Knot Floer Homology of Left Head Trefoil

Here we are finding $\text{CFK}^\infty$, which is an $\mathbb{F}[[y^m]]$-module

1. Start with a doubly-pointed Heegaard diagram.

2. Verify this actually is a Heegaard diagram for LHT...

Connect $w$ to $z$ avoiding $a$ circles, pushed into $\mathcal{H}_1$. Connect $z$ to $w$ avoiding $b$ circles, pushed into $\mathcal{H}_2$. $\mathcal{H}_1$ is the solid torus drawn, $\mathcal{H}_2$ is exterior.
3. Count holomorphic disks

\[ g = 1 \]  ( genus )
\[ k = 1 \]  ( \# boundary )
\[ d = 1 \]  ( \# a.c.m.)

We have...

- one from a to b containing \( z \)  \( \emptyset \)
- one from c to b containing \( W \)  \( \emptyset \)

So...

\[ \partial a = b \]
\[ \partial c = u_b \]

Formula:

\[ \exists x = \sum_{y \in \partial_a \cap \partial_c} \sum_{\phi \in T_z(x, y)} \# \hat{M}(\phi) \cdot u^w(\phi) \cdot \gamma \]

\[ \text{for } M(\phi) = 1 \]
Determine the relative Maslow guidings

Formula

\[ M(x) - M(y) = m(\phi) - 2 \sum_{i=1}^{k} n_{w_i}(\phi) \]

In our case,

\( n_w(\phi) = 0 \)
\( n_z(\phi) = 1 \)
\( n_w(\psi) = 1 \)
\( n_z(\psi) = 0 \)

\( M(a) - M(b) = m(\phi) - 2 n_w(\phi) \)
\[ = 1 - 0 \]
\[ = 1 \]

\( M(c) - M(b) = m(\psi) - 2 n_w(\psi) \)
\[ = 1 - 2 \]
\[ = -1 \]

\( M(a) - M(b) = M(b) - M(c) = 1 \)
Determine the relative Alexander gradings

**Formula**

\[ A(x) - A(y) = \sum_{i=1}^{k} n_{z_i}(\phi) - \sum_{i=1}^{k} n_{w_i}(\phi) \]

In our case,

\[ A(a) - A(b) = n_{z}(\phi) - n_{w}(\phi) \]

\[ = 1 - 0 \]

\[ = 1 \]

\[ A(c) - A(b) = n_{z}(\psi) - n_{w}(\psi) \]

\[ = 0 - 1 \]

\[ = -1 \]
Remove the indeterminacy of the Maslov grading by looking at \( \hat{\text{CFK}}(\mathbb{H}) \). The homology is 
\[ \hat{H}(S^3) \cong \mathbb{F} \text{ in degree 0}. \]

Since \( \hat{\text{CFK}}(\mathbb{H}) \) only looks at disks which miss \( w \), we have
\[ \text{one from } a \text{ to } b \text{ containing } z \quad \emptyset \]
\[ \text{one from } a \text{ to } b \text{ containing } w \text{ up} \]

So...
\[ da = b \]
\[ dc = \text{outside} \quad dc = 0 \]

So the homology is
\[ \text{if } b,c \text{ disjoint } \Rightarrow \hat{H} < b > \cong \hat{H} < c > \]

so we set \( c \) to have Maslov grading 0.
Use the Alexander polynomial
\[ \Delta_k(t) = t - 1 + t^{-1} \] to remove the indeterminacy in the Alexander grading, since we require

\[ \sum (-1)^d \text{rank } \hat{H}^k_d(k,r) \cdot t^r = \pm \Delta_k(t) \] \[ \text{symmetrized} \]

From the above, a was the biggest and the relative values give everything else.

Summarizing, we have

<table>
<thead>
<tr>
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<th>M</th>
<th>A</th>
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<tbody>
<tr>
<td>a</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>1</td>
<td>0</td>
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<tr>
<td>c</td>
<td>0</td>
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See the computation of \( \hat{H}^k_d \) below...
8 Plot $\text{CFk}^\infty$

$U$ decreases $\text{Alexandr}$ by 1

$\exists a = b$

$\exists c = ub$
We can specialize to $\text{CF}_K^-$.

To do this, consider the pairs $[x, i, j]$ above with $i \geq 0$

\[ j = \text{Alexander} \]

Note: $\ker$ is $F[u] < b, \text{Im} a - c$

im is $F[u] < b$

Homology is $F[u] \cong F[u] < \text{Im} a - c$
Compute $HF_k^-$ by looking at Alexander filtration... (reintroduce the \( z \)-basepoint)

- Intersection points \( x \in T_a \cap T_b \) with \( A(x) \leq j \) form a subcomplex $F(K_j, j) \subset CF_k^-$

- Filled chain homotopy type is knot invariant
- Filled isomorphism type depends on diagram

\[ \begin{align*}
  \text{Basically, each } F(k, j) \text{ is a horizontal line and we glue vertical arrows...} \\
  HF_k^- & \cong [F[u] \leftarrow ] \oplus [F[b] \rightarrow ] \\
\end{align*} \]
Can do the same for $\text{CF}_k$: compute $\text{HF}_k$ which is the homology of the associated graded w.r.t. to Alexander filtration.

Associated graded of $\text{CF}_k$ is

$$\begin{align*}
\text{F}(k,a+2)/\text{F}(k,a) & \quad \Rightarrow \quad 0 \\
\text{F}(k,a+1)/\text{F}(k,a) & \quad \Rightarrow \quad 0 \\
\text{F}(k,a)/\text{F}(k,a-1) & \quad \Rightarrow \quad \text{F} < a \\
\text{F}(k,a-1)/\text{F}(k,a-2) & \quad \Rightarrow \quad \text{F} < b \\
\text{F}(k,a-2)/\text{F}(k,a-3) & \quad \Rightarrow \quad \text{F} < c \\
\text{F}(k,a-3)/\text{F}(k,a-4) & \quad \Rightarrow \quad 0 \\
\end{align*}$$

So $\alpha$ must be 1 to agree with the Euler char. property.

$g_{\text{CF}_k} = \{ F < a \} \oplus F < b \oplus F < c \}$

$$
\begin{bmatrix}
(1,2) & (0,1) & (-1,0) & (A_m)
\end{bmatrix}
$$

$\text{HF}_k = \{ F < a,b,c \}$ since differentials on $g_{\text{CF}_k}$ are trivial.
Compare this to $\Delta_k(t) = t - 1 + t^{-1}$

\[
\Rightarrow (-1)^0 t^{-1} + (-1)^1 t^0 + (-1)^2 t^1
\]

\[
= t^{-1} - 1 + t \quad \checkmark
\]