Equivariant fundamental classes in $RO(C_2)$-graded cohomology

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Introduction to $RO(G)$-graded Cohomology

- Let $G$ be a finite group
- $G$-Top
- Given a finite-dimensional, real, orthogonal $G$-representation $V$, we can form the representation sphere $S^V = \hat{V}$
- Have equivariant suspensions $\Sigma^V X = S^V \wedge X$
- Bredon Cohomology $H^\alpha(\_; M)$
  - $\alpha \in RO(G)$, $M$ Mackey functor
  - Have suspension isomorphisms $\tilde{H}^\alpha(X; M) \cong \tilde{H}^{\alpha+V}(\Sigma^V X; M)$
The case when $G = C_2$

- $V \cong \mathbb{R}^{p,q} = \mathbb{R}^{p-q}_{\text{triv}} \oplus \mathbb{R}^q_{\text{sgn}}$
  - $RO(C_2)$-graded cohomology is a bigraded theory
  - $p$ is the "topological dimension", $q$ is the "weight"

- $H^{p,q}(X; M) := H^\mathbb{R^{p,q}}(X; M)$

- $S^{p,q} := S^\mathbb{R^{p,q}}$

- Take $M = \mathbb{Z}/2$
The cohomology of orbits in $\mathbb{Z}/2$-coefficients

The $(p, q)$ group is plotted in the box up and to the right of $(p, q)$.

\[
\mathbb{M}_2 = H^{*,*}(pt; \mathbb{Z}/2)
\]

\[
A_0 = H^{*,*}(C_2; \mathbb{Z}/2)
\]
The cohomology of a point in $\mathbb{Z}/2$-coefficients

Abbreviated pictures

$M_2 = H^{*,*}(pt; \mathbb{Z}/2)$

$A_0 = H^{*,*}(C_2; \mathbb{Z}/2)$
Prologue: $C_2$-surfaces

- Given a $C_2$-manifold, goal is to understand certain Bredon cohomology classes using equivariant submanifolds

- In 2016, Dugger classified all $C_2$-surfaces up to equivariant isomorphism using equivariant surgery.

- Computations have been done for all $C_2$-surfaces in $\mathbb{Z}/2$ and $\mathbb{Z}$ coefficients (H.)

- $C_2$-surfaces will be our examples in this talk
Q: How do we connect the algebraic answer to the geometry of the torus and its $C_2$-action?
Nonequivariant fundamental classes

- Example:

\[
\begin{align*}
[C], [C'] &\in H^1(T_1) \\
[C] \sim [C'] &= [C \cap C'] = [x], \\
[C'] \sim [C'] &= 0 = [C] \sim [C] \\
H^*(T_1) &\cong \mathbb{Z}/2[a, b]/(a^2 = b^2 = 0),
\end{align*}
\]

- More generally, for \( N^k \subset M^n \) smooth manifold, get fundamental class \([N] \in H^{n-k}(M)\)

- Defined using the Thom isomorphism theorem for the normal bundle

- If \( X \) and \( Y \) intersect transversally, then \([X] \sim [Y] = [X \cap Y]\)
$[C'] \in H^{1,1}$, $[C] \in H^{1,0}$

$[q] \in H^{2,1}$

$[C'] \sim [C] = [C' \cap C] = [q]$

$H^{*,*}(T_1^{relf}) \cong \mathbb{M}_2[a, b]/\langle a^2 = \rho \cdot a, b^2 = 0 \rangle$

$|a| = (1, 1), \quad |b| = (1, 0)$
Example 2

\[ \mathbb{R}P^2_{tw} \]

\[ [D] \in H^{1,1}, \quad [q] \in H^{2,1} \]

\[ [C] \in H^{1,??}? \]
Equivariant fundamental classes

- $X$ is a $n$-dimensional $C_2$-manifold and $Y$ is a nonfree $k$-dimensional $C_2$-submanifold
- $[Y] \in H^{n-k}(X; \mathbb{Z}/2)$

- Consider the equivariant normal bundle $E$ of $Y$ in $X$
- Over each fixed point $y \in Y^{C_2}$, $E_y \cong \mathbb{R}^{n-k,q_y}$
- Let $q$ be the maximum weight appearing over $Y^{C_2}$

**Theorem (H.)**

*We get a unique class $[Y] \in H^{n-k,q}(X; \mathbb{Z}/2)$*

- Defined by proving an equivariant version of the Thom isomorphism theorem
$[D] \in H^{1,1}$, $[q] \in H^{2,1}$

$[C] \in H^{1,1}$

$[C] \sim [D] = \tau \cdot [q]$

$\mathbb{M}_2[x, y]/(x^2 = \tau y + \rho x, xy = y^2 = 0)$,

$|x| = (1, 1), |y| = (2, 1)$
Example 3

\[ T_{2,1}^{\text{spit}} \]

\[
[C \sqcup \sigma C]_q, [D \sqcup \sigma D]_q \in H^{1,q}
\]

\[
[C \sqcup \sigma C]_r \sim [D \sqcup \sigma D]_s = [z \sqcup \sigma z]_{r+s}
\]
Conclusions

• Have a working theory of fundamental classes with a nice intersection product

Theorem (H.)

*The cohomology of all C₂-surfaces in \( \mathbb{Z}/2 \)-coefficients is generated by fundamental classes.*

• Future question(s): equivariant analog of Steenrod’s problem:

• Q: Is every class in \( H_{*,*}(X; \mathbb{Z}/2) \) an \( M_2 \)-multiple of \( f_*[M] \) for some manifold \( M \) and \( f : M \to X \)?
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Thank you!