Research Statement

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My research lies at the intersection of mathematical physics and number theory—currently the study of asymmetric random matrices and heights of integer polynomials and algebraic numbers.

Random matrix theory is the study of eigenvalue statistics of *ensembles* of matrices. An ensemble is a collection of (typically square) matrices together with a probability measure. This probability measure induces a probability measure on the corresponding sets of eigenvalues. The importance of random matrix theory lies in the fact that the eigenvalue statistics of many ensembles are useful in modelling various physical and mathematical phenomena: The statistics of discrete energy levels in atomic spectra share many features in common with the eigenvalues of certain ensembles of Hermitian matrices. Likewise, Hermitian ensembles are useful in modelling the statistics of the zeros of the Riemann zeta function and other L-functions on the critical line. These examples are the standard-bearers for the subject, but the applicability of random matrix eigenvalue statistics extends to many more phenomena in many other disciplines. This broad applicability is suggestive of a universal paradigm; a new set of theorems akin to the Central Limit Theorem.

For many ensembles, closed forms exist for certain important eigenvalue statistics. Statistics of interest include the density of eigenvalues, the probability that a given subset contains no eigenvalues, and limit laws of the largest eigenvalue as the size of the matrices go to infinity. Classical random matrix theory has been concerned with ensembles of Hermitian matrices or other classes of matrices where the eigenvalues are forced to lie on the real line (or on the unit circle). These ensembles suggest themselves as models for sequences of real numbers arising in applications. Less well-understood are *asymmetric* ensembles: ensembles of square matrices with no symmetry conditions imposed on the entries. Ensembles of real asymmetric matrices are particularly ticklish due to the fact that they have two species of eigenvalues: real and complex conjugate pairs. The eigenvalue statistics of such ensembles are therefore more complicated than those of Hermitian ensembles. Nonetheless, these ensembles have applications in such varied disciplines as statistical mechanics, financial mathematics and quantitative neuroscience. My research demonstrates yet another application of these ensembles: the study of statistical properties of algebraic numbers of bounded height.

In spite of the introduction of asymmetric ensembles relatively early in the development of random matrix theory, the understanding of these ensembles has lagged far behind our understanding of Hermitian ensembles. My research has helped close this gap by allowing certain fundamental quantities of asymmetric ensembles to be put into a form directly analogous to that for Hermitian ensembles. This line of research has resolved a critical unsolved problem posed in 1965 regarding ensembles of matrices whose entries are independent standard normal random variables (perhaps the simplest ensemble of random matrices with real entries). These results have removed a major hurdle in understanding the spectral statistics of asymmetric ensembles, and it is hopeful (probable even) that many of the beautiful theorems regarding Hermitian ensembles have analogs for asymmetric ensembles.

Turning to polynomials, a *height* of a polynomial is a measure of its complexity. More generally, heights measure the complexity of classes of arithmetically defined objects (*e.g.* integer polynomials, algebraic numbers, rational points on elliptic curves, etc.). Simple examples of (polynomial) heights are given by norms on coefficient vectors and the Hardy p-norms. Perhaps less well-known, but equally important, is the *Mahler measure*—the absolute value of the leading coefficient of a polynomial times the product of the moduli of its roots lying outside the unit circle. One of the reasons this height is important is because it is multiplicative; a handy feature! The Mahler measure of a polynomial with integer coefficients is equal to one precisely when the polynomial has all of its roots on the unit circle (with the notable exception of the monomials). In this way, the Mahler measure can be thought of as a measure of the 'cyclotomicness' of an integer polynomial. It is an old (1933) problem of D.H. Lehmer to determine whether 1 is a limit point of Mahler measures of integer polynomials. (To date no irreducible noncyclotomic polynomial in $\mathbb{Z}[z]$ has been found with Mahler measure less than 1.17). This problem is representative of a large class of problems in number theory: finding algebraic/arithmetic objects of small height.

Another general application of heights in Diophantine problems is to quantify the behavior of objects of bounded height as the bound gets big, and much of my early research revolved around this topic as applied to the Mahler measure of special classes of integer polynomials. More recently, I have considered similar questions regarding how algebraic numbers are distributed in the complex plane as a function of their height.

On first inspection, my research interests in random matrix theory may seem disjoint from those in number theory. However, they share in common the characteristic of determining the spectral (eigenvalue/root) behavior of an object (matrix/polynomial) chosen with regard to some property (probability measure/bounded height). The connection between these seemingly dissimilar areas is actually quite strong, and a key aspect of my long-term research agenda is to fully explore (and exploit) the interplay between these subject areas.

The rest of this document will explain some of my active and completed research projects. This summary is necessarily brief, but some of these topics (and many additional ones) are discussed in fuller detail in the grant proposal for my recently funded project entitled *Integrable Structure of Random Spectra Derived from Diophantine Geometry* (NSF DMS-0801243). A copy can be found at: `http://euclid.colorado.edu/∼sinclaic`.

1 Random Matrix Theory

1.1 Completed Works

1.1.1 Averages over Ginibre's Ensemble of Random Real Matrices

In [18] I give a formula for averaging certain functions over real asymmetric matrix ensembles. This work is important because the formula given is directly analogous to the formula for averages over Hermitian ensembles, and these averages lead to more nuanced eigenvalue statistics. The results of this work have as a corollary a generating function which gives the probability that a matrix in Ginibre's real ensemble ($N \times N$ matrix with real independent identically distributed (iid) normal entries) has exactly L real eigenvalues. Pertinent references include [14] and [7].

1.1.2 Correlation Functions of Ensembles of Asymmetric Real Random Matrices (*joint work with Alexei Borodin*)

Given an ensemble of $N \times N$ random matrices, the nth correlation functions give the probability density that a matrix chosen at random has n prescribed eigenvalues (in probabilistic language, this is the nth marginal probability density). In the case of asymmetric real ensembles the existence of two species of eigenvalues complicates matters, and the ℓ, m-correlation function gives the probability density that a matrix chosen at random has ℓ prescribed real eigenvalues and m prescribed complex conjugate pairs of eigenvalues. In [2], we give a closed form for the ℓ, m-correlation functions (in terms of the Pfaffian of an $2\ell \times 2m$ matrix formed by a 2×2 matrix kernel associated to the ensemble). When the ensemble in question is Ginibre's real ensemble, this resolves a 1965 question of Ginibre in his founding paper on the subject [10]. This work clarifies previous attempts of others given in [12] [1]. Nearly simultaneous with the appearance of our manuscript on the arXiv, two other groups claimed similar findings (without proofs) [9] [22]; both of these results rely heavily on my work on averages over Ginibre's real ensemble [18]. One important reference for this work is [24].

1.1.3 Scaling Limits for Ginibre's Real Ensemble (*joint work with Alexei Borodin*)

Much of the interest in random matrices are the scaling limits of eigenvalue statistics as $N \to \infty$. Most of the important statistical information can be gleaned from a kernel (in the sense of functional analysis) used in the description of the correlation functions. In [3], we give the limiting $N \to \infty$ kernel in the appropriate scaling at the edge and in the bulk of the spectrum. The scaled kernel in these various regimes gives microscopic information about the interplay between the real and complex eigenvalues of random real matrices with independent identically distributed Gaussian entries as the size of the matrix grows. This work relies on the recent paper [8].

1.2 Current Projects

1.2.1 The Limit Law of the Largest Eigenvalue in Ginibre's Real Ensemble (*joint work with Brian Rider*)

One consequence of the study of spectral statistics of ensembles of Hermitian random matrices was the discovery of a new type of random variable given by the (scaled) largest eigenvalue of a random $N \times N$ Hermitian matrix in the limit as $N \to \infty$ [23]. The distribution of this random variable, the Tracy-Widom distribution, has since been discovered in a variety of contexts beyond random matrix theory. Therefore, a natural goal is to determine distribution of the (scaled) largest (real and complex) eigenvalue of random real $N \times N$ matrices with iid Gaussian entries in the limit as $N \to \infty$. The resolution of this question now seems in reach due to my recent results reported in [3] and [18] (the former with Alexei Borodin).

1.2.2 Eigenvalue Statistics of Ensembles of $*$-Cosquare Matrices (*joint work with Francesco Mezzadri*)

Given a square complex matrix $\mathbf{A}$, the eigenvalues of the matrix $\mathbf{A}^* \mathbf{A}^{-1}$ are stabilized by the involution $z \mapsto 1/\overline{z}$. Since, this involution stabilizes the unit circle, and 'swaps' the interior and exterior of the open unit disk, the eigenvalues have either modulus 1, or form pairs of the form $(\beta, 1/\overline{\beta})$. That is, the characteristic polynomials of such matrices are *self-inversive* (see Section 2.1.4). Matrices of this type are known as $*$-cosquare matrices. By imposing a probability measure on this set of matrices we may create an ensemble of $*$-cosquare matrices. It is our goal to explore the statistics of such ensembles. As the zeros of random self-inversive polynomials have been used to model a variety of physical and number theoretic phenomena, it is our hope that the eigenvalues of ensembles of $*$-cosquare matrices will likewise find applications in various contexts.

1.2.3 Correlation Functions for $\beta = 1$ and N odd

A Pfaffian formulation for the correlation functions of $\beta = 1$ ensembles was originally derived by F. Dyson in [6]. Much later Tracy and Widom gave an elegant new derivation of these correlation functions in the case when N is even [24]. I have discovered how to extend Tracy and Widom's method to the N odd case, which is the subject of this note.

1.2.4 Eigenvalue Statistics for Mixed β Ensembles

A Pfaffian form for the partition function of a two-component electrostatic system with real particles with integer charges is given. This work is in a very formative state but it is hoped that Pfaffian correlations and their scaling limits can be derived for simple examples of such ensembles.

2 Number Theory

2.1 Completed Works

2.1.1 The Distribution of Mahler's Measures of Reciprocal Polynomials

As mentioned previously, one important theme in number theory is the identification of objects of small height. In the context of Mahler measure, the irreducible integer polynomials with minimal height are exactly the cyclotomic polynomials (polynomials with all roots on the unit circle). Since cyclotomic polynomials are real, if ζ is the root of such a polynomial, then so too is ζ^{-1}. Polynomials whose roots are stabilized by the map $z \mapsto 1/z$ are called *reciprocal*. It is known that if an irreducible integer polynomial has Mahler measure less than 1.3, then the polynomial is necessarily reciprocal [21]. Thus, when studying the range of Mahler measure of integer polynomials, reciprocal polynomials play a special role. In the paper [17] (my first published work) I give an asymptotic estimate for the number of reciprocal polynomials with Gaussian integer coefficients of fixed degree and Mahler measure bounded by T as $T \to \infty$.

2.1.2 The Range of Values of Multiplicative Functions on $\mathbb{C}[x], \mathbb{R}[x]$ and $\mathbb{Z}[x]$

In the paper [19], I give an asymptotic estimate for the number of degree N reciprocal polynomials with integer coefficients and Mahler measure at most T as $T \to \infty$. This estimate is made by computing the volume of the set of real polynomials of degree N and Mahler measure at most 1. The method of calculation of this volume is extended to other multiplicative functions on $\mathbb{R}[x]$ and $\mathbb{C}[x]$. The integrals which arise are similar to those that arise in the calculation of the partition function of real asymmetric matrices. It is this connection, and the given representation for the volumes of polynomials, which lead to my random matrix theory paper [18]. This work is based on my Ph.D. thesis and follows similar results for non-reciprocal polynomials given in [4].

2.1.3 Patterns and Periodicity in a Family of Resultants (*joint work with Kevin Hare and David McKinnon*)

In the paper [11], my coauthors and I describe patterns of divisibility of resultants of certain polynomials by prime powers. The resultants of the polynomials in question arise in the proof of the best known lower bound of the Mahler measure of an integer polynomial based on its degree [5]. It is hoped (though unlikely, given the difficulty of the problem) that our results will lead to a better lower bound for the Mahler measure as a function of degree.

2.1.4 Conjugate Reciprocal Polynomials with all Roots on the Unit Circle (*joint work with Kathleen Petersen*) and Self Inversive Polynomials with all zeros on the unit circle (*joint work with Jeff Vaaler*)

Conjugate reciprocal polynomials are those polynomials which are invariant under the reversal of their coefficient vector followed by complex conjugation. The roots of self-inversive polynomials are either on the unit circle or are members of pairs which are stabilized by the involution on $\mathbb{C}$ given by $z \mapsto 1/\overline{z}$. Self-inversive polynomials generalize conjugate reciprocal polynomials: they are exactly the set of polynomials in $\mathbb{C}[z]$ whose roots are invariant under this involution (they likewise have a description in terms of the coefficients very similar to conjugate reciprocal polynomials). In the first of this series of papers, [15], Kathleen Petersen and I show that the set of coefficient vectors of conjugate reciprocal polynomials with all roots on the unit circle and degree N is homeomorphic to the $N-1$ ball, and has isometry group isomorphic to the dihedral group of order $2N$. Moreover, we compute the Lebesgue measure of this set and find that it is equal to the volume of the $N-1$ ball of radius 2. This computation relies on an integral formula derived by Dyson in the context of random matrix theory [6].

In the second of this series, [20], Jeff Vaaler and I give sufficient conditions on the coefficient vector of a self-inversive polynomial so that it has all roots on the unit circle. These conditions are based on geometric information learned in [15]. These results are similar to recent results given in [16] and [13].

2.2 Current Projects

2.2.1 Equidistribution of Imaginary Quadratic Algebraic Numbers on the Unit Circle (*joint work with Kathleen Petersen*)

Given K, a fixed imaginary quadratic extension of $\mathbb{Q}$, a point $\beta \in K$ on the unit circle $\mathbb{T}$ is necessarily of the form $\alpha/\overline{\alpha}$ for some $\alpha \in \mathcal{O}_K$. The point α is uniquely determined if we demand it be in the upper half plane and of minimal norm. Using properties of twisted principal ideal class zeta functions, we show that the points $\beta \in K \cap \mathbb{T}$ are equidistributed with respect to the map $\alpha \mapsto \beta$. We also aim to prove similar results regarding the equidistribution of points $\beta \in K \cap \mathbb{T}$ ordered by inclusion with respect to the absolute Weil height.

2.2.2 The Distribution of the House of Polynomials with Bounded Mahler Measure (*joint work with Brian Rider*)

The *house* of a polynomial is the maximum modulus of its roots—this is the polynomial analog of the spectral radius of a square matrix. Given the connection between the root statistics of polynomials with bounded Mahler measure and the eigenvalues of asymmetric matrices, it is our aim to give the limiting distribution for the house of (real/complex) polynomials with bounded Mahler measure as the degree increases to ∞.

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