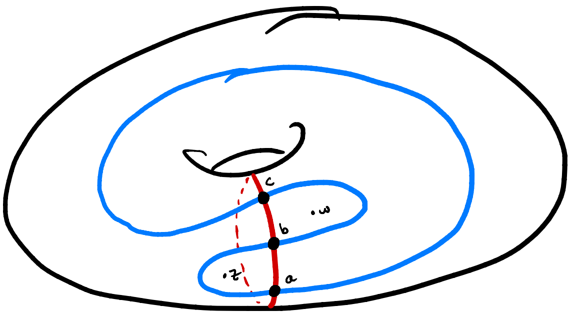
Knot Floer Homology of Left Hand Trefoil

Here we are finding CFK^∞, which is an $\mathbb{F}[u,u^{-1}]$-module

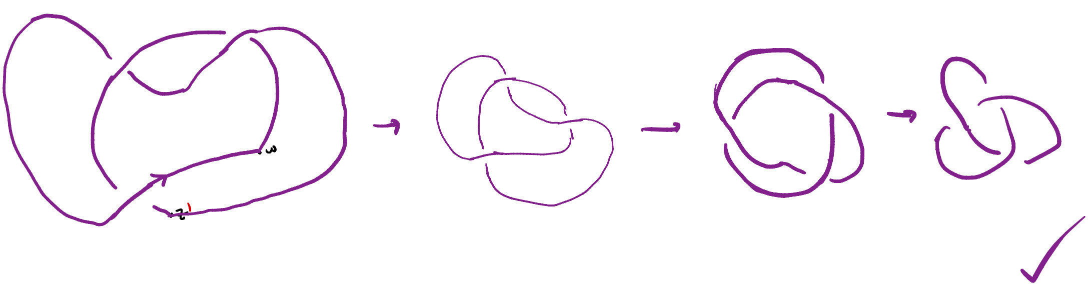
① Start with a doubly-pointed Heegaard diagram.



② Verify this actually is a Heegaard diagram for LHT...

Connect w to z avoiding α circles, pushed into $\mathcal{H}_1$. Connect z to w avoiding β circles, pushed into $\mathcal{H}_1$.

$\mathcal{H}_1$ is the solid torus drawn, $\mathcal{H}_2$ is exterior.

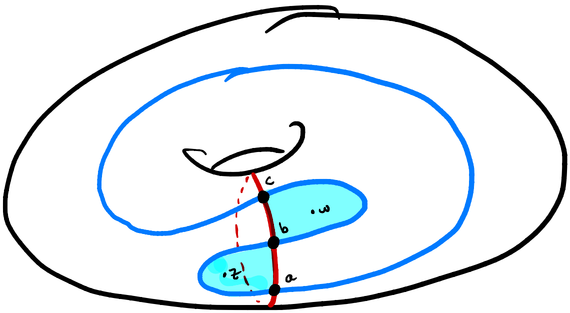


③ Count holomorphic disks

$g=1$ (genus)

$k=1$ (# basepoints)

$d=1$ (# α-curves)



We have...

- one from a to b containing z $\quad\phi$
- one from c to b containing w $\quad\psi$

So...

$$\partial a = b$$

$$\partial c = ub$$

Formula:

$$\boxed{\partial x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1}} \#\hat{M}(\phi)\, u^{n_w(\phi)} \cdot y}$$

④ Determine the relative Maslov gradings

Formula

$$\boxed{M(x)-M(y) = \mu(\phi) - 2\sum_{i=1}^{k} n_{w_i}(\phi)}$$

In our case,

$n_w(\phi)=0$ $\qquad$ $n_z(\phi)=1$

$n_w(\psi)=1$ $\qquad$ $n_z(\psi)=0$

$$M(a)-M(b) = \mu(\phi) - 2n_w(\phi)$$
$$= 1-0$$
$$= 1$$

$$M(c)-M(b) = \mu(\psi) - 2n_w(\psi)$$
$$= 1-2$$
$$= -1$$

$$\Rightarrow M(a)-M(b) = M(b)-M(c) = 1$$

(5) Determine the relative Alexander gradings

Formula

$$\boxed{A(x) - A(y) = \sum_{i=1}^{k} n_{z_i}(\phi) - \sum_{i=1}^{k} n_{w_i}(\phi)}$$

In our case,

$$A(a) - A(b) = n_z(\phi) - n_w(\phi)$$
$$= 1 - 0$$
$$= 1$$

$$A(c) - A(b) = n_z(\psi) - n_w(\psi)$$
$$= 0 - 1$$
$$= -1$$

(6) Remove the indeterminacy of the Maslov grading by looking at $\widehat{CFK}(\mathcal{H})$. The homology is $\widehat{HF}(S^3) \cong \mathbb{F}$ in degree 0.

Since $\widehat{CFK}(\mathcal{H})$ only looks at disks which miss w, we have

- one from a to b containing z $\quad \emptyset$
- ~~one from c to b containing w~~ $\quad \cancel{\emptyset}$

So...

$$\partial a = b$$

$$\cancel{\partial c = wb} \qquad \partial c = 0$$

So the homology is

$$\mathbb{F}\langle b, c\rangle / \langle b\rangle \cong \mathbb{F}\langle c\rangle$$

so we set c to have Maslov grading 0.

(7) Use the Alexander polynomial $\Delta_K(t) = t - 1 + t^{-1}$ to remove the indeterminacy in the Alexander gradings, since we require

$$\sum_{d,r} (-1)^d \operatorname{rank} \widehat{HFK}_d(K,r) \cdot t^r = \pm \Delta_K(t)$$

symmetrized

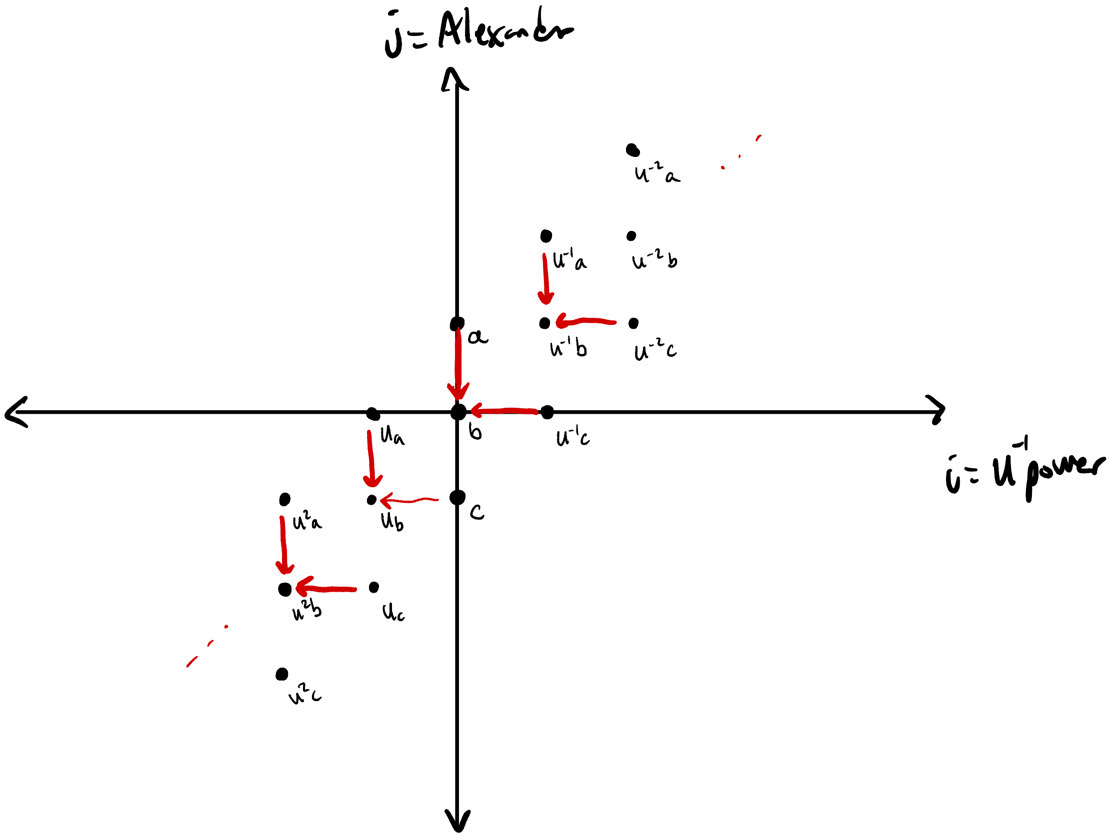
From the above, a was the biggest and the relative values give everything else....

Summarizing, we have

See the computation of $\widehat{HFK}$ below...

	M	A
a	2	1
b	1	0
c	0	-1

⑧ Plot CFK^∞



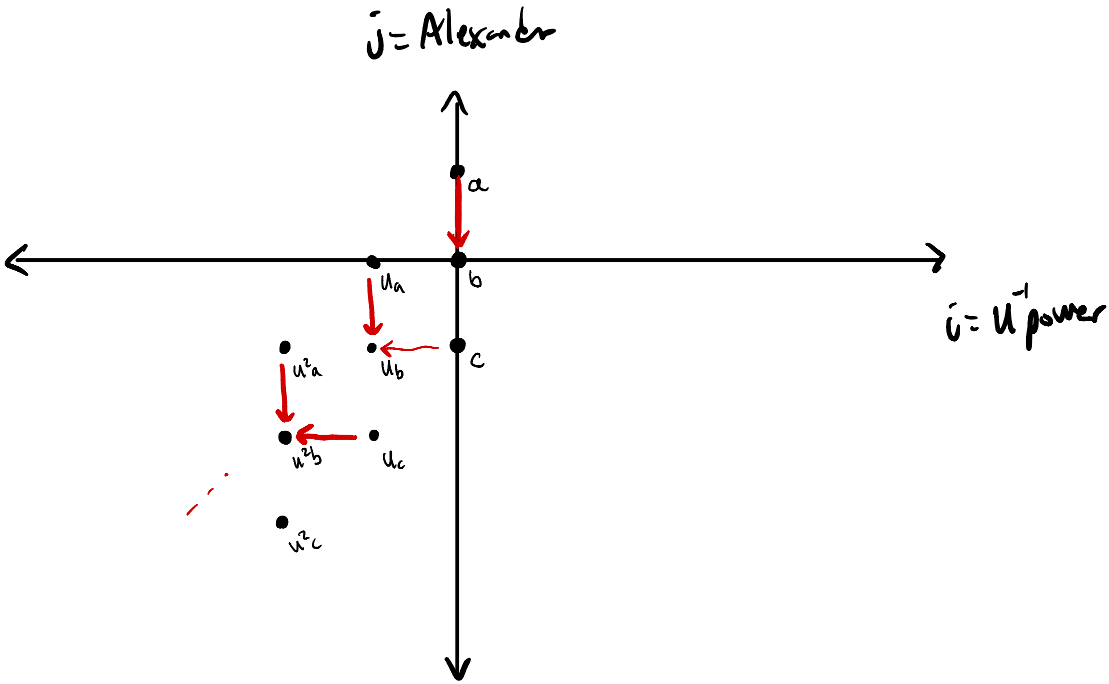
$\partial a = b$

$\partial c = ub$

U decreases Alexander by 1

⑨ We can specialize to CFK^-

To do this, consider the pairs $[x, i, j]$ above with $i \leq 0$



<u>Note:</u> ker is $\mathbb{F}[u]\langle b, Ua - c\rangle$

im is $\mathbb{F}[u]\langle b\rangle$

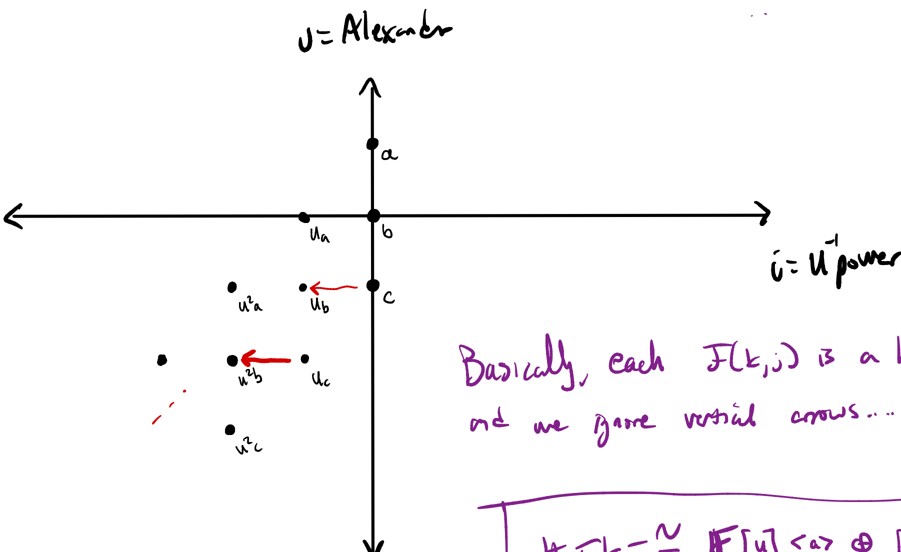
Homology is $\mathbb{F}[u] \cong \mathbb{F}[u]\langle Ua - c\rangle$

⑩ Compute HFK^- by looking at Alexander filtration... (reintroduce the z-basepoint)

- Intersection points $x \in T_\alpha \cap T_\beta$ with $A(x) \leq j$ form a subcomplex $\mathcal{F}(K, j) \subset CFK^-$

- Filtered chain homotopy type is knot invariant

- Filtered isomorphism type depends on diagram



Basically, each $\mathcal{F}(K, j)$ is a horizontal line and we ignore vertical arrows...

$$\boxed{HFK^- \cong \mathbb{F}[U]\langle a\rangle \oplus \mathbb{F}\langle b\rangle}$$

(11) Can do the same for $\widehat{CFK}$... compute $\widehat{HFK}$ which is the homology of the associated graded w.r.t. to Alexander filtration

* Pins down A-grading.

Associated graded of $\widehat{CFK}$ is

$$\begin{array}{ll}
\vdots & \vdots \\
\mathcal{F}(K,\alpha+2)/\mathcal{F}(K,\alpha) & 0 \\
\mathcal{F}(K,\alpha+1)/\mathcal{F}(K,\alpha) & 0 \\
\mathcal{F}(K,\alpha)/\mathcal{F}(K,\alpha-1) & \mathbb{F}\langle a\rangle \\
\mathcal{F}(K,\alpha-1)/\mathcal{F}(K,\alpha-2) & \mathbb{F}\langle b\rangle \\
\mathcal{F}(K,\alpha-2)/\mathcal{F}(K,\alpha-3) & \mathbb{F}\langle c\rangle \\
\mathcal{F}(K,\alpha-3)/\mathcal{F}(K,\alpha-4) & 0 \\
\vdots & \vdots
\end{array}$$

Let α be the Alexander grading of generator a...

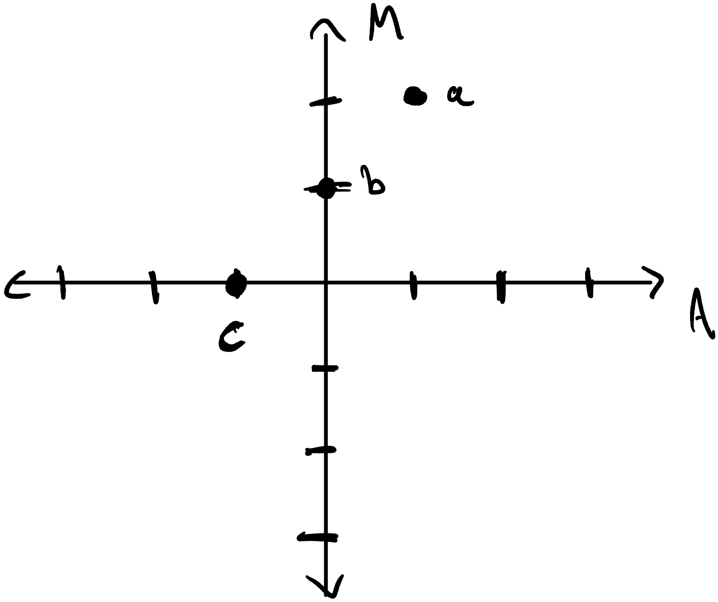
$\Downarrow$

So α must be 1 to agree with the Euler char property.

$$g\widehat{CFK} = \underset{(1,2)}{\mathbb{F}\langle a\rangle} \oplus \underset{(0,1)}{\mathbb{F}\langle b\rangle} \oplus \underset{(-1,0)}{\mathbb{F}\langle c\rangle} \qquad (A,m)$$

$\widehat{HFK} = \mathbb{F}\langle a,b,c\rangle$ since differentials on $g\widehat{CFK}$ are trivial...

(12) Compare this to $\Delta_K(t) = t - 1 + t^{-1}$



$$\Rightarrow \quad (-1)^0 t^{-1} + (-1)^1 t^0 + (-1)^2 t^1$$

$$= t^{-1} - 1 + t \qquad \checkmark$$