}(x^r) + S \) for any \( r \geq 0 \). Hence, by another induction on \( r \), the sub-bimodule of \( V/S \) generated by \( \tilde{v}(1) + S \) contains all \( \tilde{v}(x^r) + S \).

(2) This is similar to the proof of (1), or it can be deduced from (1) using Lemma 9.2(1). We just explain the first step of the argument, namely, that the relations (9.17) and (9.18) both hold for the actions of \( OH^\ell_n \) and \( OH^\ell_{n+1} \) on \( U^\ell_n \). As in the proof of (1), to prove (9.17), it suffices to check it in the case that \( f(x) = 1 \), then one can act on the right using Lemma 9.7 to get the general result. To prove it when \( f(x) = 1 \), it suffices to show that \( \tilde{v}^{(n+1)}(t)u_\alpha(1) = (-1)^\ell u_\alpha((t + x^{-1})\tilde{v}(m)(-t)) \). This follows because, in view of the signs in the definition of the left and right actions in Lemma 9.1(2) and also (9.4), both sides are equal to \((t + (-1)^d t^d x_{n+1}^d) \ldots (t + (-1)^d x_d)\)\((t + (-1)^d x_{n+2}^d)\). Similarly, to prove (9.18), it suffices to check the case \( f(x) = 1 \), which amounts to showing that \( u_\alpha(1)\tilde{v}^{(n+1)}(t)(t + (-1)^d x_d) \cdot \cdots \cdot (t + (-1)^d x_{n+2}^d) \). This follows because both sides are equal to \((t + (-1)^d x_d) \cdot \cdots \cdot (t + (-1)^d x_{n+2}^d) \).

We are nearly in position to be able to prove a fundamental result about the 2-supercategory \( OGBim^\ell \). It shows that it is rigid in the sense that all of its 1-morphisms have left and right duals. From this point onwards, we often need to work with a degree- and parity-shifted version of \( V^\ell_{n,\alpha} \) and, very occasionally, of \( U^\ell_{n,\alpha} \). These shifts are forced upon us in order to have adjunctions that are even of degree 0. Recall that \( n \# d \) denotes \( n + 1 \) \( + \cdots + (n + d - 1) \). For \( \ell = n + d + n' \) and \( \alpha \in \Lambda(k, d) \) as usual, we define

\[ V^\ell_{n,\alpha} := (\Pi^2 Q^{-1})^{n \# d} V^\ell_{n,\alpha}, \quad U^\ell_{n,\alpha} := (\Pi^2 Q^{-1})^{n \# d} U^\ell_{n,\alpha}, \]

(9.22)

recalling (2.6). To avoid ambiguity we denote the vectors \( v_{n,\alpha}(f) \) and \( u_{n,\alpha}(f) \) viewed as elements of \( V^\ell_{n,\alpha} \) and \( U^\ell_{n,\alpha} \) by \( v_{n,\alpha}(f) \) and \( \bar{u}_{n,\alpha}(f) \), respectively. In particular, \( v_{n,\alpha}(f) \) is of degree \( \deg(f) - 2(n \# d) \) and parity \( \text{par}(f) + (n \# d) \) (mod 2). All of the results involving \( V^\ell_{n,\alpha} \) and \( U^\ell_{n,\alpha} \) established so far in this section have analogs for \( V^\ell_{n,\alpha} \) and \( U^\ell_{n,\alpha} \). In some cases this involves some additional signs. We give a brief summary explaining some minor but crucial modifications as we go.

- We replace the even degree 0 \( R^\ell \)-supermodule isomorphisms from (9.2) and (9.3) with \(^7\)

\[ \Upsilon : V^\ell_{n,\alpha} \rightarrow \bar{U}^\ell_{n,\alpha}, \quad a \otimes \hat{c} \mapsto (-1)^{\text{par}(a)} \hat{c} \otimes a^*, \]

\[ \Upsilon : \bar{U}^\ell_{n,\alpha} \rightarrow V^\ell_{n,\alpha}, \quad \hat{c} \otimes a \mapsto (-1)^{\text{par}(a)} a^* \otimes \hat{c}. \]

(9.23, 9.24)

The formulae (9.5) become

\[ v_{n,\alpha}(f) \Upsilon = (-1)^{\alpha \# d \text{par}(f)} u_{n,\alpha}(\epsilon_d(f)^*), \quad \bar{u}_{n,\alpha}(g) \Upsilon = (-1)^{\alpha \# d \text{par}(g)} v_{n,\alpha}(\epsilon_d(g)^*). \]

(9.25)

Lemma 9.2 and Corollary 7.8 together show that

\[ (\hat{b}_1 u \hat{b}_2) \Upsilon = \psi^r_{n+d}(\hat{b}_1) u^r \psi^r_{n+d}(\hat{b}_2) \]

(9.26)

for \( b_1 \in O\text{Sym}_{n+d}, b_2 \in O\text{Sym}_n \) and \( u \in \bar{U}^\ell_{n,\alpha}. \)

\(^7\)In these formulae, we mean the parity of \( a \) as an element of \( O\text{Sym}_n \) or \( O\text{Sym}_{n'}, \) respectively, before the additional parity shift defining \( V^\ell_{n,\alpha} \) or \( U^\ell_{n,\alpha} \) has been applied.
The appropriate analog of $\tilde{c}_{\alpha,\alpha'}$ is the graded $(OH_n^\ell, OH_{n+d+1}^\ell)$-superbimodule isomorphism

$$c_{\alpha,\alpha'} : V_{n,d+\ell}^{\ell} \to V_{n,\alpha}^{\ell} \otimes_{OH_{n+d}^{\ell}} V_{n,d,\alpha'}^{\ell}$$

$$(9.27)$$

$$v_{n,d+\ell}(f \sigma_d(f')) \mapsto (-1)^{(n+d)(n+d')+(n+d+\text{par}(f))} v_{n,\alpha}^{\ell}(f) \otimes v_{n+d,\alpha'}^{\ell}(f')$$

for $\ell = n + d + d' + n'$, $\alpha \in \Lambda(k,d)$, $\alpha' \in \Lambda(k',d')$ and $f \in OSym_\alpha$, $f' \in OSym_{\alpha'}$. To see that this is a graded superbimodule homomorphism, consider the following commuting diagram:

$$\begin{array}{ccc}
V_{n,d+\ell}^{\ell} & \longrightarrow & V_{n,\alpha}^{\ell} \otimes_{OH_{n+d}^{\ell}} V_{n,d,\alpha'}^{\ell} \\
\downarrow \text{id} & & \uparrow \text{id} \otimes \text{id} \\
\overline{V}_{n,d+\ell}^{\ell} & \longrightarrow & \overline{V}_{n,\alpha}^{\ell} \otimes_{OH_{n+d}^{\ell}} \overline{V}_{n,d,\alpha'}^{\ell}
\end{array}$$

The vertical maps come from the identity maps on the underlying vector spaces, which are graded superbimodule isomorphisms but they are not even of degree 0. The bottom map is the graded superbimodule homomorphism already constructed in Lemma 9.6(1). It follows that the top map is a graded superbimodule isomorphism too. It remains to compute this map on $v_{n,d+\ell}(f \sigma_d(f')) \in V_{n,d+\ell}^{\ell}$ explicitly by tracing it around the other three sides of the square to see that it is exactly the map $c_{\alpha,\alpha'}$ from (9.27). The complicated sign arises because the right hand map takes $\overline{V}_{n,\alpha}^{\ell}(f) \otimes \overline{V}_{n+d,\alpha'}^{\ell}(f')$ to $(-1)^{(n+d)(n+d'+\text{par}(f))} v_{n,\alpha}^{\ell}(f) \otimes v_{n+d,\alpha'}^{\ell}(f')$ since $\text{id} : \overline{V}_{n+d,\alpha'}^{\ell} \to V_{n+d,\alpha'}^{\ell}$ is of parity $(n + d)\#d'$.

- In the special case $\alpha = (1)$, we write simply $V_{n}^{\ell}, \overline{U}_{n}^{\ell}, v_n(f)$ and $\tilde{u}_n(f)$ instead of $V_{n,\alpha}^{\ell}, \overline{U}_{\alpha,\alpha'}^{\ell}, v_{n,\alpha}^{\ell}(f)$ and $\tilde{u}_{\alpha,\alpha'}^{\ell}(f)$. Iterating (9.27) gives a graded $(OH_n^{\ell}, OH_{n+d}^{\ell})$-superbimodule isomorphism

$$c_{(1)^{d}} : V_{n,(1)^d}^{\ell} \to V_{n}^{\ell} \otimes_{OH_{n+d}^{\ell}} V_{n+1}^{\ell} \otimes_{OH_{n+d+1}^{\ell}} \cdots \otimes_{OH_{n+d+1}^{\ell}} V_{n+d-1}^{\ell}$$

$$(9.28)$$

$$v_{n,(1)^d}(x_1^{a_1} \cdots x_d^{a_d}) \mapsto (-1)^{\sum_{i=1}^{d} a_i ((n+i)(n+i)\#d+\text{par}(f))} v_{n}^{\ell}(x_1^{a_1}) \otimes \cdots \otimes v_{n+d-1}^{\ell}(x_d^{a_d}).$$

- The left action of $ONH_d$ on $\overline{V}_{n,(1)^d}^{\ell}$ induces an action on $V_{n,(1)^d}^{\ell}$. In view of the $(-1)^{(n+d-1)\text{par}(a)}$ already present in Lemma 9.7(1) and the additional $(-1)^{(n+d)\text{par}(a)}$ due to the parity shift functor, this satisfies

$$a \cdot v_{n,(1)^d}^{\ell}(f) = (-1)^{(n+d-1)\text{par}(a)} v_{n,(1)^d}^{\ell}(a \cdot f)$$

for $a \in ONH_d$ and $f \in OPol_d$.

- Lemma 9.10 can be rewritten in terms of $V_{n}^{\ell}$ and $\overline{U}_{n}^{\ell}$. We just record the following relation, which is obtained from (9.15) incorporating the parity shift into the left $OH_n^{\ell}$-supermodule structure:

$$\tilde{e}^{(n)}(t)v_n(x) = (-1)^{n+r} v_n((t + 1)^{n+r} t^{-1} x) \tilde{e}^{(n+1)}(-1)^{n+r} t).$$

$$(9.30)$$

**Theorem 9.11.** Suppose that $\ell = n + 1 + n'$. There are even degree 0 superbimodule homomorphisms

$$\text{coev}_n : OH_n^{\ell} \to V_{n}^{\ell} \otimes_{OH_{n+1}^{\ell}} U_{n}^{\ell}, \quad 1 \mapsto \sum_{r=0}^{n} v_n(x^r) \otimes u_n(1) \tilde{e}^{(n)}_{n-r},$$

$$(9.31)$$

$$\text{ev}_n : U_{n}^{\ell} \otimes_{OH_{n}^{\ell}} V_{n}^{\ell} \to OH_{n+1}^{\ell}, \quad u_n(x^r) \otimes v_n(x^s) \mapsto \begin{cases} \tilde{e}^{(n+1)}_{r+s-n} & \text{if } r + s \geq n \\ 0 & \text{otherwise}, \end{cases}$$

$$(9.32)$$

for $r, s \geq 0$. These maps satisfy $(\text{ev}_n \otimes \text{id}) \circ (\text{id} \otimes \text{coev}_n) = \text{id}$ and $(\text{id} \otimes \text{ev}_n) \circ (\text{coev}_n \otimes \text{id}) = \text{id}$. Hence, $(U_n^{\ell}, V_n^{\ell})$ is a dual pair of 1-morphisms in $OG'Bim_{\ell}$. 
Before we prove the theorem, we explain how to write down the definitions of coev$_n$ and ev$_n$ in terms of generating functions. These can be checked by computing coefficients. For the unit of adjunction coev$_n$, we have that
\[
\text{coev}_n(1) = \left[ v_n((t - x)^{-1}) \otimes u_n(1) \tilde{e}^{(n)}(t) \right]_{r-1}. \tag{9.33}
\]
The notation $[-]_r$ being used here means to take the $r'$-coefficient of the formal Laurent series inside the square brackets. Similarly, we use $[-]_c$ and $[-]_{c0}$ for the formal power series or the polynomial obtained by keeping only the strictly negative powers or weakly positive powers of $t$, respectively. Using (9.30) with $r = 0$ and also (9.18) with $f = 1$, we have that
\[
v_n((t - x)^{-1})\tilde{e}^{(n+1)}(-t) = (-1)^{n+1} \tilde{e}^{(n)}((-1)^n t) v_n(1), \tag{9.34}
\]
\[
\tilde{e}^{(n+1)}(-t) u_n((t - x)^{-1}) = (-1)^n u_n(1) \tilde{e}^{(n)}(t). \tag{9.35}
\]
Hence, (9.33) can be rewritten as
\[
\text{coev}_n(1) = (-1)^{n+1} \left[ v_n((t - x)^{-1}) \otimes \tilde{e}^{(n+1)}(-t) u_n((t - x)^{-1}) \right]_{r-1} \tag{9.36}
\]
\[
= \left[ \tilde{e}^{(n)}((-1)^n t) v_n(1) \otimes u_n((t - x)^{-1}) \right]_{r-1}. \tag{9.37}
\]
Equating coefficients in (9.36) and (9.37) gives two more formulae for coev$_n$:
\[
\text{coev}_n(1) = \sum_{r,s \geq 0} (-1)^{n-r} v_n(x^r) \otimes \tilde{e}^{(n+1)}_{n-r-s} u_n(x^s) = \sum_{r=0}^{n} (-1)^{n+1} \tilde{e}^{(n)}_{n-r} v_n(1) \otimes u_n(x^r). \tag{9.38}
\]
For the counit ev$_n$, we have the following two equivalent formulations:
\[
\text{ev}_n \left( u_n((t - x)^{-1} f(x)) \otimes v_n(g(x)) \right) = \left[ \tilde{e}^{(n+1)}(t) f(t) g(t) \right]_{c=0} \tag{9.39}
\]
\[
\text{ev}_n \left( u_n(f(x)) \otimes v_n(g(x)(t - x)^{-1}) \right) = \left[ \tilde{e}^{(n+1)}(t) f(t) g(t) \right]_{c=0} \tag{9.40}
\]
for any $f(x), g(x) \in \mathbb{F}[x]$. The following elementary identity about power series will be needed in the proof: for $f(x) \in \mathbb{F}[x]$ we have that
\[
f(x) = \left[ (t - x)^{-1} f(t) \right]_{r-1}. \tag{9.41}
\]

**Proof of Theorem 9.11.** First we show that ev$_n$ is a supermodule homomorphism. To do this, we actually give a more transparent definition of this map. We saw in the proof of Lemma 9.1 that $V_n^\ell = (\Pi Q^{-2} \varepsilon)O\text{Sym}_{n-(1,n')} \otimes O\text{Sym}_\ell$. $R_\ell$ is a free $R_\ell$-supermodule generated by vectors of the form $b \otimes 1$ for $b \in O\text{Sym}_{n-(1,n)}$, and moreover the right action of $O\text{Sym}_{n+1}$ defined for $f \in O\text{Sym}_{n+1}$ by $(b \otimes 1) \cdot f := b f \otimes 1$ is an action by $R_\ell$-supermodule endomorphisms. Similarly $U_n^\ell = R_\ell \otimes O\text{Sym}_\ell O\text{Sym}_{(0',1,n)}$ is a free $R_\ell$-supermodule generated by vectors of the form $1 \otimes \sigma_{n'}(a)$ for $a \in O\text{Sym}_{(1,n)}$, and the left action of $O\text{Sym}_{n+1}$ defined for $f \in O\text{Sym}_{n+1}$ by $f \cdot (1 \otimes \sigma_{n'}(a)) := (-1)^{n' \text{par}(f)} 1 \otimes \sigma_{n'}(f a)$ is an action by $R_\ell$-supermodule endomorphisms. Consider the $R_\ell$-supermodule homomorphism
\[
\theta : U_n^\ell \otimes R_\ell V_n^\ell \to O\text{Sym}_{n-(1,n')} \otimes O\text{Sym}_\ell R_\ell, \tag{9.42}
\]
\[
(1 \otimes \sigma_{n'}(a)) \otimes (b \otimes 1) \mapsto (-1)^{n' \text{par}(a)} \tau_1 \cdots \tau_{n} \cdot (s_{n' \cdots 1} a) b \otimes 1
\]
for $a \in O\text{Sym}_{(1,n)}$ and $b \in O\text{Sym}_{n-(1,n)}$. Note that $\theta$ is even of degree 0. We claim that it is $O\text{Sym}_n$-balanced for the natural left action of $O\text{Sym}_n$ on $V_n^\ell$ and the right action on $U_n^\ell$ defined by restriction along $\sigma_{n'+1} \circ p_{n'+1} : O\text{Sym}_n \to O\text{Sym}_{(1,n'+1),n}$, i.e., acting with $f \in O\text{Sym}_n$ takes $1 \otimes \sigma_{n'}(a)$ to $(-1)^{(n'+1) \text{par}(f)} 1 \otimes \sigma_{n'}(a \sigma_1(f))$. To see this, we have to show that
\[
(-1)^{(n'+1) \text{par}(f)}) \theta (1 \otimes \sigma_{n'}(a \sigma_1(f)) \otimes (b \otimes 1)) = (-1)^{n \text{par}(f)} \theta ((1 \otimes \sigma_{n'}(a) \otimes (f b \otimes 1))
\]
for $a \in OSym_{(1,n)}$, $b \in OSym_{(n,1)}$ and $f \in OSym_n$. This follows because $s_{n-1} \sigma_1(f) = (-1)^{(n-1)}\text{par}(f)$, so
\[
(-1)^{n+\ell} \cdot \text{par}(f) \theta \left( 1 \otimes \sigma_n\epsilon(a \sigma_1(f)) \otimes (b \otimes 1) \right) = (-1)^n \epsilon b \cdot f b \otimes 1.
\]

Now the first claim is checked, and it implies that $\theta$ induces an $R_\ell$-supermodule homomorphism
\[
\overline{\theta} : U_n^\ell \otimes_{OH_n^\ell} V_n^\ell \to OSym_{(n+1,n')} \otimes_{OSym} R_\ell.
\]

We next claim that $\overline{\theta}$ is an $(OSym_{n+1},OSym_{n+1})$-superbimodule homomorphism. This is obvious for the right actions. To see it for the left actions, we must show that
\[
\tau_1 \cdot \cdots \cdot \tau_n \cdot (s_{n-1}(f a)) \otimes 1 = f \tau_1 \cdot \cdots \cdot \tau_n \cdot (s_{n-1} a) \otimes 1
\]
for all $a \in OSym_{(1,n)}$ and $f \in OSym_{n+1}$. Using the definition (5.9), the proof of this reduces to showing for each $i = n, \ldots, 1$ that $\tau_i \cdot (s_{n-1}(f)) = 0$ for $f \in OSym_{n+1}$. It suffices to check this for $f = (t + x_1) \cdots (t + x_{n+1})$, when it is clear as $s_{n-1} f = (t - x_1) \cdots (t - x_{n+1})(t + x_1) \cdots (t + x_n)$ which is annihilated by $\partial_i$. The second claim is now proved. Then we post-compose $\overline{\theta}$ with the inverse of the isomorphism $OH_{n+1}^\ell \to OSym_{(n+1,n')} \otimes_{OSym} R_\ell$ from Theorem 7.4(1). The second claim implies that this is produces an $(OH_{n+1}^\ell,OH_{n+1}^\ell)$-superbimodule homomorphism $U_n^\ell \otimes_{OH_n^\ell} V_n \to OH_{n+1}^\ell$. Finally, we observe that this is the map $ev_n$ from the statement of the theorem because it takes $u_n(x') \otimes v_n(x)$ to

\[
(-1)^n \tau_1 \cdot \cdots \cdot \tau_n \cdot (s_{n-1}(f a)) \otimes 1 = (-1)^n \tau_1 \cdot \cdots \cdot \tau_n \cdot x_{n+1} \otimes 1 = \gamma^{(n-1)}
\]

where the final equality follows from the identity obtained by applying $\epsilon_n$ to (5.13). This completes the proof that $ev_n$ is a superbimodule homomorphism.

For $coev_n$, we define it to be the unique even degree zero homomorphism of graded right $OH_n^\ell$-supermodules taking the identity element 1 in $OH_n^\ell$ to the expression in the statement of the theorem. It is not yet clear that it is a left $OH_{n+1}^\ell$-supermodule homomorphism. We will deduce this at the end after we have checked the zig-zag identities.

To check the zig-zag identities, we take a polynomial $f(x) \in \mathbb{F}[x]$. We first show that $((\text{id} \otimes ev_n) \circ (coev_n \otimes \text{id})(v_n(f(x)))) = v_n(f(x))$. By (9.36), the map $coev_n \otimes \text{id}$ takes $v_n(f(x))$ to

\[
(-1)^{n+1}v_n((t-x)^{-1})\bar{e}^{(n+1)}(-t) \otimes u_n((t-x)^{-1}) \otimes v_n(f(x))
\]

Then we apply the second (even) map using (9.39) to get

\[
(-1)^{n+1}v_n((t-x)^{-1})\bar{e}^{(n+1)}(-t)\gamma^{(n+1)}(t)f(t)_{\tau_{n+1,0}}
\]

Since $v_n((t-x)^{-1})\bar{e}^{(n+1)}(-t) = -\bar{e}^{(n+1)}((-1)^{n+1})v_n(1)$ and, crucially, this is a polynomial in $t$, we can omit the inside square brackets. After doing that, $\epsilon$ and $\gamma$ cancel using (4.46), so we obtain

\[
\left[v_n\left((t-x)^{-1}\right)f(t)_{\tau_{n+1,1}} = v_n(f(x)),
\right.
\]

where we used (9.41) for the final equality. The check of the second zig-zag identity, namely, that $((ev_n \otimes \text{id}) \circ (\text{id} \otimes coev_n)(u_n(f(x)))) = u_n(f(x))$ is similar: applying the unit gives us

\[
(-1)^{n+1}u_n(f(x)) \otimes v_n((t-x)^{-1}) \otimes \bar{e}^{(n+1)}((-t)u_n((t-x)^{-1}))_{\tau_{n+1,1}}.
\]

Then we use (9.40) to get

\[
(-1)^{n+1}\left[\gamma^{(n+1)}(t)f(t)_{\tau_{n+1,0}}\bar{e}^{(n+1)}((-t)u_n((t-x)^{-1}))_{\tau_{n+1,1}}
\right.
\]
Again we use that \( \tilde{e}^{(n+1)}(-t)u_n(t-x)^{-1} \) is a polynomial in \( t \) to see that we can omit the inside square brackets. Then the \( \gamma \) and \( \epsilon \) cancel by the infinite Grassmannian relation, leaving us with
\[
[f(t)u_n(t-x)^{-1}]_{t=1} = u_n(f(x)).
\]

Both zig-zag identities are now proved.

It remains to show that coev\( _n \), which we have defined so that it is a right \( OH^\ell_{n} \)-supermodule homomorphism, is also a left \( OH^\ell_{n} \)-supermodule homomorphism. It suffices to show that \( \tilde{e}^{(n)} \) \( \text{coev}_n(1) = \text{coev}_n(1)\tilde{e}^{(n)} \) for all \( r \geq 1 \). We show equivalently that \( \tilde{e}^{(n)}((-1)^n t) \text{coev}_n(1) = \text{coev}_n(1)\tilde{e}^{(n)}((-1)^n t) \). By Lemma 9.10(1a), this follows if we can show that \( \tilde{e}^{(n)}((-1)^n t) \text{coev}_n(1) \otimes v_n(1) = \text{coev}_n(1)\tilde{e}^{(n)}((-1)^n t) \otimes v_n(1) \) in \( V^\ell_1 \otimes_OH^\ell_{n+1} U^\ell_n \otimes_OH^\ell_n V^\ell_1 \). The zig-zag identities imply that \( \text{id} \otimes \text{ev}_n \) is injective, so we can apply it to this equation to reduce the problem one more time to that of showing
\[
\left( \text{id} \otimes \text{ev}_n \right)\left( \tilde{e}^{(n)}((-1)^n t) \text{coev}_n(1) \otimes v_n(1) \right) = \left( \text{id} \otimes \text{ev}_n \right)\left( \text{coev}_n(1)\tilde{e}^{(n)}((-1)^n t) \otimes v_n(1) \right).
\]

By the zig-zag identities again, the left hand side is simply equal to \( \tilde{e}^{(n)}((-1)^n t)\text{coev}_n(1) = v_n(t+x)\tilde{e}^{(n+1)}(t) \). To see that the right hand side is equal to this, we commute \( \tilde{e}^{(n)}((-1)^n t) \) to the right to get
\[
\left( \text{id} \otimes \text{ev}_n \right)\left( \text{coev}_n(1) \otimes \tilde{e}^{(n)}((-1)^n t) v_n(1) \right) = \left( \text{id} \otimes \text{ev}_n \right)\left( \text{coev}_n(1) \otimes v_n((t+x)^{-1})\tilde{e}^{(n+1)}(t) \right).
\]

We do know that \( \text{ev}_n \) is a right \( OH^\ell_{n+1} \)-supermodule homomorphism, so we can now compute this using the zig-zag identity a final time to see that it equals \( v_n((t+x)^{-1})\tilde{e}^{(n+1)}(t) \). This completes the proof. \( \square \)

**Corollary 9.12.** The following diagrams of graded superfunctors commute up to even degree 0 isomorphisms:

\[
\begin{array}{ccc}
ONH^\ell_{n+1} \text{-gsMod} & \xrightarrow{(\Pi Q^2)^n \text{Res}} & ONH^\ell_n \text{-gsMod} \\
\downarrow (\omega \chi)_{n+1} & & \downarrow (\omega \chi)_n \\
OH^\ell_{n+1} \text{-gsMod} & \xrightarrow{V^\ell_1 \otimes_OH^\ell_{n+1}} & OH^\ell_n \text{-gsMod}
\end{array}
\]

\[
\begin{array}{ccc}
ONH^\ell_{n+1} \text{-gsMod} & \xrightarrow{(\Pi Q^2)^n \text{Ind}} & ONH^\ell_n \text{-gsMod} \\
\downarrow (\omega \chi)_{n+1} & & \downarrow (\omega \chi)_n \\
OH^\ell_{n+1} \text{-gsMod} & \xrightarrow{U^\ell_n \otimes_OH^\ell_n} & OH^\ell_n \text{-gsMod}
\end{array}
\]

where the vertical arrows are the equivalences of graded supercategories from Corollary 8.3.

**Proof.** Note that \( (\Pi Q^2)^n \text{Ind}_{OH^\ell_{n+1}}^{OH^\ell_n} \) is left adjoint to \( (\Pi Q^2)^n \text{Res}_{OH^\ell_n}^{OH^\ell_{n+1}} \). Also \( U^\ell_n \otimes_OH^\ell_n \) is left adjoint to \( V^\ell_1 \otimes_OH^\ell_{n+1} \) by Theorem 9.11. Hence, using the unicity of adjoints, it suffices to prove that the first square commutes. We show equivalently that the following commutes up to even degree 0 isomorphism:

\[
\begin{array}{ccc}
ONH^\ell_{n+1} \text{-gsMod} & \xrightarrow{\text{Res}} & ONH^\ell_n \text{-gsMod} \\
\downarrow (\omega \chi)_{n+1} & & \downarrow (\omega \chi)_n \\
OH^\ell_{n+1} \text{-gsMod} & \xrightarrow{V^\ell_1 \otimes_OH^\ell_{n+1}} & OH^\ell_n \text{-gsMod}
\end{array}
\]

Here, we have removed the degree- and parity-shifts on the horizontal arrows and we have replaced the equivalence \( (\omega \chi)_{n+1} = \text{Hom}_{OH^\ell_{n+1}}(ONH^\ell_{n+1}(\omega \chi)_{n+1},-) \) on the left hand vertical arrow with the quasi-inverse equivalence \( \text{ONH}^\ell_{n+1}(\omega \chi)_{n+1} \otimes_{OH^\ell_{n+1}} - \sim \text{OPol}_{n+1} \otimes_{\text{OSym}_{n+1}} \text{OH}^\ell_{n+1} \otimes_{OH^\ell_{n+1}} - \). To prove this new diagram commutes, it suffices to show that
\[
(\omega \chi)_n \text{OPol}_{n+1} \otimes_{\text{OSym}_{n+1}} \text{OH}^\ell_{n+1} = \tilde{V}^\ell_n
\]
as \((\text{OH}^\ell_n, \text{OH}^\ell_{n+1})\)-superbimodules. By Theorem 7.4, the superbimodule on the left here can be replaced by \((\omega \chi)_n \text{OPol}_{n+1} \otimes \text{OSym}_n \otimes \text{OSym}_R R^\ell\) (where \(n' = \ell - n - 1\) as usual). By Lemma 5.8, we have that \((\omega \chi)_n \text{OPol}_{n+1} \cong \text{OSym}_n(1)_1\), so this is \(\cong \text{OSym}_n(1) \otimes \text{OSym}_R R^\ell\), which is exactly the definition of \(\widetilde{V}^\ell_n\).

**Corollary 9.13.** Suppose that \(\ell = n + 1 + n'\). There are even degree 0 superbimodule homomorphisms

\[
\overline{\text{coev}}_n : \text{OH}^\ell_{n+1} \to \widetilde{U}^\ell_n \otimes \text{OH}^\ell_{n+1} \widetilde{V}^\ell_n, \quad 1 \mapsto \sum_{r=0}^{n'} (-1)^{r+(r+\ell)n'} \bar{u}_n(x') \otimes \bar{v}_n(1) \psi'_{n}(\bar{v}'_{n'-r})
\]

\[
\overline{\text{ev}}_n : \widetilde{V}^\ell_n \otimes \text{OH}^\ell_{n+1} \to \text{OH}^\ell_n, \quad \bar{v}_n(x') \otimes \bar{u}_n(x') \mapsto \begin{cases} (-1)^{r+s+n'(r+s+\ell)} \psi'_{n'+1}(\bar{v}'_{n'+1}) & \text{if } r + s \geq n' \\ 0 & \text{otherwise} \end{cases}
\]

making \((\widetilde{V}^\ell_n, \widetilde{U}^\ell_n)\) into another dual pair of 1-morphisms in \(\text{OGBim}_\ell\).

**Proof.** We define \(R^\ell\)-supermodule homomorphisms \(\overline{\text{coev}}_n\) and \(\overline{\text{ev}}_n\) so that the following diagrams commute:

\[
\begin{array}{ccc}
\text{OH}^\ell_{n+1} & \xrightarrow{\overline{\text{coev}}_n} & \widetilde{U}^\ell_n \otimes \text{OH}^\ell_{n+1} \widetilde{V}^\ell_n \\
\psi_{n+1}^- \downarrow & & \downarrow \otimes * \\
\text{OH}^\ell_n & \xrightarrow{\overline{\text{coev}}_n} & \widetilde{V}^\ell_n \otimes \text{OH}^\ell_{n+1} U^\ell_{n'} \\
\psi_{n'}^- \downarrow & & \downarrow \otimes * \\
\text{OH}^\ell_n \otimes \text{OH}^\ell_{n+1} U^\ell_{n'} & \xrightarrow{\overline{\text{ev}}_n} & \text{OH}^\ell_n
\end{array}
\]

To see that the vertical maps \(\uparrow \otimes \downarrow\) and \(\otimes \uparrow \downarrow\) in these diagrams make sense, one needs to check that they are balanced, which follows using Lemma 9.2 and (9.26). In fact, \(\overline{\text{coev}}_n\) and \(\overline{\text{ev}}_n\) defined in this way are superbimodule homomorphisms. This again follows using Lemma 9.2 and (9.26) since \(\overline{\text{coev}}_{n'}\) and \(\overline{\text{ev}}_{n'}\) are superbimodule homomorphisms. The zig-zag identities for \(\overline{\text{coev}}_n\) and \(\overline{\text{ev}}_n\) follow from the zig-zag identities for \(\text{coev}_{n'}\) and \(\text{ev}_{n'}\). Finally, to compute the explicit formulae for the maps given in the statement of the corollary, one uses (9.5), (9.25), (9.31) and (9.32) plus Corollary 7.8.

**Remark 9.14.** All of the results in this section up to but not including Corollary 9.13 can be upgraded to the extended setup of Remarks 4.14, 7.9 and 8.10. The odd Grassmannian bimodules \(\widetilde{V}^\ell_{n,0}\) and \(U^\ell_{n,0}\) are defined in just the same way as (9.1), one just tensors with \(R^\ell\) instead of \(R\). All important maps \(\text{coev}_n\) and \(\text{ev}_n\) in Theorem 9.11 in this setting are given by exactly the same formulae. However, in order to prove a version of Corollary 9.13, one needs to twist \(\widetilde{U}^\ell_n\) with the automorphism (7.7), and this causes significant problems in working with the second adjunction subsequently.

The final task in this section is to compute various mates of the endomorphisms of \(U^\ell_{(1^d),n}\) defined by the action of \(\text{ONH}_d\) from Lemma 9.7(2). Specifically, we need to work out the endomorphisms in \(\text{OGBim}_\ell\) that correspond to the diagrams (11.4) and (11.8) in the graphical calculus to be introduced later in the article. Suppose that \(\ell = n + d + n'\) for \(d \geq 0\). We define graded superalgebra homomorphisms

\[
\rho_{(1^{d})n} : \text{ONH} \to \text{End}_{\text{OH}_{n+d} \otimes \text{OH}^\ell_{n+d-1} \otimes \cdots \otimes \text{OH}^\ell_{n+1}} \overline{U}^\ell_n
\]

\[
\lambda_{n,1^{d}} : \text{ONH} \to \text{End}_{\text{OH}^\ell_{n+d} \otimes \text{OH}^\ell_{n+d-1} \otimes \cdots \otimes \text{OH}^\ell_{n+1}} (V^\ell_n \otimes \text{OH}^\ell_{n+d-1} \otimes \cdots \otimes \text{OH}^\ell_{n+1})
\]

as follows. For \(a \in \text{ONH}_d\), \(\rho_{(1^{d})n}(a)\) is defined to be the top map in the following diagram commute:

\[
\begin{array}{ccc}
U^\ell_{n+d-1} \otimes \text{OH}^\ell_{n+d-1} \otimes \cdots \otimes \text{OH}^\ell_{n+1} U^\ell_n & \xrightarrow{\rho_{(1^{d})n}(a)} & U^\ell_{n+d-1} \otimes \text{OH}^\ell_{n+d-1} \otimes \cdots \otimes \text{OH}^\ell_{n+1} U^\ell_n \\
\downarrow (\rho_{(1^{d})n})^\ell & & \downarrow (\rho_{(1^{d})n})^\ell \\
U^\ell_{(1^d),n} & \xrightarrow{u_{(1^d),n}(f) - (-1)^{p(a)}u_{(1^d),n}(f) - a} & U^\ell_{(1^d),n}
\end{array}
\]

(9.45)
Also $\lambda_{n,1}(\ell)(a)$ is the top map making the following commute:

$$
\begin{array}{c}
V_n^\ell \otimes_{OH_n^\ell} \cdots \otimes_{OH_{n-d-1}^\ell} V_{n+d-1}^\ell \\
\downarrow \lambda_{n,1}(\ell)(a) \downarrow
\end{array}
\rightarrow
\begin{array}{c}
V_n^\ell \otimes_{OH_{n+1}^\ell} \cdots \otimes_{OH_{n+d-1}^\ell} V_{n+d-1}^\ell \\
\uparrow \ell_{n,1}(\ell)
\end{array}
\tag{9.46}
$$

The vertical maps in (9.45) and (9.46) come from (9.10) and (9.28). We also remind the reader that $u_{(1^d),n}(f) \cdot a = (-1)^{d+1} \text{par}(a) u_{(1^d),n}(f \cdot a)$ and $a \cdot v_{n,(1^d)}(f) = (-1)^{n(d-1)} \text{par}(a) v_{n,(1^d)}(a \cdot f)$.

**Lemma 9.15.** For $0 \leq n < \ell$, the mate $(\text{id} \otimes \text{ev}_n) \circ (\text{id} \otimes \rho_{(1),n}(x_1) \otimes \text{id}) \circ (\text{coev}_n \otimes \text{id})$ of the $\text{par}(1^d)\text{-superbimodule endomorphism } \rho_{(1),n}(x_1) : U_n^\ell \rightarrow U_n^\ell$ under the adjunction from Theorem 9.11 is equal to $\lambda_{n,1}(1) : V_n^\ell \rightarrow V_n^\ell$.

**Proof.** Since $V_n^\ell$ is generated as a superbimodule by $v_n(1)$ by Lemma 9.10(1c), it suffices to show that

$$(\text{id} \otimes \text{ev}_n) \circ (\text{id} \otimes \rho_{(1),n}(x_1) \otimes \text{id}) \circ (\text{coev}_n \otimes \text{id})(v_n(1)) = \lambda_{n,1}(1)(v_n(1)).$$

This is a calculation from the definitions. The right hand side is $v_n(x)$ by the definition (9.46). For the left hand side, we first apply $\text{coev}_n \otimes \text{id}$ using (9.37) to get

$$\left[\hat{e}^{(n)}((-1)^{n+1}t)v_n(1) \otimes u_n((t-x)^{-1}) \otimes v_n(1)\right]_{r=1}.$$

Then we apply the odd endomorphism $\text{id} \otimes \rho_{(1),n}(x_1) \otimes \text{id}$ defined by (9.45) to get

$$\left[\hat{e}^{(n)}((-1)^{n}t)v_n(1) \otimes u_n((t+x)^{-1}) \otimes v_n(1)\right]_{r=1}.$$

Finally we apply $\text{id} \otimes \text{ev}_n$ using (9.39) with $r$ replaced by $-r$ to get

$$\left[\hat{e}^{(n)}((-1)^{n}t)v_n(1)[\bar{y}^{(n+1)}((-1)^{r+1})]_{r=1}.$$

This is equal to $\gamma_1^{(n+1)} - (-1)^{n} \hat{e}_1^{(n)} v_n(1)$. Since $\gamma_1^{(n+1)} = x_1 + \cdots + x_{n+1}$ and $\hat{e}_1^{(n)} = x_1 + \cdots + x_n$ (and $(-1)^n$ disappears on acting on $v_n(1)$ due to the parity shift) this is $v_n(x)$.

In the next lemma, the following case-free form of (9.18) will be useful:

$$\hat{e}^{(n+1)}(t)u_n((t+x)^{-1})x^r = (-1)^{n+r}u_n(x)\hat{e}^{(n)}((-1)^{r+1}t).$$

**Lemma 9.16.** For $0 < n < \ell$, the $(OH_n^\ell, OH_{n-1}^\ell)$-superbimodule endomorphism

$\sigma_n := (\text{id} \otimes \text{id} \otimes \text{ev}_{n-1}) \circ (\text{id} \otimes \rho_{(1^2),n-1}(\tau_1) \otimes \text{id}) \circ (\text{coev}_n \otimes \text{id} \otimes \text{id}) : U_{n-1}^\ell \otimes_{OH_{n-1}^\ell} V_{n-1}^\ell \rightarrow V_n^\ell \otimes_{OH_n^\ell} U_n^\ell$

maps $u_{n-1}(x^r) \otimes v_{n-1}(x^s)$ to

$$(-1)^{n+r+s+n+1}v_n(x^s) \otimes u_n(x^r) + \sum_{p=0}^{n} \sum_{q=0}^{s} (-1)^{n+p+q+r+q}v_n(x^p) \otimes u_n(x^q)\hat{e}^{(n)}_{n-p} \bar{y}_{r+s+n-q}^{(n)}$$

for any $r \geq 0$ and $0 \leq s \leq n-1$.

**Proof.** First, we show that $\rho_{(1^2),n-1}(\tau_1) \left(u_n((t-x)^{-1}) \otimes u_{n-1}(x^r)\right)$ equals

$$- u_n((t+x)^{-1})x^r \otimes u_{n-1}((t-(1)^{r+1}x)^{-1}) + \sum_{q=0}^{r-1}(-1)^{q+r+q}u_n((t+x)^{-1}x^q) \otimes u_{n-1}(x^{q-1}).$$

(9.49)
To do this, according to the definition (9.45), we first use the inverse of the map \( c'_{(1)} \) from (9.10) to pass to \( U^\ell_{(1^2)n-1} \). This maps \( u_n((t - x)^{-1}) \otimes u_{n+1}(x^r) \) to \((-1)^ru^{(1^2)n-1}((t - x)^{-1}x^r) \). The application of \( \rho_{(1^2)n-1}(\tau_1) \) takes this to \(-u^{(1^2)n-1}((t + x)^{-1}x^r \cdot \tau_1) \). This we can compute with Lemma 5.10 to get

\[
-u^{(1^2)n-1}((t + x)^{-1}x^r(t + (-1)^r x)) - \sum_{q=0}^{r-1} (-1)^{qr}u^{(1^2)n-1}((t + x)^{-1}x^r x^q x^{-q-1}).
\]

After that we apply \( c'_{(1)} \) to obtain the vector in \( U^\ell_n \otimes OH_n^U U^\ell_{n-1} \) displayed in (9.49).

Now to prove the lemma, we again calculate with generating functions. Start with the vector \( u_{n-1}(x^r) \otimes v_{n-1}(x^s) \) for \( 0 \leq s \leq n - 1 \) (this assumption on \( s \) will be crucial shortly). By (9.36), the map \( \text{coev}_n \otimes \text{id} \otimes \text{id} \) takes it to

\[
(-1)^{r-1} u_{n-1}(x^r) \otimes u_{n-1}(x^s) \cdot v_{n-1}(x^s) \big|_{r-1}. \tag{9.50}
\]

Then we apply the odd homomorphism \( \text{id} \otimes \rho_{(1^2)n-1}(\tau_1) \otimes \text{id} \) using (9.49) to obtain

\[
-(-1)^{r} u_{n-1}(x^r) \otimes u_{n-1}(x^s) \cdot v_{n-1}(x^s) \big|_{r-1} + \sum_{q=0}^{r-1} (-1)^{n+q+r+qr} u_{n-1}(x^r) \otimes u_{n-1}(x^s) \big|_{r-1} \tag{9.51}
\]

It just remains to apply \( \text{id} \otimes \text{id} \otimes \text{ev}_{n-1} \). We treat the two terms in (9.51) separately. For the first term, we have that \( \text{ev}_{n-1}(u_{n-1}((t + x)^{-1}x^s)) = (-1)^{r-1} x^{\ell + r} \big|_{r-1} \tau s \) by (9.39) (with \( r \) replaced by \( -1)^r t \)). The assumption that \( s \leq n - 1 \) means that we can omit the truncation to \( t < 0 \) here. So the first term contributes

\[
-(-1)^{n+r+s+1} u_{n-1}(x^r) \otimes u_{n-1}(x^s) \big|_{r-1}. \tag{9.54}
\]

Now we use (9.47) to rewrite this as

\[
(-1)^{n+r+s+n+1} u_{n-1}(x^r) \otimes u_{n-1}(x^s) \big|_{r-1}. \tag{9.41}
\]

The \( \varepsilon \) and \( y \) cancel by the infinite Grassmannian relation to leave

\[
(-1)^{n+r+s+n+1} u_{n-1}(x^r) \otimes u_{n-1}(x^s) \big|_{r-1} \tag{9.41} = (-1)^{n+r+s+n+1} u_{n-1}(x^r) \otimes u_{n-1}(x^s).
\]

It remains to consider the term obtained by applying \( \text{id} \otimes \text{id} \otimes \text{ev}_{n-1} \) to the second term from (9.51). Using (9.47) again, this contributes

\[
\sum_{q=0}^{r-s-n} (-1)^{n+q+r+qr} u_{n-1}(x^r) \otimes u_{n-1}(x^s) \big|_{r-1} + \sum_{q=0}^{r-s-n} (-1)^{r+s+nq+qr} u_{n-1}(x^r) \otimes u_{n-1}(x^s) \big|_{r-1}.
\]

It remains to work out the \( r \)-coefficient explicitly to complete the proof. \( \square \)

**Lemma 9.17.** For \( 0 < n < \ell \), the mate of \( \rho_{(1^2)n-1}(\tau_1) : U^\ell_n \otimes OH_n^U U^\ell_{n-1} \rightarrow U^\ell_n \otimes OH_n^U U^\ell_{n-1} \) under the adjunction from Theorem 9.11 is equal to \( \lambda_{n-1,(1^2)}(\tau_1) : V^\ell_{n-1} \otimes OH_n^V V^\ell_{n-1} \rightarrow V^\ell_{n-1} \otimes OH_n^V V^\ell_{n-1} \).
Proof. By Lemma 9.10(1b)–(1c), $V_{n-1}^\ell \otimes_{OH_n^{\ell}} V_n^\ell$ is generated as an $(OH_{n-1}^{\ell}, OH_n^{\ell + 1})$-superbimodule by the vectors $v_{n-1}(1) \otimes v_n(x^s)(0 \leq s \leq n)$. Also the mate of $P_{(1^2)_{n-1}}$ is equal to $(id \otimes id \otimes ev) \circ (id \otimes \sigma_n \otimes id) \circ (coev_{n-1} \otimes id \otimes id)$ where $\sigma_n$ is the superbimodule homomorphism described in Lemma 9.16; diagrammatically, we have rotated through $180^\circ$ by rotating by $90^\circ$ twice, see (11.4) and (11.8). Therefore it suffices to show that

$$(id \otimes id \otimes ev) \circ (id \otimes \sigma_n \otimes id) \circ (coev_{n-1} \otimes id \otimes id)(v_{n-1}(1) \otimes v_n(x^s)) = \alpha_{n-1; (1^2)}(\tau)(v_{n-1}(1) \otimes v_n(x^s))$$

for $0 \leq s \leq n$. The right hand side may be computed directly from (9.28) and (9.46). It equals

$$(c_{(1)^r})^n(v_{n-1}(1) \otimes v_n(x^s)) \equiv (1)^{n+1}(c_{(1)^r})(v_{n-1}(1) \otimes v_n(x^s))$$

so to complete the proof we must show that

$$(c_{(1)^r})^n \equiv (1)^{n+1}(c_{(1)^r}).$$

To compute the left hand side, we first use (9.37) to get

$$(coev_{n-1} \otimes id \otimes id)(v_{n-1}(1) \otimes v_n(x^s)) = \sum_{r=0}^{n-1} (-1)^{n-1} c_{(1)^r} v_{n-1}(1) \otimes u_{n-1}(1) \otimes v_n(x^s).$$

Then we apply $id \otimes \sigma_n \otimes id$ using (9.48), noting also that $\sigma_n$ is odd. Since $r \leq n-1$ in this expression, the summation over $q$ on the right hand side of (9.48) is actually an empty sum, so zero, and we get simply

$$(1)^{n+1} \sum_{r=0}^{n-1} \tilde{E}_{n-1-r} v_{n-1}(1) \otimes v_n(1) \otimes u_n(x^r).$$

Next we apply $id \otimes id \otimes ev$. We must have that $r+s \geq n$ so $r \geq n-s$, and the final expression is

$$(1)^{n+1} \sum_{r=n-s}^{n-1} \tilde{E}_{n-1-r} v_{n-1}(1) \otimes v_n(1) \tilde{y}_{s-1-r}^{(n+1)} = (1)^{n+1} \sum_{r=0}^{n-s} \tilde{E}_{r} v_{n-1}(1) \otimes v_n(1) \tilde{y}_{s-1-r}^{(n+1)}.$$

Applying $c_{(1)^r}$ as is required for (9.52), we get

$$(1)^{n+1} \sum_{r=0}^{n-s} \tilde{E}_{r} v_{n-1; (1^2)}(1) \tilde{y}_{s-1-r}^{(n+1)}.$$

There is a sign change of $(-1)^r$ due to the parity shift $(n-1)! \equiv 1 \pmod{2}$. Also we have that $\sum_{r=0}^{n-s} (-1)^r \tilde{y}_{s-1-r}^{(n+1)} = \sigma_{n-1}(\tilde{y}_{s-1}^{(2)})$ by (4.43). So this is $(1)^{n+1} v_{n-1; (1^2)}(\tilde{y}_{s-1}^{(2)})$ exactly as in (9.52). \qed

10. Singular Rouquier complex for odd Grassmannian bimodules

Throughout the section, we fix $\ell \in \mathbb{N}$. The graded $(Q, \Pi)$-2-supercategory $O\mathcal{G}Bim_{\ell}$ categorifies the locally unital $\mathbb{Z}[q, q^{-1}]$-algebra that is the image of $U_{q, \ell}(s_{2^\ell})$ in its representation on $V(-\ell)$, notation as at the end of Section 3. To make this statement precise, let $K_0(OH_n^{\ell}_{\text{ps}})$ be the split Grothendieck group of the underlying ordinary category of the $(Q, \Pi)$-supercategory of finitely generated projective graded left $OH_n^{\ell}$-supermodules. Since $OH_n^{\ell}$ is positively graded with degree zero component that is the ground field $\mathbb{F}$, this is nothing more than the free $\mathbb{Z}[q, q^{-1}]$-module generated by the isomorphism class $[OH_n^{\ell}]$ of the regular module, with the actions of $\pi$ and $q$ induced by the parity- and degree shift functors $\Pi$ and $Q$, respectively. So we can identify

$$V(-\ell) \equiv \bigoplus_{n=0}^{\ell} K_0(OH_n^{\ell}_{\text{ps}}), \quad b_n^{\ell} \equiv [OH_n^{\ell}].$$

(10.1)
Theorem 10.1. Under the identification (10.1) of $\bigoplus_{n=0}^{\ell} K_0(OH_n^\ell\text{-pgsmod})$ with $V(-\ell)$, the $\mathbb{Z}[q,q^{-1}]$-module endomorphisms induced by tensoring with odd Grassmannian bimodules correspond to endomorphisms defined by actions of elements of $U(sl_2)$ according to the following dictionary:

1. $[U_n^\ell \otimes OH_n^\ell] \equiv q^n E_{12n-\ell}$ and, more generally, $[U_{(d),n}^\ell \otimes OH_n^\ell] \equiv q^{nd} E_{d} 1_{2n-\ell}$;
2. $[V_n^\ell \otimes OH_{n+1}^\ell] \equiv q^{-(3n-1)} 1_{2n-F}$ and $[V_{n,(d)}^\ell \otimes OH_{n+d}^\ell] \equiv q^{d(3n-2d+1)} 1_{2n-\ell} E_d$.

Also, for $-\ell \leq k \leq \ell$ with $k \equiv \ell \pmod{2}$, the endomorphism $T : 1_k V(-\ell) \to 1_k V(-\ell)$ from Theorem 3.6 corresponds to the $\mathbb{Z}[q,q^{-1}]$-module homomorphism

$$T : K_0(OH_n^\ell\text{-pgsmod}) \to K_0(OH_n^{\ell'}\text{-pgsmod}), \quad [OH_n^\ell] \mapsto (-1)^n (\pi q^2)^{(n_1)} q^n k^{-nk}[OH_n^{\ell'}],$$

where $n := \frac{\ell-k}{2}$ and $n' := \frac{\ell+k}{2}$.

Proof. (1) By Lemma 9.10(2b), we have that $[U_n^\ell \otimes OH_n^\ell] = q^n [n+1]_{q,q} [OH_n^{n+1}]$. Also $EB_n^\ell = [n+1]_{q,q} b_n^\ell$. It follows that $[U_n^\ell \otimes OH_n^\ell] = q^n E_{12n-\ell}$ and $q^n E_{12n-\ell}$ define the same endomorphisms of $V(-\ell)$. For the more general assertion, take $d \geq 1$. By Lemma 9.7, we have that $[U_{(d),n}^\ell \otimes OH_n^\ell] = q^{(d)} [d]_{q,q} U_{(d),n}^\ell \otimes OH_n^\ell]$. Also $U_{(d),n}^\ell \equiv U_n^{n+d-1} \otimes OH_{n+d-1} \cdots \otimes OH_{n+1}^\ell U_{n}^\ell$ by (9.10), so we deduce using the special case already treated that

$$q^{(d)} [d]_{q,q} U_{(d),n}^\ell \otimes OH_n^\ell] = q^{nd+\ell} E_d 1_{2n-\ell} = q^{nd+\ell} [d]_{q,q} E_d 1_{2n-\ell}.$$ 

Cancelling $q^{(d)} [d]_{q,q}$ gives the required conclusion.

(2) The first step is to show that $[V_n^\ell \otimes OH_{n+1}^\ell] = q^{-(3n-1)} 1_{2n-F}$, which follows from Lemma 9.10(1a) like in the proof of (1). Hence, since $V_n^\ell = (\Pi Q^{-2})^n V_n^\ell$, we get that $[V_n^\ell \otimes OH_{n+1}^\ell] = q^{-(3n-1)} 1_{2n-F}$. The passage from this to the more general result about $V_{n,(d)}^\ell$ follows in a similar way to the argument given in (1).

Now consider the final statement about $T$. Take $k$ and $n = \frac{k}{2}$, $n' = \frac{k}{2}$ as in the statement of the theorem. We saw in (3.19) that $T(b_n^\ell) = (-1)^n n^\ell \pi^{n+n'} q^{n+n'} b_n^\ell$. Using the identification (10.1), it follows that $T([OH_n^\ell]) = (-1)^n n^\ell \pi^{n+n'} q^{n+n'} [OH_n^{\ell'}]$. On replacing $n'$ by $k+n$, this becomes the formula in the statement of the theorem.

The goal in the remainder section is to categorify $T : 1_k V(-\ell) \to 1_k V(-\ell)$ for all $-\ell \leq k \leq \ell$ with $k \equiv \ell \pmod{2}$. Let $n := \frac{k}{2}$ and $n' := \frac{k}{2} + k$ so that $2n - \ell = -k$ and $2n' - \ell = k$. Since $n + n' = \ell$, Theorem 7.4(4) shows that the graded superalgebras $OH_n^\ell$ and $OH_n^{\ell'}$ are isomorphic.

Definition 10.2. For $0 \leq d \leq n$, let

$$C_d := \begin{cases} U_{(d),n}^\ell \otimes OH_{n+d}^\ell \otimes \cdots \otimes OH_{n-d}^\ell & \text{if } d \geq -k, \\ 0 & \text{otherwise.} \end{cases}$$

The singular Rouquier complex for odd Grassmannian bimodules is the following sequence of graded $(OH_n^\ell,OH_n^{\ell'})$-superbimodules and even degree 0 superbimodule homomorphisms in $OGBim_\ell$:

$$0 \to C_n \to C_{n-1} \to \cdots \to C_d \to C_{d-1} \to \cdots \to C_1 \to C_0 \to 0 \quad (10.4)$$
where \( \partial_d = 0 \) unless \( \max(0,-k) < d \leq n \) and \( \partial_d : C_d \rightarrow C_{d-1} \) \((\max(0,-k) \leq d \leq n)\) is the even degree zero superbimodule homomorphism defined by the composition

\[
U_{(k+d)n-d}^\ell \otimes OH^\ell_{n-d} \otimes V_{n-d}(d) \xrightarrow{inc} U_{(k+d-1)n-d}^\ell \otimes OH^\ell_{n-d} \otimes V_{n-d}(1,d-1) \xrightarrow{\epsilon^{(k-1,1,d-1)}_1} U_{(k+d-1)n-d+1}^\ell \otimes OH^\ell_{n-d+1} \otimes V_{n-d+1}(1,d-1) \xrightarrow{id \otimes ev_{n-d} \otimes id} U_{(k+d-1)n-d+1}^\ell \otimes OH^\ell_{n-d+1} \otimes V_{n-d+1}(1,d-1) \xrightarrow{can} U_{(k+d-1)n-d+1}^\ell \otimes OH^\ell_{n-d+1} \otimes V_{n-d+1}(d-1).
\]

**Theorem 10.3.** The singular Rouquier complex \((10.4)\) is a chain complex with homology that is zero in all except for the top \((n\text{th})\) homological degree. Moreover, as a graded \((OH_n^\ell, OH_n^\ell)\)-superbimodule the top homology is \(\approx (\Pi Q^2)^{1/2} + n! OH_n^\ell \), viewed as a graded left \(OH_n^\ell\)-supermodule by the natural action and as a graded right \(OH_n^\ell\)-supermodule by restricting the natural right action of \(OH_n^\ell\) along some graded superalgebra isomorphism \(OH_n^\ell \rightarrow OH_n^\ell\).

The following corollary follows immediately. To formulate it, let \(K^b(OH_n^\ell \text{-pgsm})\) be the bounded homotopy supercategory of the graded supercategory finitely generated projective graded left \(OH_n^\ell\)-supermodules; in the definition of this we require that differentials and homotopies are even of degree 0 but chain maps between cochain complexes can be constructed using arbitrary morphisms in \(OH_n^\ell \text{-pgsm}\). For \(n\) and \(n'\) as in \((10.4)\), \(K_0(K^b(OH_n^\ell \text{-pgsm}))\) and \(K_0(K^b(OH_n^\ell' \text{-pgsm}))\) are identified via \((10.1)\) with \(1_{-k} V(-\ell)\) and \(1_k V(-\ell)\), respectively.

**Corollary 10.4.** Tensoring with the singular Rouquier complex \((10.4)\) \((\text{viewed now as a cochain complex rather than a chain complex as usual when working with homotopy categories})\) then taking the total complex defines a graded superequivalence \(K^b(OH_n^\ell \text{-pgsm}) \rightarrow K^b(OH_n^\ell' \text{-pgsm})\). The induced \(\mathbb{Z}[q,q^{-1}]\)-module isomorphism \(1_{-k} V(-\ell) \rightarrow 1_k V(-\ell)\) at the level of the Grothendieck groups of these categories is equal to \(q^{nk} T\) for \(T\) as in \((10.2)\).

The remainder of the section is devoted to the proof of Theorem 10.3, which will be carried out with a series of lemmas. We assume for simplicity of notation that \(k \geq 0\), although with obvious modifications the arguments work for negative \(k\) too.

**Lemma 10.5.** We have that \(\partial_{d-1} \circ \partial_d = 0 \) for \(d = 1, \ldots, n + 1\), hence, \((10.4)\) is a chain complex.

**Proof.** By the super interchange law, \(\partial_{d-1} \circ \partial_d\) factorizes as the composition first of the embedding

\[
\begin{align*}
C_d = U_{(k+d)n-d}^\ell \otimes OH^\ell_{n-d} \otimes V_{n-d}(d) & \xrightarrow{inc} U_{(k+d-2)n-d}^\ell \otimes OH^\ell_{n-d} \otimes V_{n-d}(2,d-2) \xrightarrow{\epsilon^{(k-2,2,d-2)}_1} U_{(k+d-2)n-d+2}^\ell \otimes OH^\ell_{n-d+2} \otimes V_{n-d+2}(2,d-2),
\end{align*}
\]

then the map \(id \otimes (\partial \circ \iota) \otimes id\) from there to \(C_{d-2} = U_{(k+d-2)n-d+2}^\ell \otimes OH^\ell_{n-d+2} \otimes V_{n-d+2}(d-2)\) where

\[
\iota : U_{(2)n-d}^\ell \otimes OH^\ell_{n-d} \otimes V_{n-d}(2) \xrightarrow{inc} U_{(1)n-d}^\ell \otimes OH^\ell_{n-d} \otimes V_{n-d}(1,1) \xrightarrow{\epsilon^{(1,1,1)}_1} U_{n-d+1}^\ell \otimes OH^\ell_{n-d+1} \otimes V_{n-d+1},
\]

\[
\partial : U_{n-d+1}^\ell \otimes OH^\ell_{n-d+1} \otimes OH^\ell_{n-d} \otimes V_{n-d+1} \xrightarrow{id \otimes ev_{n-d} \otimes id} U_{n-d+1}^\ell \otimes OH^\ell_{n-d+1} \otimes OH^\ell_{n-d+1} \otimes V_{n-d+1} \xrightarrow{ev_{n-d+1}} OH^\ell_{n-d+2}.
\]
Thus, we are reduced to proving that $\partial$ takes vectors in the image of $\iota$ to zero. By Lemma 9.7, the image of $\iota$ is equal to the image of the projection $\rho_1\otimes(\chi_0)\otimes\lambda_{n-d,1}(\tau_1)$ of $\lambda_{n-d,1}(\tau_1)\otimes\lambda_{n,d,1}(\tau_1)$. This projection equals 

\[ \rho_1\otimes\lambda_{n-d,1}(\tau_1)\otimes\lambda_{n-d,1}(\tau_1) = (\rho_1\otimes\lambda_{n-d,1}(\tau_1)\otimes\lambda_{n-d,1}(\tau_1)) \circ (\rho_1\otimes\lambda_{n-d,1}(\tau_1)\otimes\lambda_{n-d,1}(\tau_1)). \]

Finally, to complete the proof, we observe that $\partial \circ (\rho_1\otimes\lambda_{n-d,1}(\tau_1)\otimes\lambda_{n-d,1}(\tau_1)) = 0$ because 

\[ \partial \circ (\rho_1\otimes\lambda_{n-d,1}(\tau_1)\otimes\lambda_{n-d,1}(\tau_1)) = \partial \circ (\id\otimes\id) = \partial = 0. \]

\[ \square \]

Now we need to understand the “numerology” of (10.4). In fact, the combinatorial Lemma 3.3 derived long ago is just what we need for this. Recall the definitions of $b_{m,n}(r), c_{m,n}(r) \in \mathbb{Z}^q$, made in the statement of that lemma. The following shows that $c_{n+k,n}(d)$ is the graded superrank of $C_d$ either as a free graded right $OH^d_n$-supermodule or a free graded left $OH^d_n$-supermodule. We will also see in a bit that $b_{n+k,n}(d)$ is the graded rank of $\im \partial_d$ for $d = 0, 1, \ldots, n$.

**Lemma 10.6.** The vectors 

\[ \left\{ u_{n-d,1}(\lambda) \left( (\mathcal{S}^{k+d}_1)^* \right) \otimes v_{n-d,1}(\mu) \left( (\mathcal{S}^{d}_\mu)^* \right) \right\} \quad (\lambda, \mu) \in \Lambda^+_{(k+d)\times n} \times \Lambda^+_{d\times(n-d)} \] 

(10.5) give a basis for $C_d$ as a right $OH^d_n$-supermodule. Hence, as a graded right $OH^d_n$-supermodule, $C_d$ is free of graded superrank $c_{n+k,n}^{(d)}$. It is also free as a graded left $OH^d_n$-superbimodule with the same graded superrank.

**Proof.** Lemma 9.4 implies that it is free as a graded right $OH^d_n$-supermodule with basis (10.5). The formula for its graded superrank then follows using Corollary 3.2. It is also free as a graded left $OH^d_n$-supermodule thanks to Lemma 9.4 again. Since $OH^d_n \cong OH^d_n$, its graded superrank for $OH^d_n$ is the same as for $OH^d_n$.

**Lemma 10.7.** Suppose we are given that $\dim \im \tilde{\delta}_d \geq |b_{n+k,n}(d)|$ for $d = 1, \ldots, n$, where $|b_{n+k,n}(d)|$ denotes the natural number obtained by applying the evaluation map $\mathbb{Z}^q[q, q^{-1}] \to \mathbb{Z}$, $q \mapsto 1, \pi \mapsto 1$ to $b_{n+k,n}(d)$. Then Theorem 10.3 is true.

**Proof.** We first consider the specialized complex (10.6), showing that $\im \tilde{\delta}_d = \ker \tilde{\delta}_{d-1}$ and that it is of graded superdimension $b_{n+k,n}(d)$ for each $d = 1, \ldots, n$. This follows by induction on $d$, defining $\tilde{\delta}_0$ to be the zero map so that we can start the induction at $d = 0$. The induction base holds because $b_{n+k,n}(0) = 0$. For the induction step, take $0 \leq d < n$ and assume that $\im \tilde{\delta}_d = \ker \tilde{\delta}_{d-1}$ is of graded superdimension $b_{n+k,n}(d)$. We have that $\tilde{C}_d = \mathbb{F}_d \oplus \mathbb{Z}_d$ where $\mathbb{Z}_d := \ker \tilde{\delta}_d$ and $\mathbb{F}_d = \im \tilde{\delta}_d$ is a complementary graded superspace. By induction, $\dim \mathbb{Z}_d = b_{n+k,n}(d)$. We have that $\dim \tilde{C}_d = \dim \im \tilde{\delta}_d + \dim \ker \tilde{\delta}_d$ so, using Corollary 3.4 for the final equality, get that 

\[ |c_{n+k,n}(d)| = |b_{n+k,n}(d)| + \dim \ker \tilde{\delta}_d \geq |b_{n+k,n}(d)| + \dim \im \tilde{\delta}_{d+1} \geq |b_{n+k,n}(d)| + |b_{n+k,n}(d+1)| = |c_{n+k,n}(d)|. \]

This means that equality holds throughout, thereby proving that $\im \tilde{\delta}_{d+1} = \ker \tilde{\delta}_d$. The same sequence of equalities without evaluating at $q = \pi = 1$ now gives that $\dim \im \tilde{\delta}_{d+1} = b_{n+k,n}(d+1)$, and the argument is complete.
Next we show that \( \im \partial_d = \ker \partial_{d-1} \) and that it is free as a graded right \( \text{OH}_{n'}^{\ell} \)-supermodule of graded superrank \( b_{n+k,n}(d) \) for each \( d = 1, \ldots, n \). This is a similar induction to the one in the previous paragraph. For the induction step, we take \( 0 \leq d < n \) and assume that we have shown already that \( \im \partial_d \) is free of graded superrank \( b_{n+k,n}(d) \). Consider the short exact sequence
\[
0 \longrightarrow Z_d \longrightarrow C_d \longrightarrow \im \partial_d \longrightarrow 0
\]
where \( Z_d := \ker \partial_d \). Since \( \im \partial_d \) is free, this short exact sequence splits, so we have that \( C_d = B'_d \oplus Z_d \) where \( B'_d = \im \partial_d \) is a complement to \( Z_d \) in \( C_d \) as a graded right \( \text{OH}_{n'}^{\ell} \)-supermodule. Moreover, \( \bar{Z}_d \) from the previous paragraph is \( Z_d \otimes 1 \). As it is a summand of \( C_d \), which is free, we deduce that \( Z_d \) is a free graded right \( \text{OH}_{n'}^{\ell} \)-supermodule with \( \text{rk} \bar{Z}_d = \dim_{q,x} \bar{Z}_d = b_{n+k,k}(d+1) \). The map \( \partial_{d+1} : C_{d+1} \rightarrow Z_d \) is surjective because \( \text{id} \otimes \partial_{d+1} : \bar{C}_{d+1} \rightarrow \bar{Z}_d \) is surjective according to the previous paragraph. We deduce that \( \im \partial_{d+1} = \ker \partial_{d} \) is free of graded superrank \( b_{n+k,k}(d+1) \), and the argument is complete.

So now we have shown that \( \im \partial_d = \ker \partial_{d-1} \) is free as a graded right \( \text{OH}_{n'}^{\ell} \)-supermodule of graded superrank \( b_{n+k,n}(d) \) for \( d = 1, \ldots, n \). The same is true as a graded left \( \text{OH}_{n'}^{\ell} \)-supermodule since \( \text{OH}_{n'}^{\ell} \cong \text{OH}_{n'}^{\ell} \) and all of the numerology is the same.

To complete the proof of Theorem 10.3, it just remains to prove the assertion about the top degree homology. As \( \text{OH}_{n'}^{\ell} \cong \text{OH}_{n'}^{\ell} \), it suffices to show that it is free of graded superrank \( (\pi q^2)^{(n+1)+nk} \) both as a graded right \( \text{OH}_{n'}^{\ell} \)-supermodule and as a graded left \( \text{OH}_{n'}^{\ell} \)-supermodule. We have already shown that the image of \( \partial_n \) is free of graded superrank \( b_{n+k,k}(n) \). Hence, since \( C_n \) is free of graded superrank \( c_{n+k,n}(n) \) by Lemma 10.6, we deduce that \( \ker \partial_n \) is free of graded superrank \( c_{n+k,n}(n) - b_{n+k,k}(n) \). Thus we are reduced to showing that \( c_{n+k,n}(n) - b_{n+k,k}(n) = (\pi q^2)^{(n+1)+nk} \). This follows from the following calculation:
\[
c_{n+k,n}(n) - b_{n+k,k}(n) = (\pi q^{-2})^{(\ell)} q^{(k)n} \left[ \begin{array}{c} 2n + k \\ n \end{array} \right]_{q,x} \\
\quad - (\pi q^{-2})^{(\ell)} q^{(k)n} \sum_{s=0}^{n-1} (\pi q^{-2})^{(m(n-s-1)q^{(m-1)(s+1)} \left[ \begin{array}{c} n + k + s \\ s + 1 \end{array} \right]_{q,x}} \\
\quad = (\pi q^{-2})^{(\ell)} q^{(k)n} \sum_{s=0}^{n} (\pi q^{-2})^{(n+k)(n-s)} q^{(n+k-1)s} \left[ \begin{array}{c} n + k + s - 1 \\ s \end{array} \right]_{q,x} \\
\quad - (\pi q^{-2})^{(\ell)} q^{(k)n} \sum_{s=1}^{n} (\pi q^{-2})^{(n+k)(n-s)} q^{(n+k-1)s} \left[ \begin{array}{c} n + k + s - 1 \\ s \end{array} \right]_{q,x} \\
\quad = (\pi q^{-2})^{(n+k)n-(\ell)} = (\pi q^2)^{(n+1)+nk}.
\]
Here, we have used the definitions in Lemma 3.3 for the first equality and Corollary 3.4 for the second.

\[\square\]

**Remark 10.8.** From Theorem 10.1(1)–(2), tensoring with the \( d \)-th term \( U_{k+d,n-d}^{\ell} \otimes \text{OH}_{n-d}^{\ell} \) in the singular Rouquier complex corresponds at the level of Grothendieck groups to the \( \mathbb{Z}[q, q^{-1}] \)-module homomorphism \( 1_{-k} \mathbf{V}(-\ell) \rightarrow 1_k \mathbf{V}(-\ell) \) defined by the action of
\[
q^{(n-d)(k+d) + d(3(n-d) - 2d + 1)E^{(k+d)}} F^{(d)} 1_{2n-\ell} = q^{nk} \left( q^d E^{(k+d)} F^{(d)} 1_{2n-\ell} \right).
\]
From the original definition of the map \( T \) in Theorem 3.6, it follows that multiplication by the Euler characteristic of (10.4) corresponds to \( q^{nk} T : 1_{-k} \mathbf{V}(-\ell) \rightarrow 1_k \mathbf{V}(-\ell) \). Given that the homology of the complex vanishes in all but the top degree, it then follows by (10.2) that the top homology is \( (\pi q^2)^{(n+1)+nk} \left[ \text{OH}_{n'}^{\ell} \right] \).

This gives a reassuringly different way to check the final assertion of Theorem 10.3.
Proof of Theorem 10.3. Fix $d$ with $1 \leq d \leq n$. Define an initial pair to be $(\lambda, \mu) \in \Lambda_{(k+d)\times n}^+ \times \Lambda_{d\times(n-d)}^+$ such that $\lambda_{k+d} = d - 1 - s$ and $\mu_{d-s+1} = n - d$ for some $0 \leq s \leq d - 1$. This condition is illustrated by the following picture:

Let $I$ be the set of all initial pairs. Note that

$$|I| = |b_{n+k,n}(d)|. \quad (10.7)$$

To see this, the number of $(\lambda, \mu) \in I$ with $\lambda_{k+d} = d - 1 - s$ and $\mu_{d-s+1} = n - d$ is $|\Lambda_{(k+d-1)n-d+s+1}^+ \times \Lambda_{n-d\times n}^-|$, which is $n^{k+s}(n-d-s)!$. Summing over $s = 0, 1, \ldots, d - 1$ gives $|b_{n+k,n}(d)|$ by the original definition of this natural number.

Also define a terminal pair to be $(\kappa, \nu) \in \Lambda_{(k+d-1)\times n}^+ \times \Lambda_{d-1\times(n-d+1)}^+$ such that $k_{k+d-1} \geq d - 1 - s$ and $\nu_{d-s} < \nu_{d-s-1} = n - d + 1$ for some $0 \leq s \leq d - 1$:

Let $T$ be the set of all terminal pairs. Our final combinatorial observation is that there is a bijection

$$f : I \sim T \quad (10.8)$$

taking $(\lambda, \mu) \in I$ to $(\lambda^-, \mu^+) \in T$ where $\lambda^-$ is obtained from $\lambda$ by removing the bottom row of its Young diagram, and $\mu^+$ is obtained from $\mu$ by adding one box to the end of each row of maximal length $n - d$ then removing completely the top row.

Now we are going to make an explicit computation of the differential $\partial_d$ in terms of the basis for $\overline{C}_d = C_d \otimes_{OH_n^c} \overline{V}$ consisting of the vectors

$$w(\lambda, \mu) := u_{n-d,(k+d)}(s_{\lambda}^d)^* \otimes v_{n-d,(d)}(s_{\mu}^d)^* \in U_{(k+d),(n-d)}^c \otimes_{OH_n^c} V_{n-d,d}^c \otimes_{OH_n^c} \overline{V} \quad (10.9)$$

for $(\lambda, \mu) \in \Lambda_{(k+d)\times n}^+ \times \Lambda_{d\times(n-d)}^+$; cf. Lemma 10.6. Order pairs $(\kappa, \nu) \in \Lambda_{(k+d-1)\times n}^+ \times \Lambda_{d-1\times(n-d+1)}^+$ so that $(\kappa', \nu') < (\kappa, \nu)$ if either $|\kappa'| < |\kappa|$, or $|\kappa'| = |\kappa|$ and $\nu' \leq_{\text{lex}} \nu$. We claim for $(\lambda, \mu) \in I$ that

$$\overline{\partial}_d(w(\lambda, \mu)) = \pm w(\lambda^-, \mu^+) + \text{(a linear combination of } w(\kappa, \nu) \text{ for } (\kappa, \nu) < (\lambda^-, \mu^+)). \quad (10.10)$$

The $\pm$ here indicates some sign which we will not determine exactly. Given the claim, it follows by (10.7) and (10.8) that $|\text{im } \overline{\partial}_d| \geq |b_{n+k,n}(d)|$, so that the theorem follows by Lemma 10.7.

---

8This is all we need here, but with a little more care using also Corollary 3.2, this argument can be used to show that the coefficient of $q^r \pi'$ in $b_{n+k,n}(d)$ is equal to the number of $(\lambda, \mu) \in I$ with $|\lambda| + |\mu| = 2r$, explaining the definition of $b_{n+k,n}(d)$ itself rather than merely its evaluation at $q = \pi = 1$. 
It remains to prove (10.10). Take \((\lambda, \mu) \in \Lambda^+_n(\lambda \times n) \times \Lambda^+_d(n - d)\) and consider \(\overline{\partial}_d(w(\lambda, \mu))\). According to the definition of \(\overline{\partial}_d\), we have to apply three different maps to \(w(\lambda, \mu)\) arising from \(c'_d(\lambda \times n, (d - 1)\) and \(\text{ev}_{n-d}\). We apply these maps one by one.

- First, the map \(c'_{d(k-d)}(1)\) comes from the embedding 
  \[\text{OSym}_{d} \hookrightarrow \text{OSym}_{d-1} \sim \text{OSym}_{d-1} \otimes \mathbb{F}[x].\]
  The version of the odd Pieri formula obtained by applying \(*\) to Lemma 6.15, shows that this embedding takes \((s^{(k+d)}_{\mu})^*\) to \(\sum_{\kappa,\nu} s^{(k+d)}_{\kappa} \otimes s^{(d-1)}_{\nu}\) summing over all \(\kappa \in \Lambda^+_n(\lambda \times n)\) whose Young diagram is obtained by removing boxes from the bottoms of columns of the Young diagram of \(\lambda\), necessary including all \(\lambda_{k+d}\) boxes on its \((k+d)\)th row. We say simply \(\kappa\) obtained by removing a row strip from \(\lambda\) for this from now on.

- Next, we apply the map \(c_{(d-1)(d-1)}\), which comes from the embedding 
  \[\text{OSym}_{d-1} \hookrightarrow \text{OSym}_{d-1} \sim \mathbb{F}[x] \otimes \text{OSym}_{d-1},\]
  plus some extra signs due to the parity shift. The version of Pieri obtained by applying \(\epsilon_d\) to Lemma 6.15 using also (6.7) shows that this takes \((s^{(d+1)}_{\mu})^*\) to \(\sum_{\delta} s^{(d-1)}_{\delta} \otimes \sum_{\kappa,\nu} s^{(d-1)}_{\kappa} \otimes s^{(d-1)}_{\nu}\) summing over \(\delta \in \Lambda^+_d(n - d)\) whose Young diagram is obtained by removing boxes from the bottoms of columns of the Young diagram of \(\mu\), necessary including all \(\mu_{d}\) boxes on its \(d\)th row, to obtain the Young diagram of partition \(\delta\). We say simply \(\delta\) obtained by removing a row strip from \(\mu\) for this from now on.

- So far, we have shown that \(c'_d(\lambda \times n, (d - 1)\) and \(\text{id}\) o \((\text{inc} \otimes \text{id})\) takes \(w(\lambda, \mu)\) to 
  \[
\sum_{(\kappa, \delta)} \pm u_{(d-1)n-d-1}\left(\left(s^{(k+d-1)}_{\kappa}\right)^* \otimes v_{n-d}(s^{(d-1)}_{\delta}) \otimes v_{n-d+1;1}(s^{(d-1)}_{\delta})^*\right) \oplus 1
\]
  summing over \((\kappa, \delta) \in \Lambda^+_n(\lambda \times n) \times \Lambda^+_d(n - d)\) such that \(\kappa\) is obtained by removing a row strip from \(\lambda\) and \(\delta\) is obtained by removing a row strip from \(\mu\). Then we use the definition in Theorem 9.11 to apply \((\text{can} \otimes \text{id})\) o \((\text{id} \otimes \text{ev}_{n-d} \otimes \text{id} \otimes \text{id})\), giving 
  \[
\overline{\partial}_d(w(\lambda, \mu)) = \sum_{(\kappa, \delta)} \pm u_{(d-1)n-d-1}\left(\left(s^{(k+d-1)}_{\kappa}\right)^* \otimes \overline{\gamma}_{(n-d+1)}(\lambda \times n) \times \Lambda^+_d(n - d)\) \otimes 1.
\]

It remains to commute the elements \(\overline{\gamma}_{p}(n-d+1)(\lambda \times n) \times \Lambda^+_d(n - d)\) to the right hand side in this expression. In view of Lemma 9.4(1) and degree considerations, this will produce some linear combination of basis vectors of the form \(w(\kappa, \nu)\) for \(\nu \in \Lambda^+_d(n - d)\) with \(|\nu| = |\kappa| + (|\lambda| - |\kappa|) - (n - d)\). We just need to show that \(w(\lambda^-, \mu^+)\) appears with coefficient \(\pm 1\) and all other \(w(\lambda^+, \nu)\) that arise satisfy \((\nu, \nu) < (\lambda^-, \mu^+)\). This is clearly the case if \(|\kappa| < |\lambda^-|\), so we may assume from now on that \(\kappa\), like \(\lambda^+\), is obtained from \(\lambda\) by removing the minimal number of boxes, i.e., just its bottom row. So we have that \(\kappa = \lambda^+\) and \(|\lambda| - |\kappa| = \lambda_{k+d}\), which equals \(d - s - 1\) for a unique \(0 \leq s < d\). Also let \(p := (d - s + 1) + (|\mu| - |\nu|) - (n - d)\) for short and consider 
  \[
\overline{\gamma}_{p}(n-d+1)v_{n-d+1;1}(s^{(d-1)}_{\delta})^* \otimes 1,
\]
  By Theorem 6.2, we have that \(\overline{\gamma}_{p}(n-d+1) = \overline{\gamma}_{p}(n-d+1)\) plus a sum of other \(\overline{\gamma}_{p}(n-d+1)\) for partitions \(\tau\) with \(|\tau| = p\) and \(\text{ht}(\tau) > 1\). Also \(s^{(d+1)}_{(p)} = e_{(p)}\) and \(\text{ht}(\tau) > 1\). Also \(s^{(d+1)}_{(p)} = e_{(p)}\). So by Lemma 9.5, we deduce that 
  \[
\overline{\gamma}_{p}(n-d+1)v_{n-d+1;1}(s^{(d-1)}_{\delta})^* \otimes 1 = \pm v_{n-d+1;1}(s^{(d-1)}_{\delta})^* \otimes 1 + (\ast)\]
where $(\ast)$ is a linear combination of terms of the form $v_{n-d+1;\delta}(s^{(d-1)}_{\delta}) \otimes 1$ for partitions $\tau$ with $|\tau| = p$ and $\text{ht}(\tau) > 1$. We can compute all of these products of Schur polynomials using the odd Littlewood-Richardson rule; see the discussion in the last paragraph of Section 6. Remembering also that $v_{n-d+1;\delta}(s^{(d-1)}_{\delta}) \otimes 1 = 0$ unless $\nu \in \Lambda^+_{(d-1) \times (n-d+1)}$ by Lemma 9.5 again, we obtain a linear combination of basis vectors $v_{n-d+1;\delta}(s^{(d-1)}_{\delta}) \otimes 1$ for $\nu \in \Lambda^+_{(d-1) \times (n-d+1)}$ obtained from $\delta$ by adding $p$ boxes in particular ways. If $p < d - s - 1$ then we cannot have $\nu = \mu^\ast$ since that has $d - s + 1$ boxes in the rightmost column, whereas that column is empty in $\delta$. Now suppose that $p = d - s - 1$; then $\delta$ is $\mu$ with all $(n - d)$ boxes in its first row removed. In this case, the leading term of $v_{n-d+1;\delta}(e^{(d-1)}_{\mu^\ast} s^{(d-1)}_{\delta}) \otimes 1$ computed via the odd Littlewood-Richardson rule does produce $\pm v_{n-d+1;\delta}(s^{(d-1)}_{\mu^\ast}) \otimes 1$ when a column strip of $p$ boxes is added at the top right of the Young diagram of $\delta$. All other basis vectors coming from this leading term are of the form $v_{n-d+1;\delta}(s^{(d-1)}_{\mu^\ast}) \otimes 1$ for $\nu < \mu^\ast$. The basis vectors coming from the lower terms $v_{n-d+1;\delta}(s^{(d-1)}_{\mu^\ast}) \otimes 1$ for $\tau$ with $|\tau| = p$ and $\text{ht}(\tau) > 1$ must also be of the form $v_{n-d+1;\delta}(s^{(d-1)}_{\mu^\ast}) \otimes 1$ for $\nu < \mu^\ast$ since for $\text{ht}(\tau) > 1$ the odd Littlewood-Richardson rule does not allow all $p$ boxes to be added to the same column of the Young diagram of $\delta$. \end{proof}

**Remark 10.9.** The proof of Theorem 10.3 also works in the extended setting of Remarks 4.14, 7.9, 8.10 and 9.14 without any significant differences. The approach taken in this section can also be “reverse engineered” to give yet another approach to the singular Rouquier complex in the purely even setting. The matrix describing the differential $\partial_k$ in terms of the natural Schur bases as computed in the proof of Theorem 10.3 is genuinely different in the odd case compared to the even case; this is not just a matter of some extra signs. This is because in the even case the correction terms $(*$) in (10.11) do not arise since in that case there is no such thing as the dual complete symmetric polynomial $y^{(n-d-1)}_p$—one simply replaces it with $h^{(n-d-1)}_p$ throughout.

### 11. The reduced odd 2-category $\mathcal{U}(\mathfrak{s}_2)$ and its Grothendieck Ring

The following definition originated in [EL] and was reformulated in the present terms in [BE2]. We are using the string calculus for strict monoidal supercategories, our conventions being the ones from [BE1]. In particular, $f \circ g$ is vertical composition ($f$ on top of $g$) and $f \otimes g$ or simply $fg$ is horizontal composition ($f$ to the left of $g$).

**Definition 11.1.** The (non-reduced) odd $\mathfrak{s}_2$ 2-category is the (strict) graded 2-supercategory $\widetilde{\mathcal{U}}(\mathfrak{s}_2)$ with object set $\mathbb{Z}$, generating 1-morphisms $E1_k = 1_{k+2} E : k \to k + 2$ and $1_k F = F1_{k+2} : k + 2 \to k$ for each $k \in \mathbb{Z}$ whose identity 2-morphisms are represented graphically by $\uparrow k = k + 2$ and $\downarrow k = k + 2$, respectively, and generating 2-morphisms

\begin{align}
\uparrow^k : E1_k \Rightarrow E1_k & \quad \downarrow^k : E^2 1_k \Rightarrow E^2 1_k \\
\bigcup^k : FE1_k \Rightarrow 1_k & \quad \bigcap^k : 1_k \Rightarrow EF1_k
\end{align}

(11.1)

which are odd of degree 2, odd of degree $-2$, even of degree $k + 1$, and even of degree $1 - k$, respectively. Then there are three families of relations. First we have the odd nil-Hecke relations (in the standard formulation rather than the modified version from (5.1) to (5.6)):

\begin{align}
\delta_k & = 0 \\
\phi_k & + \psi_k = \delta_k \\
\phi^k & + \psi^k = \delta^k
\end{align}

(11.2)
Next we have the **right adjunction relations** asserting that $Q^{-k-1}F_{1+k+2}$ is right dual to $E_{1_k}$ in the $(Q, \Pi)$-envelope of $\hat{U}(sl_2)$:

\[
\bigcup^k = \bigcup^k \quad \quad \quad \quad k \bigcap = k
\]  
\[
(11.3)
\]

Finally there are some **inversion relations**. To formulate these, we first introduce new 2-morphisms

\[
\bigotimes^k := \bigotimes^k : EF_{1_k} \to FE_{1_k}.
\]  
\[
(11.4)
\]

Then, denoting powers of the dot generator by labelling it with a natural number, we require that the following (not necessarily homogeneous) 2-morphisms are isomorphisms:

\[
\begin{pmatrix}
\bigotimes^k \\
\bigotimes^k \\
\bigotimes^k \\
\bigotimes^k \\
\bigotimes^k \\
\end{pmatrix} : EF_{1_k} \sim FE_{1_k} \oplus 1_{++}^k \quad \quad \text{for } k \geq 0 \quad (11.5)
\]

\[
\begin{pmatrix}
\bigotimes^k \\
\bigotimes^k \\
\bigotimes^k \\
\bigotimes^k \\
\bigotimes^k \\
\end{pmatrix} : EF_{1_k} \oplus 1_{--}^{k+1} \sim FE_{1_k} \quad \quad \text{for } k \leq 0. \quad (11.6)
\]

The morphisms depicted in (11.5) and (11.6) represent a $(k+1) \times 1$ matrix and a $1 \times (1-k)$ matrix of 2-morphisms in $\hat{U}(sl_2)$, respectively, i.e., they are 2-morphisms in the additive envelope of $\hat{U}(sl_2)$. Saying that they are isomorphisms means that there are some further generating 2-morphisms in $\hat{U}(sl_2)$ which provide the matrix entries of two-sided inverses to these morphisms.

The defining relations (11.1) to (11.3), (11.5) and (11.6) look quite innocent but they imply many further relations. In order to record some of these, we first need to introduce some further shorthand for generating 2-morphisms whose existence is provided by the inversion relation. First, we have the leftward crossing and the leftward cups and caps

\[
\bigotimes^k : FE_{1_k} \Rightarrow EF_{1_k}
\]

\[
\bigcup^k : 1_kE_{1} \Rightarrow 1_k
\]

\[
\bigcap^k : 1_k \Rightarrow FE_{1_k}
\]

which are defined as follows.

- We let $\bigotimes^k : FE_{1_k} \Rightarrow EF_{1_k}$ be the **negation** of the leftmost entry of the $1 \times (k+1)$ matrix that is the two-sided inverse of (11.5) if $k \geq 0$, or the **negation** of the top entry of the $(-k-1) \times 1$ matrix that is the two-sided inverse of (11.6) if $k \leq 0$.

- We let $\bigcup^k$ be the rightmost entry of the $1 \times (k+1)$ matrix that is the two-sided inverse of (11.5) if $k > 0$, or $(-1)^{k+1} \bigcup^k$ if $k \leq 0$.

- We let $\bigcap^k$ be the bottom entry of the $(-k-1) \times 1$ matrix that is the two-sided inverse of (11.6) if $k < 0$, or $(-1)^{k+1} \bigcap^k$ if $k \geq 0$.

Finally, we have the downward dot and the downward crossing, which are the right mates of the upward dot and the upward crossing:

\[
\begin{pmatrix}
\bigcup^k \\
\bigcup^k \\
\bigcup^k \\
\bigcup^k \\
\bigcup^k \\
\end{pmatrix} : 1_kF \Rightarrow 1_kF
\]

\[
\begin{pmatrix}
\bigcap^k \\
\bigcap^k \\
\bigcap^k \\
\bigcap^k \\
\bigcap^k \\
\end{pmatrix} : 1_kF^2 \Rightarrow 1_kF^2.
\]  
\[
(11.8)
\]
The following table summarizes the parity and degree information about all of the 2-morphisms defined thus far.

<table>
<thead>
<tr>
<th>Generator</th>
<th>Degree</th>
<th>Parity</th>
<th>Generator</th>
<th>Degree</th>
<th>Parity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \downarrow^k )</td>
<td>2</td>
<td>1</td>
<td>( \downarrow^k )</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( \times^k )</td>
<td>-2</td>
<td>( \bar{1} )</td>
<td>( \times^k )</td>
<td>0</td>
<td>( \bar{1} )</td>
</tr>
<tr>
<td>( \times^k )</td>
<td>-2</td>
<td>( \bar{1} )</td>
<td>( \times^k )</td>
<td>0</td>
<td>( \bar{1} )</td>
</tr>
<tr>
<td>( \times^k )</td>
<td>( k + 1 )</td>
<td>0</td>
<td>( \times^k )</td>
<td>( k + 1 )</td>
<td>( \bar{1} )</td>
</tr>
<tr>
<td>( \times^k )</td>
<td>( 1 - k )</td>
<td>0</td>
<td>( \times^k )</td>
<td>( k + 1 )</td>
<td>( \bar{1} )</td>
</tr>
</tbody>
</table>

The following relations are derived from the defining relations in [BE2].

- **Downward odd nil-Hecke relations.**
  \[
  k = 0 \quad k \times = k \quad k \text{ + } k = k \text{ + } k = -k \tag{11.10}
  \]

- **Left adjunction relations.**
  \[
  k \bigcup = k \quad \bigcup = (-1)^{k+1} \bigcup k
  \tag{11.11}
  \]

- **Infinite Grassmannian relation.** Recall that \( \hat{R} \) is the largest supercommutative quotient of \( OSym \) described explicitly in Theorem 4.11. For each \( k \in \mathbb{Z} \), there is a graded superalgebra homomorphism
  \[
  \hat{\beta}_k : \hat{R} \to \text{End}_{\mathbb{U}(sl_2)}(1_k), \quad \hat{e}_r \mapsto r + k - 1 \bigcup k \text{ if } r \geq 1 - k, \quad \hat{h}_r \mapsto k \bigcup r - k - 1 \text{ if } r \geq k + 1. \tag{11.12}
  \]

Following Lauda’s convention from [L1, L2], we introduce new shorthands for endomorphisms of \( 1_k \) called “fake bubbles”: we have clockwise bubbles decorated by \( r + k - 1 \) dots on their left boundary for all \( r \leq -k \) which denote \( \hat{\beta}_k(\hat{e}_r) \) if \( r \geq 0 \) or 0 if \( r < 0 \), and we have counterclockwise bubbles decorated by \( r - k - 1 \) dots on their right boundary for all \( r \leq k \) which denote \( \hat{\beta}_k(\hat{h}_r) \) if \( r \geq 0 \) or 0 if \( r < 0 \).

- **Centrality of the odd bubble.** The “odd bubble” \( \bigotimes_k \) is shorthand for \( \hat{\beta}_k(\hat{o}) \in \text{End}_{\mathbb{U}(sl_2)}(1_k) \).
  So it is \( \bigotimes_k \) if \( k \geq 0 \) or \( \bigotimes_k \) if \( k \leq 0 \). These are odd 2-morphisms whose square is zero. Moreover, they are strictly central:
  \[
  \bigotimes \bigcup = \bigcup \bigotimes \quad \bigotimes \downarrow = \downarrow \bigotimes \tag{11.13}
  \]

- **Pitchfork relations.**
  \[
  \bigotimes \bigcup = \bigcup \bigotimes \quad \bigotimes \downarrow = \downarrow \bigotimes \tag{11.14}
  \]

- **Dot slides.**
  \[
  \bigotimes \bigcup = k \bigotimes \bigcup \quad \bigotimes \downarrow = k \bigotimes \downarrow \tag{11.15}
  \]

\[
\bigotimes \bigcup = (-1)^{k+1} \bigotimes \bigcup + (-1)^k 2 \bigotimes \bigcup \quad \bigotimes \downarrow = (-1)^{k+1} \bigotimes \downarrow + 2 \bigotimes \downarrow
\tag{11.17}
\]
Definition 11.2. The reduced odd \( \mathfrak{sl}_2 \)-2-category is the (strict) graded 2-supercategory \( \mathfrak{U}(\mathfrak{sl}_2) \) with the same objects and 1-morphisms as in \( \widehat{\mathfrak{U}}(\mathfrak{sl}_2) \), but with 2-morphisms defined by imposing the additional relations \( \otimes k = 0 \) for all \( k \in \mathbb{Z} \). (This is a reasonable thing to consider due to the centrality of the odd bubbles discussed above.)

We will use the same diagrams to denote 2-morphisms in the reduced version as in the non-reduced case, relying on context to avoid ambiguity. Note in particular that the homomorphism \( \beta_k : R \to \text{End}_{\mathfrak{U}(\mathfrak{sl}_2)}(1_k) \) from (11.12) factors through the quotient \( R \) of \( \widehat{R} \) described in Corollary 4.12, that is, the usual ring of symmetric functions, to induce a homomorphism

\[ \beta_k : R \to \text{End}_{\mathfrak{U}(\mathfrak{sl}_2)}(1_k) \]

for any \( k \in \mathbb{Z} \). The following theorem is an odd analog of [L2, Th. 4.12]. Our proof is shorter since we are using the more efficient presentation of Definition 11.1 (although afterwards there is still work to do to determine the images of the leftward cups and caps).

Theorem 11.3. Fix \( \ell \geq 0 \). There is a graded 2-superfunctor \( \Psi_\ell : \mathfrak{U}(\mathfrak{sl}_2) \to \text{OGBim}_\ell \) with the following properties.

1. On objects, \( \Psi_\ell \) takes \( 2n - \ell \) to the graded superalgebra \( \text{OH}_n^{\ell} \) for \( 0 \leq n \leq \ell \). All other objects of \( \mathfrak{U}(\mathfrak{sl}_2) \) go to the trivial graded superalgebra.

2. On generating 1-morphisms, \( \Psi_\ell \) takes \( E_{2n-\ell} \) to the graded superbimodule \( Q^{-n}U_n^{\ell} \) and \( 1_{2n-\ell}F \) to the graded superbimodule \( Q^{3n-\ell+1}V_n^{\ell} \), respectively, both for \( 0 \leq n \leq \ell \). All other generating 1-morphisms necessarily go to trivial graded superbimodules.

3. On generating 2-morphisms, \( \Psi_\ell \) takes

\[ \sum_{r,k=0}^{2n-\ell} \left( \rho_{(1),n}(x_1) : Q^{-n}U_n^{\ell} \to Q^{-n}U_n^{\ell} \right) \]

and

\[ \sum_{r,k=0}^{2n-\ell} \left( -\rho_{(2),n}(\tau_1) : Q^{-n-1}U_{n+1}^{\ell} \otimes_{\text{OH}_n^{\ell}} Q^{-n}U_n^{\ell} \to Q^{-n-1}U_{n+1}^{\ell} \otimes_{\text{OH}_n^{\ell}} Q^{-n}U_n^{\ell} \right) \]

\[ (0 \leq n \leq \ell - 1) \]
Here, $\rho_{(1 \cdots n)(a)}$ is the superbimodule endomorphism from (9.45), and $\text{ev}_n$ and $\text{coev}_n$ are as in Theorem 9.11; these maps are being applied to superbimodules which are shifted in degree but of the same parities as before. All other generating 2-morphisms are taken to zero.

Proof. Note to start with that the assignments in (3) are superbimodule homomorphisms of the correct degrees and parities; cf. (11.9). Viewing $O\tilde{G}Bim_{\rho}$ as a strict graded 2-supercategory as explained in Remark 9.9, we will simply construct $\Psi_\ell$ as a strict graded 2-superfunctor by checking that the defining relations from Definition 11.1 are all satisfied. There are three sets of relations, (11.2), (11.3) and (11.5)–(11.6), plus at the end we also need to check that the images of all odd bubbles are zero since we are considering $\mathcal{U}(sl_2)$ rather than $\widetilde{\mathcal{U}}(sl_2)$.

The right adjunction relations (11.3) follow immediately from Theorem 9.11.

Let us check the odd nil-Hecke relations from (11.2). The formulation of these relations in (11.2) differs by signs from the formulation in (5.1) to (5.6). This discrepancy is explained by the signs in formula (9.10). To be clear about this, for $a \in ONH_d$, let $\rho_{(1 \cdots n)(a)}$ be the $(OH^\ell_{n+d}, OH^\ell_n)$-superbimodule endomorphism from (9.45) viewed now as an endomorphism of the degree-shifted $Q^{-n-d}U^\ell_{n+d-1} \otimes OH^\ell_{n+d-1}$, $\cdots \otimes OH^\ell_n Q^nU^\ell_n$, for $0 \leq d \leq n$ and $0 \leq n \leq \ell - d$. The definition from (3) implies more generally that

$$\Psi_\ell \left( \sum_{\iota(i)} \iota(1) \cdots \iota(d) \right) = (-1)^{j-1} \rho_{(1 \cdots n)(a)}(x_j),$$

(11.23)

and

$$\Psi_\ell \left( \sum_{\iota(i)} \iota(1) \cdots \iota(d) \right) = (-1)^{j-1} \rho_{(1 \cdots n)(a)}(x_j).$$

(11.24)

We check (11.24), leaving the easier (11.23) to the reader. We must show that

$$\text{id} \otimes \cdots \otimes \rho_{(1 \cdots n)(j+1)}(\tau_1) \otimes \cdots \otimes \text{id} = (-1)^{j-1} \rho_{(1 \cdots n)(j)}(\tau_j).$$

We do this by checking that both sides take the same value on $u_{n+d-1}(x^{e_1}) \otimes \cdots \otimes u_n(x^{e_n})$ for any $\kappa \in \mathbb{N}^d$. Let $\sum_{\iota(i)} \iota(1) \cdots \iota(d) \otimes$ denote summation over $\kappa' \in \mathbb{N}^d$ with $\kappa'_i = \kappa_j$ for $i \neq j, j + 1$. Suppose that $x_{j+1}^{e_{j+1}} \cdots x_j^{e_j} \cdot \tau_j = \sum_{\kappa'} c_{\kappa'} x_{j+1}^{e_{j+1}} \cdots x_j^{e_j}$. Then, using (9.10), the left hand side gives

$$\sum_{\kappa'} (-1)^{1+(\kappa_j-\kappa'_j) + \cdots + \kappa_d} c_{\kappa'} u_{n+d-1}(x^{e_j}) \otimes \cdots \otimes u_n(x^{e_1}).$$

and the right hand side gives

$$(-1)^{j-1} \sum_{\kappa'} (-1)^{d-1+(d-j)(\kappa_j-\kappa'_j) + \cdots + \kappa_d} c_{\kappa'} u_{n+d-1}(x^{e_j}) \otimes \cdots \otimes u_n(x^{e_1}).$$

These are equal because $\kappa_j - \kappa'_j + \kappa_{j+1} - \kappa_{j+1}' = 1$ whenever $c_{\kappa'} \neq 0$.

With (11.23) and (11.24) in hand, the relations (11.2) are easily checked. For example, to check the length three braid relation, we must show that

$$\rho_{(1 \cdots n)(\tau_2)} \circ (- \rho_{(1 \cdots n)(\tau_1)}) \circ \rho_{(1 \cdots n)(\tau_2)} = (- \rho_{(1 \cdots n)(\tau_1)}) \circ \rho_{(1 \cdots n)(\tau_2)} \circ (- \rho_{(1 \cdots n)(\tau_1)}).$$

Translating to $U^\ell_{(1 \cdots n)}$ using the isomorphism $c_{(1 \cdots n)}$, the left hand side becomes the map $u_{(1 \cdots n)}(f) \mapsto (-1)^{\text{par}(f)} u_{(1 \cdots n)}(f \cdot \tau_2 \tau_1 \tau_2)$ and the right hand side becomes $u_{(1 \cdots n)}(f) \mapsto (-1)^{\text{par}(f)} u_{(1 \cdots n)}(f \cdot \tau_1 \tau_2 \tau_1)$. These are equal due to the sign in the relation (5.5). To check the third relation in (11.2), we must show that

$$\rho_{(1 \cdots n)(x_1)} \circ (- \rho_{(1 \cdots n)(\tau_1)}) + \rho_{(1 \cdots n)(\tau_1)} \circ \rho_{(1 \cdots n)(x_2)} = \rho_{(1 \cdots n)(\text{id})}.$$
The left hand side corresponds to the map $u_{(1^2)}(f) \mapsto u_{(1^2)}(f \cdot \tau_1 x_1 - f \cdot x_2 \tau_1)$, which equals $u_{(1^2)}(f)$ by (5.6).

Next we check (11.5) and (11.6). There is nothing to do if $\ell = 0$ (the zero map is an isomorphism between zero superbimodules!) so assume that $\ell > 0$. Take a weight $k = 2n - \ell$ of $V(-\ell)$ for some $0 \leq n \leq \ell$. Setting $n' := \ell - n - 1$, we have that $k = n - n' - 1$. We will ignore grading shifts since they play no role in this place.

We first check (11.5), so $k \geq 0$ or, equivalently, $n \geq n' + 1$. We need to show that the superbimodule homomorphism $f$ defined by the $(n - n') \times 1$ matrix

$$
\begin{pmatrix}
\sigma_n & \text{ev}_{n-1} & \cdots & \text{ev}_{n-1} \circ \rho_{(1^2),n}(x_2)^{n-n' - 2} \\
\end{pmatrix}^T : U_{n-1} \otimes_{OH_n} V_{n-1}^\ell \to V_n^\ell \otimes_{OH_n} U_n^\ell \oplus (OH_n^\ell)^{(n-n' - 1)}
$$

is an isomorphism where $\sigma_n$, the image of the rightward crossing, is the superbimodule homomorphism described explicitly in Lemma 9.16, or the zero map in the extremal case $n = \ell$, $n' = -1$. By Lemma 9.10, the domain of $f$ is free as a right $OH_n^\ell$-supermodule with basis $\{u_{n-1}(x^r) \otimes v_{n-1}(x^s) \mid 0 \leq r \leq n' + 1, 0 \leq s \leq n - 1\}$, and the codomain of $f$ is free as a right $OH_n^\ell$-supermodule with basis $\{v_n(x^r) \otimes u_n(x^s) \mid 0 \leq r \leq n', 0 \leq s \leq n\} \cup \{b_1, \ldots, b_{n-n' - 1}\}$ where $b_i$ is the identity element in the $i$th copy of $OH_n^\ell$. Both of these sets are of size $nn' + 2n$, so it suffices to show that $f$ is surjective. To prove this, since $OH_n^\ell$ is graded local, it is enough to show that the homomorphism $\bar{f} := f \otimes 1$ obtained by applying the functor $- \otimes_{OH_n^\ell} \mathbb{F}$ is surjective. Let $uv(r, s)$ denote $u_{n-1}(x^r) \otimes v_{n-1}(x^s) \otimes 1$ and $vu(s, r)$ denote $v_n(x^r) \otimes u_n(x^s) \otimes 1$. Thus, the domain of $\bar{f}$ has linear basis $\{uv(r, s) \mid 0 \leq r \leq n' + 1, 0 \leq s \leq n - 1\}$ and the codomain has linear basis $\{vu(s, r) \mid 0 \leq r \leq n', 0 \leq s \leq n\} \cup \{b_1 \otimes 1, \ldots, b_{n-n' - 1} \otimes 1\}$. By (9.48) and Theorem 9.11, we have that

$$f(uv(r, s)) = \pm vu(n, r + s - n) \pm b_{n-r-s} \otimes 1 \pm vu(s, r)$$

(11.25)

for $0 \leq r \leq n' + 1$ and $0 \leq s \leq n - 1$, where the first term should be interpreted as zero if $r + s - n < 0$ and the middle term should be interpreted as zero if $n - r - s < 1$ or $n - r - s > n - n' - 1$. Note also that the last term $vu(s, r)$ is zero for $r > n'$ by degree considerations. The argument is completed with the following observations.

- We get the vectors $vu(n, r)$ for $0 \leq r \leq n'$ from the images of the basis vectors $uv(n' + 1, s)$ for $n - n' - 1 \leq s \leq n - 1$. Indeed, in the formula (11.25) for $f(uv(n' + 1, s))$ for these values of $s$, the second and third terms on the right hand side are both zero.
- Modulo the span of vectors already obtained, we get the vectors $b_1 \otimes 1, \ldots, b_{n-n' - 1} \otimes 1$ from the images of the basis vectors $uv(n' + 1, s)$ for $0 \leq s \leq n - n' - 2$. Indeed, in the formula for $f(uv(n' + 1, s))$ for these values of $s$ the third term on the right hand side is zero.
- Modulo the span of vectors already obtained, we get the vectors $vu(s, r)$ for $0 \leq r \leq n'$ and $0 \leq s \leq n - 1$ from the images of the remaining basis vectors $uv(r, s)$ for these values of $r$ and $s$.

Now consider (11.6), so $k \leq 0$ and $n' \geq n - 1$. We need to show that the superbimodule homomorphism $f$ defined by the $1 \times (n' + n + 2)$ matrix

$$
\begin{pmatrix}
\sigma_n & \text{coev}_n & \cdots & \rho_{(1^2),n}(x_1)^{n-n'} \circ \text{coev}_n \\
\end{pmatrix}^T : U_{n-1} \otimes_{OH_n} V_{n-1}^\ell \oplus (OH_n^\ell)^{(n-n' + 1)} \to V_n^\ell \otimes_{OH_n} U_n^\ell
$$

is an isomorphism, where $\sigma_n$ is as in Lemma 9.16 or the zero map in the extremal case $n = \ell$, $n' = -1$. By Lemma 9.10, the domain of $f$ is free as a right $OH_n^\ell$-supermodule with basis $\{u_{n-1}(x^r) \otimes v_{n-1}(x^s) \mid 0 \leq r \leq n' + 1, 0 \leq s \leq n - 1\} \cup \{b_1, \ldots, b_{n-n' - 1}\}$, where $b_i$ is the identity element in the $i$th copy of $OH_n^\ell$, and the codomain of $f$ is free as a right $OH_n^\ell$-supermodule with basis $\{v_n(x^r) \otimes u_n(x^s) \mid 0 \leq r \leq n', 0 \leq s \leq n\}$. Both of these sets are of size $nn' + n' + 1$, so it suffices to show that $f$ is surjective. Again, we apply $- \otimes_{OH_n^\ell} \mathbb{F}$ and show that the resulting map $\bar{f} := f \otimes 1$ is surjective. Let $uv(r, s) := u_{n-1}(x^r) \otimes v_{n-1}(x^s) \otimes 1$ and $vu(s, r) := v_n(x^r) \otimes u_n(x^s) \otimes 1$ for short. So the domain of $\bar{f}$ has linear basis $\{uv(r, s) \mid 0 \leq r \leq n' + 1, 0 \leq s \leq n - 1\}$.
where \( p \) (mod 2). The inverse of the map identity maps on the underlying vector spaces.

**Lemma 11.4.** These are both even of degree 0. To prove the lemma, we must show that \( \bar{v}u(r + s - n) \) as zero if \( r + s - n < 0 \) and \( s \). The proof is completed by the following.

- We get \( \bar{v}u(n, r) \) for \( 0 \leq r \leq n' - n \) from the images of the vectors \( b_i \otimes 1 \) for \( i = 1, \ldots, n' - n + 1 \).
- We get \( \bar{v}u(n, r) \) for \( n' - n + 1 \leq r \leq n' \) from the images of the vectors \( \bar{v}u(n' + 1, r) \) for \( 0 \leq r \leq n - 1 \). This uses the observation that \( \bar{v}u(r, n' + 1) = 0 \).
- Modulo the span of vectors already obtained, we get the remaining \( \bar{v}u(s, r) \) for \( 0 \leq s \leq n - 1 \) and \( 0 \leq r \leq n' \) from the images of the vectors \( \bar{v}u(s, r) \) for the same values of \( r \) and \( s \).

Finally, we consider the images of the odd bubbles. For \( 0 \leq n \leq \ell \), the image \( \Psi(e)(f) \) of \( f \in \text{End}_{\text{Bimod}}(1_{2n-\ell}) \) is an \((\text{OH}_{\ell}^{n}, \text{OH}_{\ell}^{n})\)-superbimodule endomorphism of \( \text{OH}_{\ell}^{n} \). Evaluating at \( 1 \in \text{OH}_{\ell}^{n} \) gives a bijection between such endomorphisms and the supercenter \( Z(\text{OH}_{\ell}^{n}) \) consisting of all (not necessarily homogeneous) elements that supercommute with all other elements of \( \text{OH}_{\ell}^{n} \). This means that to show that \( \bar{v}u(\otimes 2n-\ell) = 0 \), it suffices to show that \( Z(\text{OH}_{\ell}^{n}) \) contains no odd element of degree 1. By Theorem 7.4, the odd degree 1 component of \( \text{OH}_{\ell}^{n} \) is zero if \( n = 0 \) or \( n = \ell \), and it \( 0 < n < \ell \) then it one-dimensional, spanned by the element \( \bar{s}(n) = \bar{s}_1(n) \). Thus, it suffices to show for \( 0 < n < \ell \) that \( \bar{s}_1(n) \) is not in the supercenter of \( \text{OH}_{\ell}^{n} \). In fact, for \( 0 < n < \ell \), we have that \( \bar{s}_1(n) = \bar{s}_1(n) + \bar{s}_1(n) \), which is non-zero, so indeed \( \bar{s}_1(n) \) is not supercentral. For the last statement here, we used the Schur basis for \( \text{OH}_{\ell}^{n} \). From Theorem 7.4: if \( n' = 1 \) then \( \bar{s}(n) = \bar{s}(n) = \bar{s}(n) \), \( \bar{s}(n) + \bar{s}(n) \neq 0 \); if \( n' > 1 \) and \( n = 1 \) then \( \bar{s}(n) + \bar{s}(n) = \bar{s}(n) \neq 0 \); if \( n, n' > 0 \) then \( \bar{s}(n) + \bar{s}(n) \) are linearly independent so their sum is non-zero.

In the next lemma, we give explicit descriptions of the images of the leftward cups and caps under the graded 2-superfunctor \( \Psi_{\ell} \) from Theorem 11.3, that is, the superbimodule homomorphisms

\[
\text{coev}_{n} : = \Psi_{\ell}(\bigcup_{2n-\ell+2}) : \text{OH}_{n+1}^{\ell} \to Q^{-n}U_{n}^{\ell} \otimes \text{OH}_{n}^{\ell} Q^{3n-\ell+1}V_{n}^{\ell}, \quad (11.27)
\]

\[
\text{ev}_{n} : = \Psi_{\ell}(\bigcap_{2n-\ell}) : Q^{3n-\ell+1}V_{n}^{\ell} \otimes \text{OH}_{n+1}^{\ell} \to Q^{-n}U_{n}^{\ell} \quad (11.28)
\]

for \( 0 \leq n \leq \ell - 1 \) (these maps are zero for all other \( n \)). Let \( n' \) be defined so that \( \ell = n + 1 + n' \). Then, by (11.9), \( \text{coev}_{n} \) is a superbimodule homomorphism of degree \( n' - n \) and parity \( n' - n \) (mod 2), and \( \text{ev}_{n} \) is of degree \( n - n' \) and parity \( n - n' \) (mod 2). Recall also the maps \( \tilde{v}n \) and \( \tilde{v}n \) from Corollary 9.13.

**Lemma 11.4.** For \( \ell = n + 1 + n' \) as above, we have that

\[
\text{coev}_{n} = (-1)^{(n'+1)n'p^{-1} \otimes q_n) \circ \tilde{v}n, \quad \text{ev}_{n} = (-1)^{(n+1)n} q_n \circ \tilde{v}n \circ (q_n^{-1} \otimes p_n).
\]

where \( p_n : Q^{-n}U_n \to \tilde{U}n \) and \( q_n : \tilde{V}_n \to Q^{3n-\ell+1}V_n \) are the superbimodule isomorphisms that are the identity maps on the underlying vector spaces.

**Proof.** Note that \( p_n \) is of degree \( n - 2n' \) and parity \( n' \) (mod 2), and \( q_n \) is of degree \( n - \ell + 1 \) and parity \( n \) (mod 2). The inverse of the map \( p_n^{-1} \otimes q_n \) is \(-1)^{n} p_n^{-1} \otimes q_n^{-1} \). With this in mind, we let

\[
\tilde{v}n := (-1)^{(n'+1)n'} q_n \circ \text{coev}_{n}, \quad \tilde{v}n := (-1)^{(n+1)n} p_n^{-1} \circ \text{ev}_{n}.
\]

These are both even of degree 0. To prove the lemma, we must show that \( \tilde{v}n = \tilde{v}n \) and \( \tilde{v}n = \tilde{v}n \).

We first show that \( \tilde{v}n \) and \( \tilde{v}n \) are the counit and unit of an adjunction. This follows from the left adjunction relations (11.11):

\[
(\text{id} \otimes \tilde{v}n) \circ (\tilde{v}n \otimes \text{id}) = (-1)^{(n'+1)n'}(\text{id} \otimes \text{ev}_{n}) \circ (\text{id} \otimes q_n \otimes p_n^{-1}) \circ (p_n \otimes q_n^{-1} \otimes \text{id}) \circ (\text{coev}_{n} \otimes \text{id})
\]
\[ \begin{aligned}
&= (-1)^{n+n'} (\text{id} \otimes \text{ev}_n') \circ (p_{n'} \otimes \text{id} \otimes \text{id}) \circ (\text{id} \otimes p_{n'}^{-1}) \circ (\text{coev}_n' \otimes \text{id}) \\
&= (-1)^{f-1} p_{n'} \circ (\text{id} \otimes \text{ev}_n') \circ (\text{coev}_n' \otimes \text{id}) \circ p_{n'}^{-1} = \text{id},
\end{aligned} \]

\[ (\text{ev}_n \otimes \text{id}) \circ (\text{id} \otimes \text{coev}_n) = (-1)^{(c)} (\text{id} \otimes \text{id}) \circ (q_n \otimes p_{n'}^{-1} \otimes \text{id}) \circ (\text{id} \otimes p_{n'} \otimes q_n^{-1}) \circ (\text{id} \otimes \text{coev}_n) \]

\[ = (\text{ev}_n' \otimes \text{id}) \circ (\text{id} \otimes \text{id} \otimes q_n^{-1}) \circ (q_n \otimes \text{id} \otimes \text{id}) \circ (\text{id} \otimes \text{coev}_n') \]

\[ = q_n^{-1} \circ (\text{ev}_n' \otimes \text{id}) \circ (\text{id} \otimes \text{coev}_n') \circ q_n = \text{id}. \]

So now we have two adjunctions making \((\hat{V}_n, \hat{U}_n')\) into a dual pair, one \(A_1\) with unit \(\text{ev}_n\) and counit \(\text{coev}_n\) just constructed, and the other \(A_2\) with unit \(\text{ev}_n\) and counit \(\text{coev}_n\) coming from Corollary 9.13. Any such adjunction \(A\) induces an even degree 0 \((\text{OH}_n, \text{OH}_n')\)-superbimodule isomorphism \(\alpha : \hat{V}_n \xrightarrow{\sim} \text{Hom}_{\text{OH}_{n+1}}(\hat{U}_n, \text{OH}_n')\). So from \(A_1\) and \(A_2\) we get isomorphisms \(\alpha_1\) and \(\alpha_2\), hence, an even degree 0 automorphism \(\alpha^{-1}_2 \circ \alpha_1\) of \(\hat{V}_n\). By Lemma 9.10(1c), \(\hat{V}_n\) is cyclic generated by the vector 1 which is even of degree 0, and \(\text{dim}(\hat{V}_n)_{0,0} = 1\). So we must have that \(\alpha^{-1}_2 \circ \alpha_1 = c \text{id} \); for \(c \in \mathbb{P}^X\).

The argument in the previous paragraph shows that \(\text{coev}_n = c \text{ coev}_n\) and \(\text{ev}_n = c^{-1} \text{ev}_n\) for some \(c \in \mathbb{P}^X\). It remains to show that \(c = 1\). For this, we look at the “bottom bubble”: we must have that \(\Psi_n(n-n'\bigcup n-n'+1) = \text{id}\) if \(n \geq n\) and \(\Psi_n(n-n'1 \bigcup n-n') = \text{id}\) if \(n \leq n\). Now we compute these superbimodule endomorphisms explicitly in the two cases using that \(\text{coev}_n = c \text{ coev}_n\) and \(\text{ev}_n = c^{-1} \text{ev}_n\), to deduce that \(c = 1\).

Here is the computation in the case \(n \geq n'.\) We need to apply \(\text{ev}_n \circ (\rho(1);x^r) \otimes \text{id}) \circ \text{coev}_n\) to \(1 \in \text{OH}_{n+1}\) using that \(\text{coev}_n = (-1)^{(c)}(n+1)n'c \text{ (p}_{n'}^{-1} \otimes q_n) \circ \text{coev}_n\), and know that the result equals 1. Applying \(\text{coev}_n\) to 1 using the formula for that in Corollary 9.13 gives

\[ \sum_{r=0}^{n'} (-1)^{(c)}(n')^{(r)} \mu_n(x^{r}) \otimes \psi_{n+1}(\phi_{n'-r}). \]

Then we scale by \((-1)^{(c)}(n+1)n'c\) and apply \((p_{n'}^{-1} \otimes q_n)\) to get

\[ c \sum_{r=0}^{n'} (-1)^{(c)}(n+1)n' \otimes (n')^{(r)} \mu_n(x^{r}) \otimes \psi_{n+1}(\phi_{n'-r}). \]

This is \(\text{coev}_n'(1)\). The sign simplifies to \((-1)^{(c)}(n')^{(r)} = (n'-r) + (n'-r).\) Then we apply \(\rho(1);x^r \otimes \text{id}\) (the dots on the left boundary of the bubble) using (9.45) to get

\[ c \sum_{r=0}^{n'} (-1)^{(c)}(n')^{(r)} \mu_n(x^{r+n'+r}) \otimes \psi_{n+1}(\phi_{n'-r}). \]

Finally we apply \(\text{ev}_n\) using the formula from (9.32). The sum collapses just to the \(r = n'\) term, and we are left with

\[ (-1)^{(c)}c^{(c')} + (c')^{(n-n')}c = 1. \]

The sign on the left hand side is just 1, and it follows that \(c = 1\).

The case \(n \leq n'\) is a similar calculation\(^{10}\) for the bubble \(n-n'1 \bigcup n-n'.\) This is carried out in the proof of the next corollary—the result established there for \(r = 0\) is exactly what is needed to show that \(c = 1\) in this situation. \(\square\)

\(^{10}\) The case \(n = n'\) gets double checked this way, giving some confidence that the signs are correct!
Corollary 11.5. For $\ell = n+n'$, $k = 2n-\ell$ and $r \geq 0$, $\Psi_{\ell}( k \otimes r \otimes_{r-k-1} )$ is the element of $\text{End}_{\text{OH}_n^\ell \otimes \text{OH}_n^{n'}}(\text{OH}_n^k)$ defined by left multiplication by

$$
\sum_{s=0}^{r} (-1)^{n'(r-s)+r} \psi_n^f \bar{e}^n_{s}(\bar{\gamma}_{r-s}^n) e^n_s \in \text{OH}_n^k.
$$

(11.29)

If $r = 0$ this element is equal to 1. In the special case $n = 0, n' = \ell$ this is $\gamma_{r}^n = (-1)^{(\ell)} \bar{h}_{r}$. Also, for any $r \geq 0$, the image of this element under the homomorphism $\alpha_n^f : \text{OH}_n^k \to \text{R}_{n'}$ from Lemma 7.3 is $\gamma_{r}^{n'} = (-1)^{(\ell)} \bar{h}_{r}$.  

Proof. For the proof, it is more convenient to work with $n'$ defined from $\ell = n+1+n'$ as in Lemma 11.4, so $r-k-1 = r+n'-n$ and

$$
\Psi_{\ell}( k \otimes r \otimes_{r-k-1} ) = \text{ev}'_n \circ (\text{id} \otimes \rho(1) \otimes \alpha(x)^{r+n'-n}) \circ \text{coev}_n.
$$

By (9.31), we have that $\text{coev}_n(1) = \sum_{s=0}^{n} v_n(x^s) \otimes u_n(1) \bar{e}^n_{s}$. Then we apply $\text{id} \otimes \rho(1) \otimes \alpha(x)^{r+n'-n}$ to obtain

$$
\sum_{s=0}^{n} (-1)^{(n'+s-n)+(r+n'-n)+(s)} v_n(x^s) \otimes u_n(x^{r+n'-n}) \bar{e}^n_{s}.
$$

Then we apply $\text{ev}'_n$ using Lemma 11.4 and Corollary 9.13. First we have to scale by $(-1)^{(n')}^{(n+1)} n'$ and apply $q_n^{-1} \otimes p_n'$ to get

$$
\sum_{s=0}^{n} (-1)^{(n'+s-n)+(r+n'-n)+(s)} v_n(x^s) \otimes u_n(x^{r+n'-n}) \bar{e}^n_{s}.
$$

Then the application of $\bar{\text{ev}}_n$ gives

$$
\sum_{s=n-r}^{n} (-1)^{(n+1)+(n)} + (r-n)(n+s) + (r+s+n) \psi_n^f \bar{e}^n_{n+1}(\bar{\gamma}_{r+s-n}^n) e^n_s.
$$

Some final simplifying, replacing $\bar{e}^n_{n+1}(\bar{\gamma}_{r+s-n}^n)$ with $(-1)^{(n+1)} \bar{e}^n_{n+1}$ and remembering to replace $n' + 1$ with $n'$ at the end, gives (11.29). 

It is trivial to see that (11.29) equals 1 when $r = 1$. To apply $\alpha_n^f$ to (11.29) for the final assertion, one immediately gets $(-1)^{n'} \alpha_n^f(\psi_n^f(\bar{\gamma}_r^n))$. From (7.4), $\alpha_n^f(\psi_n^f(e^n_{s}(\bar{\gamma}_{r-s}^n))) = e^n_s(\bar{\gamma}_r^n)$. Hence, $(-1)^{n'} \alpha_n^f(\psi_n^f(e^n_{s}(\bar{\gamma}_{r-s}^n))) = (-1)^{n'} \gamma_r^n e^n_s(\bar{\gamma}_r^n)$. This also equals $(-1)^{(\ell)} \bar{h}_{r}$ by (4.38), since the anti-automorphism of $\text{R}_{n'}$ induced by $*$ is the identity. When $n = 0, n' = \ell$, $\alpha_n^f$ is the identity, so this special case has also been addressed. 

\[\square\]

The results so far in this section have an application to prove the non-degeneracy of the reduced $\text{sl}_2$ 2-category $\text{Ut}(\text{sl}_2)$. This asserts that the 2-morphism spaces in $\text{Ut}(\text{sl}_2)$ have the expected graded dimensions. The result may be formulated as follows. For any $k, \ell \in \mathbb{Z}$ and 1-morphisms $X, Y \in \text{Hom}_{\text{Ut}(\text{sl}_2)}(k, \ell)$ (i.e., words consisting of $m$ letters $E$ and $n$ letters $F$ such that $\ell = k + 2m - 2n$) we view the 2-morphism space $\text{Hom}_{\text{Ut}(\text{sl}_2)}(X, Y)$ as a graded right $R$-supermodule so that $\hat{c} \in R$ acts by horizontally composing on the right with $\beta_k(\hat{c})$. 

Theorem 11.6. For $k, \ell \in \mathbb{Z}$ and $X, Y \in \text{Hom}_{\text{Ut}(\text{sl}_2)}(k, \ell)$, the 2-morphism space $\text{Hom}_{\text{Ut}(\text{sl}_2)}(X, Y)$ is free as a graded right $R$-supermodule with basis given by a set of representatives for equivalence classes of decorated reduced $(X, Y)$-matchings in the sense defined in [BE2, Sec.8]. In particular, $\beta_k : R \to \text{End}_{\text{Ut}(\text{sl}_2)}(1_k)$ is an isomorphism for all $k \in \mathbb{Z}$. 

Proof. The “easy” step in the proof is to show that \( \text{Hom}_{U(\mathfrak{sl}_2)}(1, \ell) \) is spanned as a right \( R \)-supermodule by the 2-morphisms that are the representatives for equivalence classes of decorated reduced \((X, Y)\)-matchings. This is proved by exhibiting an explicit straightening algorithm going by induction on the number of crossings. See [BE2, Th. 8.1], which simply cites [KL, Prop. 3.11] as the argument is the same as in the purely even setting, or [DEL] for a more systematic treatment. Note the straightening algorithm requires all of the relations described above, including the alternating braid relation.

The “hard” step is to establish the linear independence. By a standard reduction, which is again the same as in the ordinary even setting as in [KL, Rem. 3.16], it suffices to treat the case that \( X = Y = E^d \) for some \( d \geq 0 \). In this case, the decorated reduced \((X, Y)\)-matchings consist of \( d \) strings oriented from bottom to top decorated with some dots close to the top boundary. We index them by pairs \((\kappa, \omega)\) for \( \kappa \in \mathbb{N}^d \) and \( \omega \in S_d \). For such a pair, the corresponding 2-morphism \( f(\kappa, \omega) \) has \( \kappa_i \) dots at the top of the \( i \)th string, with the strings below arranged so that they represent some reduced expression for \( \omega \).

We will also assume from now on that \( k \leq 0 \). The proof when \( k \geq 0 \) is similar, working with clockwise bubbles which correspond to \( \hat{c}_r \in R \) rather than the counterclockwise ones which correspond to \( \hat{h}_r \in R \). Consider some linear relation

\[
  f := \sum_{\kappa \in \mathbb{N}^d, \omega \in S_d} f(\kappa, \omega)\beta_\ell(\hat{c}_{\kappa, \omega}) = 0
\]

for \( \hat{c}_{\kappa, \omega} \in R \). Each \( \hat{c}_{\kappa, \omega} \) is an \( R \)-linear combination of basis vectors \( \hat{h}_\lambda \) of \( R \) for \( \lambda \) in some finite set \( P_\kappa \) of partitions. Pick \( 0 \leq n \leq \ell \) with \( k = 2n - \ell \) in such a way that \( n \) and \( \ell - n \) are both very large relative to \( |\kappa| \) and \( |\omega| \) for all \( \lambda \in P_\kappa, \kappa \in \mathbb{N}^\ell \) with \( \hat{c}_{\kappa, \omega} \neq 0 \) for some \( \omega \in S_n \). Then we apply the 2-superfunctor \( \Psi_\ell \) to \( f \) to obtain the relation

\[
  \Psi_\ell(f) := \sum_{\kappa \in \mathbb{N}^d, \omega \in S_d} \Psi_\ell(f(\kappa, \omega))\Psi_\ell(\beta_\ell(\hat{c}_{\kappa, \omega})) = 0
\]

in \( \text{End}_{OH_n} U_{(1^\ell)n} \). Conjugating with the isomorphism \( c'_{(1^\ell)n} \) from (9.10), we get from \( \Psi_\ell(f) \) a superbimodule endomorphism \( \tilde{f} = 0 \) of \( U_{(1^\ell)n} \). Using (11.23) and (11.24), it follows that

\[
  \tilde{f} = \sum_{\kappa \in \mathbb{N}^d, \omega \in S_d} \pm \tau_\kappa \beta_\ell(\hat{c}_{\kappa, \omega})
\]

for some signs, where this is being viewed as an endomorphism of the free right \( OH_n \)-superbimodule \( U_{(1^\ell)n} \) using the right action of \( ONH_n \) from Lemma 9.7(2). By the large choice of \( n \) and \( \ell \), the endomorphisms defined by each \( \tau_\kappa \beta_\ell(\hat{c}_{\kappa, \omega}) \) are linearly independent; cf. the proof of Theorem 5.1. We deduce that \( \Psi_\ell(\beta_\ell(\hat{c}_{\kappa, \omega})) = 0 \) for all \( \kappa \) and \( \omega \).

It remains to show that \( \Psi_\ell(\beta_\ell(\hat{c}_{\kappa, \omega})) = 0 \) implies that \( \hat{c}_{\kappa, \omega} = 0 \) for sufficiently large \( n \) and \( \ell \). Assume that \( \Psi_\ell(\beta_\ell(\hat{c}_{\kappa, \omega})) = 0 \). Remembering that \( \hat{c}_{\kappa, \omega} \) is an \( R \)-linear combination of \( \hat{h}_\lambda \) for \( \lambda \) with \( |\lambda| \) small, this follows on evaluating at \( 1 \in OH_n \) then applying the homomorphism \( \alpha_n^\ell : OH_n \rightarrow \text{End}_{R_{\ell-n}} \). The point here is that, by the final assertion in Corollary 11.5 which computes \( \alpha_n^\ell(\Psi_\ell(\beta_\ell(\hat{h}_\lambda))(1)) \), we have that

\[
  \alpha_n^\ell(\Psi_\ell(\beta_\ell(\hat{h}_\lambda))(1)) = \pm \delta_{\ell-n} \hat{h}_\lambda.
\]

These elements of \( R_{\ell-n} \) are linearly independent for small \( \lambda \), so we can conclude that the coefficients of all \( \hat{h}_\lambda \) in \( \hat{c}_{\kappa, \omega} \) are zero. \( \square \)

The following corollary is well known; see also [BE2, Th. 11.7] for the explicit definition of the isomorphism. We just note a different convention for \((q, \pi)\)-integers is used in [BE2, Sec. 9] compared to (3.1). This accounts for the difference in the defining relation [BE2, (9.2)] for \( U_{q,\pi}(\mathfrak{sl}_2) \) compared to the relation (3.11) being used for it here.
Corollary 11.7. The split Grothendieck ring of the underlying ordinary category of the graded super Karoubi envelope $\text{gsKar}(\mathcal{U}(sl_2))$ is isomorphic as a $\mathbb{Z}^n[q, q^{-1}]$-algebra to the integral form $U_{q,\pi}(sl_2)$ of $U_{q,\pi}(sl_2)$ defined at the end of Section 3. Under the isomorphism, the isomorphism classes of the 1-morphisms $E1_k$ and $F1_k$ correspond to the elements of $U_{q,\pi}(sl_2)$ denoted by the same notation.

Proof. See [BE2, Th. 12.1], which explains how to deduce this from the non-degeneracy given by Theorem 11.6.

Remark 11.8. Theorem 11.6 is not new, indeed, the non-degeneracy of the non-reduced odd $sl_2$ 2-category $\tilde{U}(sl_2)$ has already been established in [DEL] by a completely different technique. Also a version of Corollary 11.7 already appeared in [EL]. The proof of Theorem 11.6 given here is in the same spirit as the proof of non-degeneracy of the ordinary $sl_2$ 2-category given in [L1, Prop. 8.2] and the more general proof of non-degeneracy of $sl_n$ given in [KL].

Remark 11.9. Theorem 11.3 can be upgraded to the extended setting of Remarks 4.14, 7.9, 8.10, 9.14 and 10.9. However Lemma 11.4 does not carry over, and this is needed for the subsequent Theorem 11.6. For this reason, we are not able to prove non-degeneracy of the non-reduced $\tilde{U}(sl_2)$ by this method.

For $d \geq 1$ and $k \in \mathbb{Z}$, there are graded superalgebra homomorphisms

$$\rho_d^{(k)} : ONH_d \to \text{End}_{\tilde{U}(sl_2)}(E_d^1 1_k)^{\text{op}}, \quad (11.30)$$

$$x_i \mapsto (-1)^{i-1} \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
1 \\
\vdots \\
\vdots \\
\vdots \\
1
\end{array}, \quad \tau_j \mapsto -(-1)^{j-1} \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
1 \\
\vdots \\
\vdots \\
\vdots \\
1
\end{array},$$

$$\lambda_d^{(k)} : ONH_d \to \text{End}_{\tilde{U}(sl_2)}(1_k F_d^1), \quad (11.31)$$

$$x_i \mapsto (-1)^{d-i} \begin{array}{c}
1 \\
\vdots \\
\vdots \\
1 \\
\vdots \\
\vdots \\
\vdots \\
1
\end{array}, \quad \tau_j \mapsto -(-1)^{d-j} \begin{array}{c}
1 \\
\vdots \\
\vdots \\
1 \\
\vdots \\
\vdots \\
\vdots \\
1
\end{array}.$$

This follows from the relations (11.2) and (11.10), with the signs in (11.30) and (11.31) accounting for the difference between these and our preferred relations for $ONH_n$ from (5.1) to (5.6). Another consequence of Theorem 11.6 is that both $\rho_d^{(k)}$ and $\lambda_d^{(k)}$ are injective.

Remark 11.10. On comparing with (11.23) and (11.24), it follows that the composition of (11.30) for $k = 2n - \ell$ with the homomorphism $\text{End}_{\tilde{U}(sl_2)}(E_d^1 1_k)^{\text{op}} \to \text{End}_{\text{OH}_{2n-\ell} \cdot \text{OH}_{2n-\ell}}(U_{n+\ell}^{k \cdot \ell} \cdot \text{OH}_{n+\ell})^{\text{op}}$ induced by the 2-superfunctor $\Psi_\ell$ is equal to the anti-homomorphism $\rho_{(1^\ell)\ell^\ell}$ from (9.43). One can check similarly that the composition of (11.31) for $k = 2n - \ell$ with the homomorphism induced by $\Psi_\ell$ is equal to the homomorphism $\lambda_{n,(1^\ell)\ell^\ell}$ from (9.44).

To conclude the section, we explain how to define divided powers. In $\text{gsKar}(\mathcal{U}(sl_2))$, there are 1-morphisms

$$E^{(d)}_1 k := \left( Q^{(d)} E^1_{k}, Q^{(d)} P^{(k)}_d ((\chi \omega)_d) \right) : k \to k + 2d, \quad (11.32)$$

$$F^{(d)}_1 k := \left( Q^{(d)} F^1_{k}, Q^{(d)} \lambda^{(k-2d)}_d ((\omega \chi)_d) \right) : k \to k - 2d. \quad (11.33)$$

By Lemma 5.7 plus (5.32), we have that

$$E^{d}_1 k \cong \bigoplus_{w \in S_d} \Pi^{(w)} Q^{(w)-(\ell)} E^{(d)}_1 k, \quad F^{d}_1 k \cong \bigoplus_{w \in S_d} \Pi^{(w)} Q^{(w)-(\ell)} F^{(d)}_1 k. \quad (11.34)$$

It follows that $E^{(d)}_1 k$ and $F^{(d)}_1 k$ categorify the divided powers (3.12) and (3.13), i.e., the isomorphism classes of the former 1-morphisms under the isomorphism from Corollary 11.7 give the latter elements of $U_{q,\pi}(sl_2)$. 

Lemma 11.11. In $\text{gsKar}(\mathfrak{U}(sl_2))$, the 1-morphism $Q^{-d(k+2d-1)}F(d)_{1,2k+2d}$ is right dual to $E(d)_{1,k}$, and the 1-morphism $\Pi^{(k+1)}Q^{(k-2d+1)}E(d)_{1,2k-2d}$ is right dual to $F(d)_{1,k}$.

Proof. We first show that $Q^{-d(k+2d-1)}F(d)_{1,2k+2d}$ is right dual to $E(d)_{1,k}$. By (11.3), $Q^{-k-1}F1_{k+2}$ is right dual to $E1_k$ in the $(Q, \Pi)$-envelope $\text{gsKar}(\mathfrak{U}(sl_2))$. Hence, $Q^{-d(k+2d)}F(d)_{1,2k+2d}$ is right dual to $E(d)_{1,k}$. From this, we deduce that $Q^{-d(k+2d-1)}Q^{(k)}F(d)_{1,2k+2d}$ is right dual to $Q^{(k)}E(d)_{1,k}$. By definition, $E(d)_{1,k}$ is the summand of $Q^{(k)}E(d)_{1,k}$. From this, we can observe that $Q^{-d(k+2d-1)}Q^{(k)}F(d)_{1,2k+2d}$ is right dual to $Q^{(k)}E(d)_{1,k}$.

The proof that $\Pi^{(k+1)}Q^{(k-2d+1)}E(d)_{1,2k-2d}$ is right dual to $F(d)_{1,k}$ is similar to the proof of Lemma 11.11 instead of (11.3). Consider first the case $d = 1$. If $k$ is odd then the leftward cups and caps are even and (11.11) plus the information about degrees in (11.9) shows that $Q^{k-1}1_{1,k}$ is right dual to $F1_k$. If $k$ is even there is a sign in (11.11), but this disappears when one replaces $E1_k$ with $\Pi Q^{-k-1}E1_k$, viewing the appropriately degree- and parity-shifted leftward cups and caps as even degree 0 2-morphisms $1_k \Rightarrow \Pi Q^{-k-1}E1_k$ and $F1\Pi Q^{-k-1}E1_k \Rightarrow 1_{1,k}$, respectively, thanks to the definition of horizontal composition in the $(Q, \Pi)$-envelope as defined in [BE2, Def. 6.10]. We are not going to repeat this definition here, but note a similar phenomenon occurred in $\text{OGr}2$ in the second paragraph of the proof of Lemma 11.4. For general $d$, one deduces from the $d = 1$ case that $\Pi^{(k+1)}Q^{(k-2d+1)}E(d)_{1,2k-2d}$ is right dual to $F(d)_{1,k}$ just as before. Hence, $\Pi^{(k+1)}Q^{(k-2d+1)}E(d)_{1,2k-2d}$ is right dual to $Q^{(k)}E(d)_{1,k}$.

The proof that $\Pi^{(k+1)}Q^{(k-2d+1)}E(d)_{1,2k-2d}$ is right dual to $F(d)_{1,k}$ requires $\Pi^{(k+1)}Q^{(k-2d+1)}E(d)_{1,2k-2d}$ to be the right mate of $Q^{(k)}E(d)_{1,k}$.

12. Some graded 2-representation theory

In this section, we develop some 2-representation theory of the reduced $sl_2$ 2-category $\mathfrak{U}(sl_2)$ from Definition 11.2. We work throughout in the grading setting, but all the definitions and results here have analogs with the $\mathbb{Z}$-grading forgotten. Let $\text{gsCat}$ be the strict graded 2-supercategory of graded supercategories, graded superfunctors and graded supernatural transformations; see [BE1, Sec. 6]. The following is modelled on [R1, Def. 5.1.1].

Definition 12.1. By a graded 2-representation $\mathcal{V}$ of $\mathfrak{U}(sl_2)$, we mean a strict graded 2-supercategorifier $\mathcal{V} : \mathfrak{U}(sl_2) \rightarrow \text{gsCat}$. Decoding the definition, $\mathcal{V}$ consists of the following data:

- a graded supercategory $\mathcal{V}$ with a given decomposition into weight subcategories $\mathcal{V} = \bigsqcup_{k \in \mathbb{Z}} \mathcal{V}_k$ (or $\mathcal{V} = \bigoplus_{k \in \mathbb{Z}} \mathcal{V}_k$ when $\mathcal{V}$ is additive);
- graded superfunctors $E : \mathcal{V} \rightarrow \mathcal{V}$ and $F : \mathcal{V} \rightarrow \mathcal{V}$ such that $E|_{\mathcal{V}_k} : \mathcal{V}_k \rightarrow \mathcal{V}_{k+2}$ and $F|_{\mathcal{V}_k} : \mathcal{V}_k \rightarrow \mathcal{V}_{k-2}$ for each $k \in \mathbb{Z}$;
- graded supernatural transformations $x : E \Rightarrow E$ and $\tau : E^2 \Rightarrow E^2$ which are odd of degrees 2 and $-2$, respectively;
- inhomogeneous graded supernatural transformations $\eta : \text{Id} \Rightarrow FE$ and $\varepsilon : EF \Rightarrow \text{Id}$ whose restrictions $\eta : \text{Id}|_{\mathcal{V}_k} \Rightarrow FE|_{\mathcal{V}_k}$ and $\varepsilon : EF|_{\mathcal{V}_{k+2}} \Rightarrow \text{Id}|_{\mathcal{V}_{k+2}}$ are even of degrees $k + 1$ and $-k - 1$, respectively.

Then there are the axioms:

- the relations from (11.2) hold: $\tau^2 = 0$, $(xE) \circ \tau + \tau \circ (Ex) = (Ex) \circ \tau + (xE) \circ \tau = E^2$ and $(\tau E) \circ (E\tau) \circ (\tau E) = (E\tau) \circ (\tau E) \circ (E\tau)$;
• \( \eta \) and \( \varepsilon \) satisfy the zig-zag relations: \( (\varepsilon F) \circ (F \eta) = F \) and \( (E \varepsilon) \circ (\eta E) = E \) (equivalently, they define units and counits of adjunctions making \( Q^{-k-1}F|_{\mathcal{W}_{k+2}} \) into a right adjoint to \( E|_{\mathcal{W}_{k}} \) for each \( k \in \mathbb{Z} \);

• letting \( \sigma := (FE) \circ (F\tau F) \circ (\eta EF) : EF \Rightarrow FE \) be the image of the rightward crossing under \( \mathcal{W} \), the following inhomogeneous matrices of supernatural transformations are isomorphisms:

\[
\begin{pmatrix}
\sigma & \varepsilon & \varepsilon \circ (xF) & \cdots & \varepsilon \circ (xF^{k-1})
\end{pmatrix}^T : EF|_{\mathcal{W}_{k}} \Rightarrow FE|_{\mathcal{W}_{k}} \oplus \text{Id}|_{\mathcal{W}_{k}}^k \\
(\sigma \eta & (Fx) \circ \eta & \cdots & (Fx)^{k-1} \circ \eta)
\]

\( : EF|_{\mathcal{W}_{k}} \oplus \text{Id}|_{\mathcal{W}_{k}}^{(k)} \Rightarrow FE|_{\mathcal{W}_{k}} \) for \( k \leq 0 \)

• letting \( \sigma' : FE|_{\mathcal{W}_{k}} \Rightarrow EF|_{\mathcal{W}_{k}} \) be the inverse of \( \sigma : EF|_{\mathcal{W}_{k}} \Rightarrow FE|_{\mathcal{W}_{k}} \), \( \eta' : \text{Id}|_{\mathcal{W}_{k}} \Rightarrow EF|_{\mathcal{W}_{k}} \) be the rightmost entry of the inverse of the above matrix if \( k > 0 \), and \( \varepsilon' : FE|_{\mathcal{W}_{k}} \Rightarrow \text{Id}|_{\mathcal{W}_{k}} \) be the bottom entry this inverse matrix if \( k < 0 \), the supernatural transformations \( \varepsilon \circ (Fx)^k \circ \eta' : \text{Id}|_{\mathcal{W}_{k}} \Rightarrow \text{Id}|_{\mathcal{W}_{k}} \) for all \( k > 0 \), \( \varepsilon' \circ (Fx)^{-k} \circ \eta : \text{Id}|_{\mathcal{W}_{k}} \Rightarrow \text{Id}|_{\mathcal{W}_{k}} \) for all \( k < 0 \), and \( \varepsilon \circ \sigma' \circ \eta : \text{Id}|_{\mathcal{W}_{k}} \Rightarrow \text{Id}|_{\mathcal{W}_{k}} \) (the images of the odd bubbles) are zero.

**Remark 12.2.** The final axiom in Definition 12.1 is difficult to formulate, but in practice it is easily checked by showing that some generating object for \( \mathcal{W} \) has no non-zero odd endomorphisms of degree 1. (A generating object \( M \) means an object such that any other object of \( \mathcal{W} \) is isomorphic to an object obtained from \( M \) by applying a sequence of the superfunctors \( E \) and \( F \).)

There are natural notations of sub-2-representations, quotient 2-representations, and morphisms of graded 2-representations. The latter definition, which is the super analog of [R1, Def. 2.3], is equivalent to the following, which is similar to the formulation adopted in [CR, Sec. 5.2.1]; the terminology being used is the same as in [BD, Def. 4.6] (and actually goes back to Ben Webster).

**Definition 12.3.** Let \( \mathcal{W} \) and \( \mathcal{W}' \) be two graded 2-representations of \( \mathcal{U}(\mathfrak{sl}_2) \). A **strongly equivariant graded superfunctor** \( \Omega : \mathcal{W} \to \mathcal{W}' \) is a graded superfunctor such that \( \Omega|_{\mathcal{W}_{k}} : \mathcal{W}_{k} \to \mathcal{W}'_{k} \) for each \( k \in \mathbb{Z} \), plus a degree 0 even graded supernatural isomorphism \( \zeta : E\Omega \Rightarrow \Omega E \), such that the following holds:

- the supernatural transformation \( (F\Omega \varepsilon) \circ (F\zeta F) \circ (\eta \Omega F) : \Omega F \Rightarrow F\Omega \) is invertible;
- we have that \( (\Omega \varepsilon x) \circ \zeta = \zeta \circ (x\Omega) \);
- we have that \( (\Omega \tau) \circ (\tau \Omega) \circ (E\zeta) = (\zeta E) \circ (E\zeta) \circ (\tau \Omega) \).

A strongly equivariant graded super equivalence is a strongly equivariant graded superfunctor which is also a superequivalence of supercategories.

**Remark 12.4.** For strongly equivariant graded equivalences, the first axiom in Definition 12.3 actually holds automatically; see [BD, Rem. 4.8] where this is explained. Also in [BD], the diagrammatic interpretation of these definitions is discussed, which we still find helpful.

**Remark 12.5.** There is an obvious way to make the composition of two strongly equivariant graded superfunctors into a strongly equivariant graded superfunctor in its own right. Also the identity functor \( \text{Id} \) is strongly equivariant with \( \zeta := 1_E \). So there is a category \( \mathcal{R}ep(\mathcal{U}(\mathfrak{sl}_2)) \) consisting of graded 2-representations and strongly equivariant graded superfunctors.

Usually, the graded supercategories \( \mathcal{W}_{k} \) in a graded 2-representation \( \mathcal{W}' \) will have some extra structure, such as being additive or II-complete. We are mainly interested here in what we call graded Karoubian 2-representations. By definition, this means a graded 2-representation \( \mathcal{W}' \) such that, for each \( k \in \mathbb{Z} \), the weight subcategory \( \mathcal{W}_{k} \) is additive and II-complete, and the underlying ordinary category \( \mathcal{W}'_{k} \) is idempotent complete. Any graded 2-representation \( \mathcal{W}' \) can be upgraded to a Karoubian graded 2-representation by passing to its graded super Karoubi envelope \( \text{gsKar}(\mathcal{W}') \).

Given a graded Karoubian 2-representation \( \mathcal{W}' \), the underlying graded 2-superfunctor from \( \mathcal{U}(\mathfrak{sl}_2) \) to \( \mathcal{W}' \) extends canonically to a graded 2-superfunctor from the graded super Karoubi envelope \( \text{gsKar}(\mathcal{U}(\mathfrak{sl}_2)) \) to \( \mathcal{W}' \).
to $\mathcal{V}'$. The direct sum over all $k \in \mathbb{Z}$ of the images under this graded 2-superfunctor of the 1-morphisms $E^{(d)}_1$ and $E^{(d)}_k$ from (11.32) and (11.33) give graded superfunctors

$$E^{(d)}, F^{(d)} : \mathcal{V} \to \mathcal{V}'.$$

By (11.34), we have that

$$E^d \cong \bigoplus_{w \in S_d} \Pi^{\ell(w)} Q^{2\ell(w) - (\frac{d}{2})} E^{(d)}, \quad F^d \cong \bigoplus_{w \in S_d} \Pi^{\ell(w)} Q^{2\ell(w) - (\frac{d}{2})} F^{(d)}.$$  \hspace{2cm} (12.2)

Lemma 11.11 implies that $Q^{-d(k+2d-1)} F^{(d)}|_{\mathcal{V}_{k+2d}}$ is right adjoint to $E^{(d)}|_{\mathcal{V}_k}$ and $\Pi^{d(k+1)} Q^{d(k-2d+1)} E^{(d)}|_{\mathcal{V}_{k-2d}}$ is right adjoint to $F^{(d)}|_{\mathcal{V}_k}$, with units and counits of adjunction that are defined by images of 2-morphisms in $\text{gsKar}(\mathbb{L}(sl_2))$.

A graded 2-representation $\mathcal{V}$ is said to be integrable if $E$ and $F$ are locally nilpotent, i.e., for any $k \in \mathbb{Z}$ and any $M \in \mathcal{V}_k$ there is some $n \geq 0$ such that $E^n M = F^n M = 0$. Also, for $\ell \in \mathbb{N}$, a lowest weight object of weight $-\ell$ means an object $M \in \mathcal{V}_{-\ell}$ such that $FM = 0$.

**Example 12.6.** Suppose that $\ell \in \mathbb{N}$. By Theorem 11.3, there is an integrable graded Karoubian 2-representation

$$OH^\ell_{\text{pgsmod}} := \bigoplus_{n=0}^{\ell} OH^\ell_n \text{-pgsmod}$$  \hspace{2cm} (12.3)

with the weight $k$ subcategory $(OH^\ell_{\text{pgsmod}})_k$ equal to $OH^\ell_n \text{-pgsmod}$ if $k = 2n - \ell$ for $0 \leq n \leq \ell$, or the trivial (zero) graded supercategory otherwise. Other data is as follows.

- The graded superfunctors $E$ and $F$ are $Q^{-n} U^\ell_n \otimes_{OH^\ell_n} -$ on the weight subcategory $OH^\ell_n \text{-pgsmod}$, and $Q^{\ell-2n-1} V^\ell_n \otimes_{OH^\ell_{n+1}} -$ on the weight subcategory $OH^\ell_{n+1} \text{-pgsmod}$, respectively.
- The supernatural transformation $\lambda$ is defined by $Q^{-n} \rho^{(1)a}(x_1) \otimes \text{id} \in \text{gsEnd}(Q^{-n} U^\ell_n \otimes_{OH^\ell_n} -)$, and $\tau$ is defined by $-Q^{-1} \rho^{(1)a}(\tau_1) \otimes \text{id} \in \text{gsEnd}(Q^{-n-1} U^\ell_{n+1} \otimes_{OH^\ell_{n+1}} Q^{-n} U^\ell_n \otimes_{OH^\ell_n} -)$ for all $n$.
- The supernatural transformations $\eta$ and $\varepsilon$ are given by the appropriate counit and unit from Theorem 9.11 (viewed as a superbimodule homomorphism between superbimodules that are shifted in degree but not in parity compared to before).
- The homomorphisms induced by (11.30) and (11.31) are equal to (9.43) and (9.44) thanks to Remark 11.10.
- For $0 \leq n + d \leq \ell$, we have that $E^{(d)}|_{OH^\ell_{\text{pgsmod}}} \cong Q^{-dn} U^\ell_{(d)n} \otimes_{OH^\ell_n} -$ and $F^{(d)}|_{OH^\ell_{\text{pgsmod}}} \cong Q^{-(\ell-3n-2d+1)} V^\ell_{(d)n} \otimes_{OH^\ell_{n+d}} -$; cf. Theorem 10.1.

We point out also by Theorem 10.1 that the split Grothendieck group $K_0(OH^\ell_{\text{pgsmod}})$ is naturally identified with the $U_{q,\sigma}(sl_2)$-module $V(-\ell)$, and $OH^\ell_0$ is a lowest weight object of weight $-\ell$.

Now we come to one of the key constructions introduced by Rouquier in [R1] in the purely even case, the construction of cyclic quotients. For any $\ell \in \mathbb{Z}$, there is a graded 2-representation $R(\ell)$ with

$$R(\ell)_k := \text{Hom}_{\mathcal{L}(sl_2)}(\ell, k)$$  \hspace{2cm} (12.4)

for $k \in \mathbb{Z}$, viewed as a graded 2-representation of the graded 2-supercategory $\mathcal{L}(sl_2)$ in an obvious way. For example, the graded superfunctor $E|_{R(\ell)_k} : R(\ell)_k \to R(\ell)_{k+2}$ is defined by horizontally composing on the left with the 1-morphism $E_1$, and the supernatural transformation $x : E|_{R(\ell)_k} \Rightarrow E|_{R(\ell)_k}$ is induced by the 2-endomorphism $\frac{3}{2} : E_1 \Rightarrow E_1$. The graded 2-representation $R(\ell)$ has the following universal property.

**Lemma 12.7.** Given any graded 2-representation $\mathcal{V}$ and any $M \in \mathcal{V}_\ell$ there is a canonical strongly equivariant graded superfunctor $\omega_M : R(\ell) \to \mathcal{V}$ taking the object $1_\ell$ of $R(\ell)_\ell$ to $M$. 
We define the universal graded 2-representation of lowest weight \(-\ell \in \mathbb{Z}\), denoted \(\mathcal{V}(-\ell)\), to be the quotient 2-representation \(\mathcal{R}(-\ell)/I\), where \(I\) here is the sub-2-representation of \(\mathcal{R}(-\ell)\) generated by \(-\ell\). We denote the lowest weight object of \(\mathcal{V}(-\ell)\) arising from the object \(1_{-\ell} \in \mathcal{R}(-\ell)_{-\ell}\) by \(\overline{1}_{-\ell}\), and call this the canonical lowest weight object. It is a generating object for \(\mathcal{V}(-\ell)\). The identity endomorphism of \(\overline{1}_{-\ell}\) is equal to the image of the bottom bubble \(-\ell \circ \overline{1}\), i.e., the image of \(1 \in R\) under the homomorphism (11.22). If \(\ell < 0\), this bubble is not a fake bubble, so it belongs to \(I\). This shows that the graded supercategory \(\mathcal{V}(-\ell)\) is trivial if \(\ell < 0\). Thus, \(\mathcal{V}(-\ell)\) is only interesting if \(\ell \in \mathbb{N}\), i.e., it is a dominant weight for \(sl_2\). The following, the universal property of \(\mathcal{V}(-\ell)\), follows immediately from Lemma 12.7 and the universal property of quotients.

**Lemma 12.8.** Let \(\mathcal{V}\) be any graded 2-representation of \(\mathcal{U}(sl_2)\), \(\ell \in \mathbb{N}\) and \(M \in \mathcal{V}'_{-\ell}\) be a lowest weight object. The superfunctor \(\omega_M : \mathcal{R}(-\ell) \rightarrow \mathcal{V}\) from Lemma 12.7 induces a strongly equivariant graded superfunctor \(\hat{\Omega}_M : \mathcal{V}(-\ell) \rightarrow M\) taking \(\overline{1}_{-\ell}\) to \(M\).

There is a more sophisticated version of Lemma 12.8, which is analogous to [R1, Prop. 5.6]. To formulate this, we need one more preliminary lemma.

**Lemma 12.9.** The homomorphism\(^{12} \beta_{-\ell} : R \rightarrow \text{End}_{\mathcal{R}(-\ell)}(1_{-\ell})\) from (11.22) induces an isomorphism \(\tilde{\beta}_{-\ell} : R_{\ell} \rightarrow \text{End}_{\mathcal{R}(-\ell)}(\overline{1}_{-\ell})\).

**Proof.** The bubble \(r_{-\ell-1} \circ \cdots \circ (-\ell)\) belongs to \(I\) for \(r > \ell\). Since the composition of \(\beta_{-\ell}\) with the canonical map \(\text{End}_{\mathcal{R}(-\ell)}(1_{-\ell}) \rightarrow \text{End}_{\mathcal{R}(-\ell)}(\overline{1}_{-\ell})\) takes \(e_r \in R\) to the the image of this bubble, which is zero, we deduce that this homomorphism factors through the quotient \(R_{\ell}\) of \(R\) to induce \(\tilde{\beta}_{-\ell}\). Moreover, \(\tilde{\beta}_{-\ell}\) is surjective by a special case of the “easy” part of Theorem 11.6.

To show that \(\tilde{\beta}_{-\ell}\) is also injective, we use the following diagram of graded supercategories and superfunctors:

\[
\begin{array}{ccc}
\mathcal{V}(-\ell)_{-\ell} & \xrightarrow{\Psi_{\ell}} & \mathcal{V}(-\ell)_{-\ell} \\
\cong \circ \downarrow \uparrow & & \circ \downarrow \uparrow \\
\mathcal{R}(-\ell)_{-\ell} & \xrightarrow{\Omega_{\mathcal{R}(-\ell)}} & \mathcal{R}(-\ell)_{-\ell} \\
\end{array}
\]

Here, the top map comes from Theorem 11.3, the left hand vertical superfunctor is given by evaluating on the object \(\overline{1}_{-\ell}\), and the right hand vertical superfunctor is given by tensoring with the lowest weight object \(OH_{0}^\ell\). The way the bottom superfunctor \(\Omega_{OH_{0}^\ell}\) is defined in Lemma 12.8 ensures that this diagram commutes strictly. It follows that the middle square in the following diagram commutes:

\[
\begin{array}{ccc}
R & \xrightarrow{\beta_{-\ell}} & \text{End}_{\mathcal{R}(-\ell)}(1_{-\ell}) \\
\downarrow \text{can} & & \downarrow \text{can} \\
R_{\ell} & \xrightarrow{\beta_{-\ell}} & \text{End}_{\mathcal{R}(-\ell)}(\overline{1}_{-\ell}) \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{End}_{\mathcal{R}(-\ell)}(1_{-\ell}) & \xrightarrow{\Psi_{\ell}} & \text{End}_{\mathcal{R}(-\ell)}(OH_{0}^\ell) \\
\downarrow \phi \circ \text{id} & & \downarrow \phi \circ \text{id} \\
\text{End}_{\mathcal{R}(-\ell)}(\overline{1}_{-\ell}) & \xrightarrow{\phi \circ 1} & OH_{0}^\ell = R_{\ell} \\
\end{array}
\]

Corollary 11.5 with \(n = 0, n' = \ell\) shows that the composition \(R \rightarrow R_{\ell}\) around the northeast boundary of this diagram maps \(\hat{h}_{r}\) to \(\gamma_{r}^{(\ell)}\). Hence, the composition \(R_{\ell} \rightarrow R_{\ell}\) of the three maps at the bottom of the diagram maps \(\hat{h}_{r}\) to \(\gamma_{r}^{(\ell)}\), so it is an isomorphism. This implies that \(\tilde{\beta}_{-\ell}\) is injective. \(\square\)

\(^{12}\)In fact, \(\beta_{-\ell}\) is itself an isomorphism thanks to Theorem 11.6, but this result is not needed here.
Any morphism space $\text{Hom}_{\mathcal{R}(\ell)}(X, Y)$ in $\mathcal{R}(\ell)$ can be viewed as a right $R$-supermodule so that $c \in R$ acts by horizontally composing on the right with $\beta(\ell)(c)$. This induces a structure of right $R_{\ell}$-supermodule on any morphism space $\text{Hom}_{\mathcal{R}(\ell)}(X, Y)$; cf. the first paragraph of the proof of Lemma 12.9. Given a graded $R_{\ell}$-superalgebra $A$, we let $\mathcal{V}(\ell) \otimes_{R_{\ell}} A$ be the graded supercategory with the same objects as $\mathcal{V}(\ell)$ and morphism spaces $\text{Hom}_{\mathcal{V}(\ell) \otimes_{R_{\ell}} A}(X, Y) := \text{Hom}_{\mathcal{V}(\ell)}(X, Y) \otimes_{R_{\ell}} A$. This is naturally a graded $2$-representation of $\mathcal{U}(\mathfrak{sl}_2)$ in its own right.

**Theorem 12.10.** Let $\mathcal{V}$ be any graded $2$-representation of $\mathcal{U}(\mathfrak{sl}_2)$, $\ell \in \mathbb{N}$ and $M \in \mathcal{V}_{\ell}$ be any lowest weight object. The strongly equivariant graded superfunctor $\Omega_M : \mathcal{V}(\ell) \to \mathcal{V}$ from Lemma 12.8 extends to a fully faithful strongly equivariant graded superfunctor $\Omega_M \otimes \text{id} : \mathcal{V}(\ell) \otimes_{R_{\ell}} A \to \mathcal{V}$.

**Proof.** The graded superfunctor $\Omega_M$ extends to $\Omega_M \otimes \text{id}$ by the universal property of tensor product. To see that the resulting graded superfunctor is fully faithful, we must show that it defines an isomorphism $\text{Hom}_{\mathcal{V}(\ell) \otimes_{R_{\ell}} A}(X, Y) \to \text{Hom}_{\mathcal{V}}(X, Y)$ for objects of any weight subcategory of $\mathcal{V}(\ell) \otimes_{R_{\ell}} A$. This is clear if $X = Y = \mathbb{T}_{\ell}$. The result in general then follows by the (now standard) technique explained in the proof of [R1, Lem. 5.4, Prop. 5.6]. \qed

**Corollary 12.11.** For $\ell \in \mathbb{N}$, let $\text{OH}^{\ell}$-pgsmod be the graded Karoubian $2$-representation from Example 12.6, and let $\text{gsKar}(\mathcal{V}(\ell))$ be the graded super Karoubi envelope of $\mathcal{V}(\ell)$, which is another graded Karoubian $2$-representation. The strongly equivariant graded superfunctor $\Omega_{\text{OH}^{\ell}} : \mathcal{V}(\ell) \to \text{OH}^{\ell}$-pgsmod associated to the lowest weight object $\text{OH}^{\ell}_{0}$ induces a strongly equivariant graded superequivalence $\Xi_{\ell} : \text{gsKar}(\mathcal{V}(\ell)) \to \text{OH}^{\ell}$-pgsmod.

**Proof.** In view of Lemma 12.9 and Theorem 12.10, $\Omega_{\text{OH}^{\ell}}$ is fully faithful. Since $\text{OH}^{\ell}$-pgsmod is Karoubian and II-complete, this extends by the universal properties of Karoubi and II-envelopes to give a fully faithful strongly equivariant graded superfunctor $\Xi_{\ell} : \text{gsKar}(\mathcal{V}(\ell)) \to \text{OH}^{\ell}$-pgsmod. To see that $\Xi_{\ell}$ is a graded superequivalence, it remains to check that it is dense. This follows because

$$E^{(n)} \text{OH}^{\ell}_{0} \simeq U^{\ell}_{(n):0} \otimes_{\text{OH}^{\ell}} \text{OH}^{\ell}_{0} \simeq \text{OH}^{\ell}_{n},$$

the last isomorphism following since $U^{\ell}_{(n):0}$ is free of rank $1$ as a graded left $\text{OH}^{\ell}_{n}$-supermodule by Lemma 9.4(2). \qed

We record one more basic lemma, which is analogous to the first part of [R1, Lem. 5.2].

**Lemma 12.12.** Let $\mathcal{V}$ be an integrable Karoubian graded $2$-representation of $\mathcal{U}(\mathfrak{sl}_2)$. Let $N$ be an object of $\mathcal{V}_{k}$ for some $k \in \mathbb{Z}$. If $\text{Hom}_{\mathcal{V}_{k}}(E^{n}M, N) = 0$ for all $n \in \mathbb{N}$, $n \geq 0$ such that $k = 2n - \ell$ and all lowest weight objects $M \in \mathcal{V}_{\ell}$, then $N = 0$.

**Proof.** Suppose that $N \neq 0$. By integrability, there exists $n \geq 0$ such that $F^{n}N \neq 0$ and $F^{n+1}N = 0$. This means that $M := F^{n}N$ is a non-zero lowest weight object of $\mathcal{V}_{\ell}$ for $\ell = k - 2n \in \mathbb{N}$. By assumption, we have that $\text{Hom}_{\mathcal{V}_{k}}(E^{n}M, N) = 0$. Hence, by adjunction,

$$\text{End}_{\mathcal{V}_{k}}(M) = \text{Hom}_{\mathcal{V}_{k}}(M, F^{n}N) \simeq \text{Hom}_{\mathcal{V}_{k}}(E^{n}M, N) = 0.$$

It follows that $1_{M} = 0$, so $M = 0$, which is a contradiction. \qed

**Remark 12.13.** There is more still to be done here. For example, Rouquier continues in [R1, Sec. 5.1.4] to construct a Jordan-Hölder series in an arbitrary integrable Karoubian $2$-representation, and this result assuredly carries over to our setting. There is also a good theory of *locally finite Abelian $2$-representations of $\mathcal{U}(\mathfrak{sl}_2)$*, including an analog of [CR, Prop. 5.20] which implies that the irreducible objects of such a $2$-representation can be given the structure of a crystal in the sense of Kashiwara. It would be worthwhile to extend [CR, Th. 5.27] (which is a special case of Rouquier’s “control by $K_{0}$”
from [R1, Th. 5.22]) to this setting. This would pave the way to more applications involving representations of the supergroup $Q(n)$ and the Lie superalgebra $\mathfrak{sl}_n(\mathbb{C})$. In the ordinary case, an alternative approach by-passing control by $K_0$ was developed in [BSW], which we expect should also have an interesting and non-trivial super analog. Another direction we would like to investigate further is to extend Theorems 11.3 and 11.6 from odd $\mathfrak{sl}_2$ to the super Kac-Moody 2-category associated to “odd $\mathfrak{s}_2^{2n+1}$”, thereby giving an odd analog of the 2-representation of the Kac-Moody 2-category of $\mathfrak{sl}_n$ constructed in [KL].

13. Odd analog of the Rickard complex

Let $\mathcal{V}'$ be a graded 2-supercategory. The notation $\text{Ch}^b(\mathcal{V})$ denotes the graded supercategory of bounded cochain complexes and chain maps in $\mathcal{V}$; differentials in a cochain complex are assumed to be even of degree 0 but we allow chain maps whose components are inhomogeneous. Also $K^b(\mathcal{V})$ is the homotopy category, which is a graded supercategory with the same objects as $\text{Ch}^b(\mathcal{V})$ and morphisms that are chain homotopy equivalence classes of chain maps; chain homotopies are again required to be even of degree 0. If $\mathcal{V}'$ is an integrable graded Karoubian 2-representation of $\mathfrak{U}(\mathfrak{sl}_2)$ as in the previous section, both $\text{Ch}^b(\mathcal{V}')$ and $K^b(\mathcal{V}')$ are themselves integrable graded Karoubian 2-representations of $\mathfrak{U}(\mathfrak{sl}_2)$ in a natural way.

Fix $k \in \mathbb{Z}$. The odd Rickard complex $\Theta_k$, so-called because it is the odd analog of the complex in [CR, Sec. 6.2] which was introduced originally by Rickard in the context of symmetric groups, is the following cochain complex in $\text{Ch}(\text{Hom}_{\text{gsKar}(\mathfrak{U}(\mathfrak{sl}_2))}(-, k))$:

$$
\begin{align*}
&\cdots \to Q^d E^{(k+d)} F^{(d)} 1_{-k} \xrightarrow{\bar{\partial}^d} Q^{d-1} E^{(k+d-1)} F^{(d-1)} 1_{-k} \to \cdots \to E^{(k)} 1_{-k} \to 0 \to \cdots \quad \text{if } k \geq 0 \\
&\cdots \to Q^d E^{(k+d)} F^{(d)} 1_{-k} \xrightarrow{\bar{\partial}^d} Q^{d-1} E^{(k+d-1)} F^{(d-1)} 1_{-k} \to \cdots \to Q^{-k} F^{(-k)} 1_{-k} \to 0 \to \cdots \quad \text{if } k \leq 0,
\end{align*}
$$

where in both cases $E^{(k+d)} F^{(d)} 1_{-k}$ is in cohomological degree $-d$. The differential

$$
\partial^{-d} : E^{(k+d)} F^{(d)} 1_{-k} \to E^{(k+d-1)} F^{(d-1)} 1_{-k}
$$

is the composition first of the “inclusion” of $Q^d E^{(k+d)} F^{(d)} 1_{-k} \to Q^{d+3d-2} E^{(k+d-1)} F^{(d-1)} 1_{-k}$ as a summand\(^{13}\) of $E^{(k+d-1)} F^{(d-1)} 1_{-k}$, then $Q^{d+3d-2} E^{(k+d-1)} F^{(d-1)} 1_{-k} \to Q^{d-1} E^{(k+d-1)} F^{(d-1)} 1_{-k}$. Note this is even of degree 0 as required. The following checks that it is a cochain complex.

**Lemma 13.1.** We have that $\partial^{-1} \circ \partial^{-d} = 0$ for all $d$.

**Proof.** Ignoring gradings for brevity, it suffices to show that the composition

$$
E^{(2)} F^{(2)} 1_{-k-2d+4} \xrightarrow{\text{inc}} E^{2} F^{2} 1_{-k-2d+4} \xrightarrow{\text{E} \text{E}F} E^{1} F^{1} 1_{-k-2d+4} \xrightarrow{\varepsilon F} E^{1} F^{1} 1_{-k-2d+4}
$$

is zero. The identity endomorphism of $E^{(2)} F^{(2)} 1_{-k-2d+4}$ is $\rho_2^{(k-2d)}(x_1) \lambda_2^{(k-2d)}(r_1 x_1) = (\rho_2^{(k-2d)}(r_1) \lambda_2^{(k-2d)}(x_1)) \circ (\rho_2^{(k-2d)}(x_1) \lambda_2^{(k-2d)}(x_1))$.

The composition of this with $\varepsilon \circ (\text{E} \text{E}F)$ is zero:

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\begin{scope}
\node (a) at (0,0) {$x$};
\node (b) at (0,1) {$x$};
\draw (a) to (b);
\end{scope}
\begin{scope}[xshift=-1cm]
\node (c) at (0,0) {$x$};
\node (d) at (0,1) {$x$};
\draw (c) to (d);
\end{scope}
\end{tikzpicture}
\end{array}
\quad =
\quad \begin{array}{c}
\begin{tikzpicture}
\begin{scope}
\node (a) at (0,0) {$x$};
\node (b) at (0,1) {$x$};
\draw (a) to (b);
\end{scope}
\begin{scope}[xshift=-1cm]
\node (c) at (0,0) {$x$};
\node (d) at (0,1) {$x$};
\draw (c) to (d);
\end{scope}
\end{tikzpicture}
\end{array}
\quad =
\quad 0.
\end{align*}
\]

\footnote{\textit{The idempotent endomorphism defining $Q^{3d-2} E^{(k+d-1)} F^{(d-1)} 1_{-k}$ as a summand of $Q^d E^{k-2d} F^d 1_{-k}$ decomposes as the sum of two mutually orthogonal idempotents, one of which is the idempotent defining $Q^{d-1} E^{(k+d-1)} F^{(d-1)} 1_{-k}$.}}
Remark 13.2. Note Lemma 13.1 plus Theorem 11.3 implies Lemma 10.5. So the proof of that lemma was actually unnecessary (as, by association, was Lemma 9.17) but we included it to make Section 10 independent of the subsequent material.

Suppose now that \( \mathcal{V} \) is an integrable graded Karoubian 2-representation of \( \mathfrak{U}(\mathfrak{sl}_2) \). Given any cochain complex \( C \in \text{Ch}^b(\mathcal{V}_k) \), we can apply the complex of graded superfunctors that is the image under \( \mathcal{V} \) of the odd Rickard complex \( \Theta_k \) to obtain a double complex. The associated total complex is again bounded thanks to the integrability assumption. This construction defines a graded superfunctor \( \text{Ch}^b(\mathcal{V}_k) \to \text{Ch}^b(\mathcal{V}_k) \). Passing to the quotient \( K^b(\mathcal{V}) \) of \( \text{Ch}^b(\mathcal{V}) \), we obtain from this a graded superfunctor

\[
\mathcal{V}(\Theta_k) : K^b(\mathcal{V}_k) \to K^b(\mathcal{V}_k).
\]

Theorem 13.5. Let \( \mathcal{V} \) be the graded 2-representation \( \mathcal{O}H^f \)-pgsmod from Example 12.6. The image of the odd Rickard complex \( \Theta_k \) under \( \mathcal{V} \) recovers the singular Rouquier complex from (10.4) shifted globally in degree by an application of \( Q^{nk} \).

Proof. This follows using the explicit identification of the divided powers \( E(d) \) and \( F(d) \) as endofunctors of \( \mathcal{V} \) explained in Example 12.6.

Corollary 13.4. For \( \ell \in \mathbb{N}, \) \( (\text{gsKar}(\mathcal{V}(\ell)))((\Theta_k)) : K^b(\text{gsKar}(\mathcal{V}(\ell))_k) \to K^b(\text{gsKar}(\mathcal{V}(\ell))_k) \) is a graded superequivalence inducing \( T : 1_{\otimes} \mathcal{V}(-\ell) \to 1_k \mathcal{V}(-\ell) \) at the level of the Grothendieck groups.

Proof. This follows from Lemma 13.3 together with Corollary 10.4 and Corollary 12.11.

The proof of the following theorem is based on the argument in [R1, Th. 5.18], the main step really being [R1, Lem. 5.5]. This was itself a generalization of [CR, Th. 6.4] which constructed equivalences between bounded derived categories of locally finite Abelian 2-representations.

Theorem 13.5. Let \( \mathcal{V} \) be an integrable graded Karoubian 2-representation of \( \mathfrak{U}(\mathfrak{sl}_2) \). For \( k \in \mathbb{Z} \), the graded superfunctor \( \mathcal{V}(\Theta_k) : K^b(\mathcal{V}_k) \to K^b(\mathcal{V}_k) \) induced by the odd Rickard complex is a graded superequivalence.

Proof. By Lemma 11.11, the 1-morphism \( Q^dE^{d(k+d)}F^{d}1_{\otimes}k \) has a right dual in \( \text{gsKar}(\mathfrak{U}(\mathfrak{sl}_2)) \). Hence, we can form the right dual \( \Theta^k \) to \( \Theta_k \), which is a cochain complex in \( \text{Ch}(\text{Hom}_{\text{gsKar}(\mathfrak{U}(\mathfrak{sl}_2))}(k, -k)) \). The 1-morphism in the \( d \)th cohomological degree of \( \Theta^k \) is the right dual of the 1-morphism in the \( -(d) \)th cohomological degree of \( \Theta_k \), and the differentials in \( \Theta^k \) are the right mates of the corresponding differentials in \( \Theta_k \). Let \( \Theta^k \circ \Theta_k \) and \( \Theta_k \circ \Theta^k \) be the total complexes associated to the double complexes obtained by composing these cochain complexes. The complex \( \Theta_k \) is bounded above, and \( \Theta^k \) is bounded below, but neither is bounded. Consequently, in each cohomological degree, the total complexes \( \Theta^k \circ \Theta_k \) and \( \Theta_k \circ \Theta^k \) involve infinite direct sums of 1-morphisms in \( \text{gsKar}(\mathfrak{U}(\mathfrak{sl}_2)) \), so in fact, one needs to pass to a completion of this graded \( (\otimes, \Pi) \)-supercategory for it to make sense. This does not cause issues since, on a given object in an integrable graded Karoubian 2-representation, the superfunctors arising from all but finitely many of the summands of these infinite direct sums are zero.

Like \( \Theta_k \), the complex \( \Theta^k \) defines a graded superfunctor denoted \( \mathcal{V}(\Theta^k) : K^b(\mathcal{V}_k) \to K^b(\mathcal{V}_k \otimes \mathcal{V}_k) \). Moreover, \( \mathcal{V}(\Theta^k) \) is right adjoint to \( \mathcal{V}(\Theta_k) \), with counit and unit of adjunction denoted

\[
\mathcal{V}(\varepsilon) : \mathcal{V}(\Theta^k) \circ \mathcal{V}(\Theta_k) \Rightarrow \text{Id}_{K^b(\mathcal{V}_k)}, \quad \mathcal{V}(\eta) : \text{Id}_{K^b(\mathcal{V}_k)} \Rightarrow \mathcal{V}(\Theta^k) \circ \mathcal{V}(\Theta_k).
\]

This is explained in more detail in [CR, Sec. 4.1.4]. As the notation \( \mathcal{V}(\varepsilon) \) and \( \mathcal{V}(\eta) \) suggests, if we identify \( \mathcal{V}(\Theta_k) \circ \mathcal{V}(\Theta_k) \) with \( \mathcal{V}(\Theta_k \circ \Theta_k) \) and \( \mathcal{V}(\Theta^k) \circ \mathcal{V}(\Theta_k) \) with \( \mathcal{V}(\Theta^k \circ \Theta_k) \) then these even degree 0 supernatural transformations are induced by corresponding chain maps denoted simply by \( \varepsilon : \Theta_k \circ \Theta^k \Rightarrow 1_k \) and \( \eta : 1_{\otimes} \rightarrow \Theta_k \circ \Theta^k \) between cochain complexes in the completion of \( \text{gsKar}(\mathfrak{U}(\mathfrak{sl}_2)) \). Although not needed here, these chain maps can be seen quite explicitly; the matrix coefficients of their components
are 2-morphisms in gsKar(\(\mathfrak{U}(sl_2)\)) that arise from the counits and units defining the duality between the 1-morphisms \(Q^d E^{(k+d)}F^{(d)}1_{-\bar{k}}\) and their right duals.

To prove the theorem, it suffices to show that \(\mathcal{V}(\epsilon)\) and \(\mathcal{V}(\eta)\) are isomorphisms. We just explain the argument to see this in the case of \(\mathcal{V}(\epsilon)\), since the case of \(\mathcal{V}(\eta)\) is similar. The even degree 0 graded supernatural transformation \(\mathcal{V}(\epsilon)\) is an isomorphism if and only if \(\text{Cone}(\mathcal{V}(\epsilon)C) = 0\) for all \(C \in K^b(\mathcal{V}_k^e)\).

Now we observe that

\[
\text{Cone}(\mathcal{V}(\epsilon)C) = \mathcal{V}(Z)(C)
\]

where \(Z := \text{Cone}(\epsilon)\) is the cone of \(\epsilon : \Theta_k \circ \Theta^k \Rightarrow 1_k\). Thus, it suffices to show that the graded superfunctor \(\mathcal{V}(Z) : K^b(\mathcal{V}_k^e) \to K^b(\mathcal{V}_k^e)\) is zero.

Consider \(K^b(\mathcal{V})\) as an integrable Karoubian graded 2-representation in its own right. In this paragraph, we show that \(\mathcal{V}(Z)(E^{n\bar{1}}C) = 0\) in \(K^b(\mathcal{V}_k^e)\) for all \(n \geq 0\) such that \(k = 2n - \ell\), and all lowest weight objects \(C \in K^b(\mathcal{V}_k^e)\). To do this, we apply Lemma 12.8 (with \(\mathcal{V}\) replaced by \(K^b(\mathcal{V})\)) to get a strongly equivariant graded superfunctor \(\Omega_C : \mathcal{V}(-\ell) \to K^b(\mathcal{V})\) taking \(1_{-\ell}\) to \(C\) by the general discussion in [CR, Sec. 4.1.4] again. We must show that \(\text{Cone}(\mathcal{V}(\epsilon)C)\) is the weight lattice; we have that \(\text{End}_{K^b(\mathcal{V}_k^e)}(\mathcal{V}(Z)) = 0\) as required.

To complete the proof, we let \(\mathcal{V}(Z)^\circ\) be a right adjoint to \(\mathcal{V}(Z) : K^b(\mathcal{V}_k^e) \to K^b(\mathcal{V}_k^e)\), which exists by the general discussion in [CR, Sec. 4.1.4] again. We must show that \(\mathcal{V}(Z)(D) = 0\) for any \(D \in K^b(\mathcal{V}_k^e)\), which we do by showing that \(\mathcal{V}(Z)^\circ(\mathcal{V}(Z)(D)) = 0\); this is sufficient since it implies that \(\text{End}_{K^b(\mathcal{V}_k^e)}(\mathcal{V}(Z)(D)) = 0\). Using Lemma 12.12, we just need to show that

\[
\text{Hom}_{K^b(\mathcal{V}_k^e)}\left(E^{n\bar{1}}C, \mathcal{V}(Z)^\circ(\mathcal{V}(Z)(D))\right) = 0
\]

for \(C\) and \(n\) as in the previous paragraph. This follows because by adjunction we have that

\[
\text{Hom}_{K^b(\mathcal{V}_k^e)}\left(E^{n\bar{1}}C, \mathcal{V}(Z)^\circ(\mathcal{V}(Z)(D))\right) \cong \text{Hom}_{K^b(\mathcal{V}_k^e)}\left(\mathcal{V}(Z)(E^{n\bar{1}}C), \mathcal{V}(Z)(D)\right)
\]

which is zero by the previous paragraph. \(\square\)

14. APPLICATION TO REPRESENTATIONS OF SPIN SYMMETRIC GROUPS

Theorem 13.5 can be applied to obtain graded superequivalences between homotopy/derived categories of supermodules over the cyclotomic quiver Hecke superalgebras from [KKT, KKO1, KKO2]. In explaining this, we will mainly cite [KKO2, Sec. 8] which presents the results needed to do this rather concisely. However, we need to reverse the roles of \(E\) and \(F\) compared to [KKO2] to be consistent with our convention for \(\mathfrak{U}(sl_2)\) in Section 11, in which we preferred lowest weight modules to highest weight modules.

Fix a Cartan superdatum \((A, P, \Pi, \Pi^\circ)\) as in [KKO2, Sec. 4.1]. So:

- \(I\) is an index set with given decomposition \(I = I_{\text{even}} \uplus I_{\text{odd}}\);
- \(A = (a_{i,j})_{i,j \in I}\) is a symmetrizable Cartan matrix such that \(a_{i,j} \in 2\mathbb{Z}\) for all \(i \in I_{\text{odd}}, j \in I\);
- \(P\) is the weight lattice;
- \(\Pi = \{\alpha_i \mid i \in I\}\) is the set of simple roots;
\begin{itemize}
    \item \( \Pi' = \{ h_i \mid i \in I \} \) is the set of simple coroots.
\end{itemize}

Let \( d_i (i \in I) \) be positive integers chosen so that \( d_i a_{i,j} = d_j a_{i,j} \) for all \( i, j \in I \). Let \( P^+ \) be the corresponding set of dominant weights and \( Q^+ := \bigoplus_{i \in I} \mathbb{N} \alpha_i \) be the non-negative part of the root lattice. Finally, let \( W < \text{Aut}(P) \) be the Weyl group.

Let \( k = \bigoplus_{d \geq 0} k_d \) be a positively graded commutative ground ring with \( k_0 = \mathbb{F} \) (our usual algebraically closed ground field) and \( \dim_k k_d < \infty \) for all \( d \). We view \( k \) as a purely even graded \( \mathbb{F} \)-superalgebra. Given any \( \alpha \in Q^+ \), there is a corresponding \textit{quiver Hecke superalgebra} \( R_{\alpha} \) which is defined by generators and relations as in [KKO2, Sec. 8.1]; the definition depends on an additional choice of parameters as explained in [KKO2]. Let \( R^d_{\alpha} \) be the deformed cyclotomic quotient from [KKO2, Def. 8.10] associated to a dominant weight \( \lambda \in P^+ \) and a choice of monic polynomials \( a_i^d (i \in I) \) as in [KKO2, (8.12)]. We are interested in the graded \((Q, \Pi)\)-supercategory

\[
R^d_{\alpha}: \bigoplus_{\alpha \in Q^+} R^d_{\alpha}-\text{pgsmod.} \quad (14.1)
\]

The constructions in [KKO2, Sec. 8.3] make \( R^d_{\alpha}-\text{pgsmod} \) into a “supercategorification” of the integrable lowest weight module \( V(-\lambda) \) for the covering quantum group \( U_{q,\pi}(g) \) with the given Cartan superdatum. From this, it can be seen that \( R^d_{\alpha}-\text{pgsmod} \) has the structure of a graded 2-representation of the corresponding graded Kac-Moody 2-supercategory as defined in [BE2], with the Grothendieck group \( K_0 (R^d_{\alpha}-\text{pgsmod}) \) being identified with the Kostant \( \mathbb{Z}[q,q^{-1}] \)-form for \( V(-\lambda) \).

To be more precise, we focus now on some fixed \( i \in I \) and consider the corresponding \( sl_2 \)-subalgebra of \( U_{q,\pi}(g) \). In this generality, we actually need to work now with \( q_i := d_i^q \) and the grading shift functor \( \varphi_i := Q^+ \) rather than \( q \) and \( Q \) used in previous sections. This means that when \( d_i > 1 \) definitions such as Definition 12.1 earlier in the paper should be modified by replacing \( Q \) with \( Q_i \) and scaling all degrees by \( d_i \) too, e.g., \( x \) and \( \tau \) are now of degrees \( 2d_i \) and \( -2d_i \) rather than of degrees 2 and \( -2 \). There are graded superfunctors

\[
E_i : R^d_{\alpha}-\text{pgsmod} \to R^d_{\alpha}-\text{pgsmod}, \quad F_i : R^d_{\alpha}-\text{pgsmod} \to R^d_{\alpha}-\text{pgsmod}.
\]

In terms of the induction and restriction functors denoted \( F_i \) and \( E_i \) in [KKO2, Sec. 8.3], our \( E_i \) is \( F_i^d = \bigoplus_{\alpha \in Q^+} F_i^d |_{R^d_{\alpha}-\text{pgsmod}} \) and our \( F_i \) is \( \bigoplus_{\alpha \in Q^+} Q^+ |_{R^d_{\alpha}-\text{pgsmod}} \). As well as switching the roles of \( E \) and \( F \) we have incorporated an additional grading shift into the restriction functors compared to [KKO2]. This is needed because [KKO2] does not follow the standard conventions for covering quantum groups. It ensures that the graded supernatural transformations \( \varepsilon : E_i |_{R^d_{\alpha}-\text{pgsmod}} = \text{Id}_{R^d_{\alpha}-\text{pgsmod}} \) and \( \eta : \text{Id}_{R^d_{\alpha}-\text{pgsmod}} \Rightarrow F_i |_{R^d_{\alpha}-\text{pgsmod}} \) defined on a graded supermodule by exactly the same underlying functions as for the natural adjunction between restriction and induction are of the correct degree to match the degrees of the rightward cups and caps in (11.9) (also now scaled by \( d_i \)). Also in [KKO2, Sec. 8.3], one finds the definition of graded supernatural transformations \( x : E_i \Rightarrow E_i \) of degree \( 2d_i \) and \( \tau : E_i^2 \Rightarrow E_i \) of degree \( -2d_i \), both of which are even if \( i \in I_{\text{even}} \) and odd if \( i \in I_{\text{odd}} \). (A further complication is that the language of supercategory, superfunctor and supernatural transformation is used differently in [KKO2] compared to here, but the appropriate translation is easy to make; see the table at the end of the introduction in [BE1].)

This construction makes \( R^d_{\alpha}-\text{pgsmod} \) into a graded integrable Karoubian 2-representation of the ordinary \( sl_2 \) 2-category from [L1, R1] if \( i \) is even, or of our reduced odd \( sl_2 \) 2-category \( \mathfrak{U}(sl_2) \) as in Definition 12.1 if \( i \) is odd (with the modified convention for degrees when \( d_i > 1 \)). The last statement is not stated explicitly in [KKO2]—the relevant place is [KKO2, Th. 8.13] but one has to work through the proof which goes back to [KK, Th. 5.2] to see that the isomorphisms are given by the appropriate matrices of supernatural transformations needed to check the difficult relations (11.5) and (11.6). In the
odd case, the fact that the odd bubbles act as zero (as required by the final axiom in Definition 12.1) follows because they are zero on the generating lowest weight subcategory $R_0^l$ as that is purely even.

The following theorem now follows from [R1, Th. 5.18] if $i$ is even, with the graded superequivalence being induced by the even analog of the Rickard complex, or our Theorem 13.5 if $i$ is odd.

**Theorem 14.1.** In the above setup, for $\alpha \in Q^+$ such that $V(-\lambda)_{\alpha-\lambda} \neq 0$, there is a graded superequivalence $K^h(R_0^l$-pgsmod) $\to K^h(R_{\alpha-(\beta,\alpha-\lambda)\alpha}^l$-pgsmod) categorifying the action of the $i$-th generator of the braid group of $W$ on $V(-\lambda)$.

There is also a dual version of this theorem with $R^l$-pgsmod replaced with

$$R^l$-gsmod := $\bigoplus_{\alpha \in Q^+} R^l_\alpha$-gsmod, \quad (14.2)$$

where for a graded superalgebra $A$ we write $A$-gsmod for the graded supercategory of graded left $A$-supermodules that are finite-dimensional over the ground field $\mathbb{F}$. The underlying ordinary category is a locally finite Abelian $(Q, \Pi)$-category. The results from [KKO2, Sec. 8.3] show that this categorifies the dual Kostant $\mathbb{Z}[q, q^{-1}]$-form for $V(-\lambda)$. For fixed $i \in I$ again, $R^l$-gsmod can be made into a graded 2-representation of the even or reduced odd $sl_2$ 2-category exactly as above. Since we are now in an Abelian setting, it makes sense to consider $D^b(R^l$-gsmod), the graded super analogue of the usual bounded derived category of an Abelian category. Formally, this is defined to be the localization of $K^b(R^l$-gsmod) at even degree 0 quasi-isomorphisms. The graded superequivalence of homotopy categories obtained as in [R1, Th. 5.18], or our Theorem 13.5 in the case that $i$ is odd, immediately implies the following.

**Theorem 14.2.** In the above setup, for $\alpha \in Q^+$ such that $V(-\lambda)_{\alpha-\lambda} \neq 0$, the even or odd Rickard complex $\Theta_{\alpha-(\beta,\alpha-\lambda)}$ induces a graded superequivalence $D^b(R_\alpha$-gsmod) $\to D^b(R_{\alpha-(\beta,\alpha-\lambda)\alpha}^l$-gsmod) with quasi-inverse induced by the right adjoint $\Theta_{\alpha-(\beta,\alpha-\lambda)}$ of this complex.

For a graded superalgebra $A$, we write $A \otimes C_1$ for the graded superalgebra obtained by tensoring with the rank one Clifford superalgebra generated by an odd degree 0 involution. There is also a variation of Theorem 14.2 with $R^l$-gsmod replaced by

$$R^l \otimes C_1$-gsmod := $\bigoplus_{\alpha \in Q^+} R^l_\alpha \otimes C_1$-gsmod. \quad (14.3)$$

This can be made into a graded 2-representation which also categorifies the dual Kostant $\mathbb{Z}[q, q^{-1}]$-form for $V(-\lambda)$, just as $R^l$-gsmod did earlier. In particular, for each $i \in I$, we can make $R^l \otimes C_1$-gsmod into a graded 2-representation of the even or reduced odd $sl_2$ 2-category exactly as above. This follows by the construction explained in the next paragraph.

There is a general notion of the Clifford twist $\mathcal{A}^{CT}$ of a graded supercategory $\mathcal{A}$, which goes back to [KKT, Lem. 2.3]. By definition, this is the graded supercategory whose objects are pairs $(X, \phi)$ for $X \in \mathcal{A}$ and an odd degree 0 involution $\phi \in \text{End}_\mathcal{A}(X)$. A morphism $f : (X, \phi) \to (Y, \theta)$ is a morphism $f : X \to Y$ in $\mathcal{A}$ such that $\theta \circ f = (-1)^{\text{par}(f)} \phi \circ f$. Degree and parity of morphisms in $\mathcal{A}^{CT}$ are induced by the ones for $\mathcal{A}$. There are obvious ways to define the Clifford twist $F^{CT} : \mathcal{A}^{CT} \to \mathcal{B}^{CT}$ of a graded superfunctor $F : \mathcal{A} \to \mathcal{B}$, and also the Clifford twist $\alpha^{CT} : F^{CT} \Rightarrow G^{CT}$ of a graded supernatural transformation $\alpha : F \Rightarrow G$ between two graded superfunctors. This makes $\mathcal{CT}$ into a strict graded 2-superfunctor $\mathcal{CT} : \text{gsCat} \to \text{gsCat}$. Now if $\mathcal{F}'$ is any graded 2-representation of the even or the reduced odd 2-supercategory $\Pi(sl_2)$, its Clifford twist $\mathcal{F}^{\mathcal{CT}}$ can be made into a graded 2-representation in its own right, with the required graded superfunctors $E$ and $F$ on $\mathcal{F}^{\mathcal{CT}}$ being the Clifford twists of the ones for $\mathcal{F}'$, and all of the required graded supernatural transformations being the Clifford twists of the one for $\mathcal{F}'$ too. If $\mathcal{F}'$ is integrable and Karoubian then so is $\mathcal{F}^{\mathcal{CT}}$.

Now the same general machinery used to prove Theorem 14.2 gives the following variation.
Theorem 14.3. In the above setup, for \( \alpha \in Q^+ \) such that \( V(-\lambda)_{\alpha \rightarrow i} \neq 0 \), the even or odd Rickard complex \( \Theta_{(h, \lambda - \alpha)} \) induces a graded superequivalence \( D^b(R^l_{\alpha} \otimes C_1\text{-}gsmod) \to D^b(R^l_{-\alpha, h, \lambda - \alpha, i} \otimes C_1\text{-}gsmod) \) with quasi-inverse induced by the right adjoint \( \Theta_{(h, \lambda - \alpha)} \) of this complex.

Assume henceforth that the ground field \( \mathbb{F} \) is algebraically closed of odd characteristic \( p = 2l + 1 \), and that the Cartan superdatum fixed above is of type \( A^{(2)}_l \), with the shortest simple root \( \alpha_0 \) being odd and all other simple roots being even. We consider the cyclotomic quiver Hecke superalgebras \( R^l_\alpha \) for \( \alpha \in Q^+ \) and \( \lambda := \Lambda_0 \), taking the ring \( \mathbb{k} \) to be the ground field \( \mathbb{F} \), and all other choices made as explained in [KLi, Sec. 3.1]. Let \( R^l_\alpha \otimes C_1 \) be the superalgebra tensor product of \( R^l_\alpha \) with the rank one Clifford superalgebra generated by an odd involution. Now we forget both the \( \mathbb{Z} \)- and \( \mathbb{Z}/2 \)-gradings on \( R^l_\alpha \) and \( R^l_\alpha \otimes C_1 \) to view them as ordinary finite-dimensional algebras. For such an algebra \( A \), we write \( A\text{-mod} \) for the Abelian category of finite-dimensional left \( A \)-modules and \( D^b(A\text{-mod}) \) for its ordinary bounded derived category.

In view of [KLi, Lem. 3.1.39], the following proves [KLi, Conj. 2].

Theorem 14.4. Suppose that \( \alpha, \beta \in Q^+ \) are such that \( \alpha - \lambda \) and \( \beta - \lambda \) are weights of \( V(-\lambda) \) in the same \( W \)-orbit. The categories \( D^b(R^l_\alpha\text{-}mod) \) and \( D^b(R^l_\beta\text{-}mod) \) are equivalent as are \( D^b(R^l_\alpha \otimes C_1\text{-}mod) \) and \( D^b(R^l_\beta \otimes C_1\text{-mod}) \).

Proof. Since the simple reflections generate \( W \), it suffices to prove the theorem in the special case that \( \alpha - \lambda \) is a weight of \( V(-\lambda) \) and \( \beta = \alpha - (h_i, \alpha) \alpha_i \) for some \( i \in I \). The graded superequivalences in Theorems 14.2 and 14.3 are obtained by taking the derived tensor product with the complex of graded superbimodules arising from the appropriate Rickard complex. Similarly, the quasi-inverse graded superequivalences are obtained from the right adjoint of this complex. Now we are forgetting both the \( \mathbb{Z} \)- and \( \mathbb{Z}/2 \)-gradings, viewing these complexes of graded superbimodules as complexes of ordinary bimodules. The resulting complexes define functors between the ordinary derived categories. Since they are quasi-inverse with all gradings present, they are obviously quasi-inverse without these gradings. \( \square \)

Corollary 14.5. Broué’s Abelian Defect Group Conjecture holds for double covers of symmetric and alternating groups over any algebraically closed field of positive characteristic.

Proof. See [KLi, Th. 5.4.12] where this is deduced from [KLi, Conj. 2]. \( \square \)

References


[ELV] M. Ebert, A. Lauda and L. Vera, Derived superequivalence for spin symmetric groups and odd $\mathfrak{sl}_2$-categorifications, preprint.


