Multiplicative Distance Functions

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January 4, 2006

1 Axiomatizing Mahler Measure

Definition 1.1. A function $\Phi : \mathbb{C}[x] \to [0, \infty)$ is a *multiplicative distance function* if

1. Φ is continuous on finite dimensional subspaces of $\mathbb{C}[x]$.
2. $\Phi(f) = 0$ iff $f = 0$,
3. $\Phi(wf) = |w|\Phi(f)$ for all $f \in \mathbb{C}[x]$ and $w \in \mathbb{C}$, and
4. $\Phi(fg) = \Phi(f)\Phi(g)$ for all $f, g \in \mathbb{C}[x]$.

There exists a function $\phi : \mathbb{C} \to (0, \infty)$ such that if

$$f(x) = a \prod_{n=1}^{N} (x - \gamma_n) \qquad \text{then} \qquad \Phi(f) = |a| \prod_{n=1}^{N} \phi(\gamma_n). \tag{1.1}$$

We call ϕ the *root function* of Φ.

The continuity of Φ implies that

$$\phi(\gamma) \to |\gamma| \qquad \text{as} \qquad |\gamma| \to \infty. \tag{1.2}$$

Moreover the converse is true, if ϕ is continuous and satisfies this asymptotic formula, then the function Φ formed as in (1.1) is a multiplicative distance function.

Theorem 1.1. *A continuous function $\phi : \mathbb{C} \to [0, \infty)$ is the root function of a multiplicative distance function iff it satisfies (1.2).*

Arguably the simplest allowable root function is $\gamma \mapsto \max\{1, |\gamma|\}$, putting Mahler measure squarely at the center of this theory.

2 Examples of Multiplicative Distance Functions

Theorem 1.1 gives us a method for creating multiplicative distance functions. Here we focus on two methods of creating new multiplicative distance functions, closely related to Mahler measure.

Let $G(x) \in \mathbb{C}[x, 1/x]$ be a fixed Laurent polynomial, and let $\mu : \mathbb{C}[x] \to [0, \infty)$ be Mahler measure. We may create a new multiplicative distance function by pulling back Mahler measure through the map $\mathbb{C}[x] \to \mathbb{C}[G(x)]$ induced by G. That is, we define $G^*\mu$ is the multiplicative distance function defined by the commutative diagram

$$\begin{tikzcd} \mathbb{C}[x] \arrow[r, "G"] \arrow[d, "G^*\mu"'] & \mathbb{C}[G(x)] \arrow[dl, "\mu"] \\ {[0,\infty)} & \end{tikzcd}$$

That is, $G^*\mu(f) = \mu(f \circ G)$. A particularly important example arises when $G(x) = x + 1/x$. Since $\mathbb{C}[x + 1/x]$ is the algebra of reciprocal Laurent polynomials, we call this multiplicative distance function the *reciprocal Mahler measure*, and denote it by ρ. The root function of ρ is given by

$$\gamma \mapsto \max\left\{1, \left|\frac{\gamma + \sqrt{\gamma^2 - 4}}{2}\right|\right\} \max\left\{1, \left|\frac{\gamma - \sqrt{\gamma^2 - 4}}{2}\right|\right\}.$$

Question 1. Can Dobrowolski's argument be made to work for ρ? If so, does this yield an improvement for the lower bound for Mahler measures of reciprocal polynomials in $\mathbb{Z}[x]$?

A related family of multiplicative distance functions is formed by pulling back Mahler's measure through the polynomial

$$G(x) = x + t/x \qquad t \in [0, 1].$$

When $t = 0$ we recover Mahler measure; when $t = 1$ we recover the reciprocal Mahler measure.

Another method for generalizing Mahler measure is to consider multiplicative distance functions formed from equilibrium potentials on compact subsets of $\mathbb{C}$. In particular, let $K \subset \mathbb{C}$ be compact, simply connected and of capacity (transfinite diameter) equal to 1. If K does not consist of a single point, there is a special measure ν_K supported on ∂K called the equilibrium measure. The multiplicative distance function P_K is then defined to be

$$P_K(f) = \exp\left\{\int_{\partial K} \log|f(z)|\, d\nu_K(z)\right\}.$$

Theorem 2.1. *If $f(x) \in \mathbb{C}[x]$ is monic then $P_K(f) = 1$ iff f has all of its roots in K.*

If, additionally $f(x) \in \mathbb{Z}[x]$, we call f a K-*tomic* polynomial. We might call $K \subseteq \mathbb{C}$ *arithmetic* if K has capacity 1 and there exist infinitely many K-tomic polynomials. That is, if there are infinitely many complete sets of conjugate algebraic integers in K.

If D is the closed unit disk, then $P_D = \mu$. This is Jensen's formula. Clearly there are infinitely many D-tomic (*i.e.* cyclotomic) polynomials. Similarly, if $I = [-2, 2]$ on the real axis in the complex plane, then there are infinitely many I-tomic polynomials.

Question 2. If K is not symmetric about the x-axis, then K is not arithmetic. Are there any other conditions which preclude a set K from being arithmetic? Can one find any sets K which are provably arithmetic (or non-arithmetic)? If K is arithmetic, is ∂K also arithmetic?

3 Generalizations of Jensen's Formula

In some cases multiplicative distance function formed from equilibrium measures are equal to multiplicative distance functions formed from pullbacks of Mahler measure through Laurent polynomials. We restrict our attention to those Laurent polynomials $G(x)$ whose highest positive power of x is 1. Thus, let $p(x) \in \mathbb{C}[x]$ be a monic polynomial of degree $M + 1$ and consider $G(x) = p(x)/x^M$.

Theorem 3.1. *If $G'(x)$ does not vanish on $\mathbb{C} \setminus D$. Then, $G^*\mu = P_K$ where K is the complement in $\mathbb{C}$ of $G(\mathbb{C} \setminus D)$.*

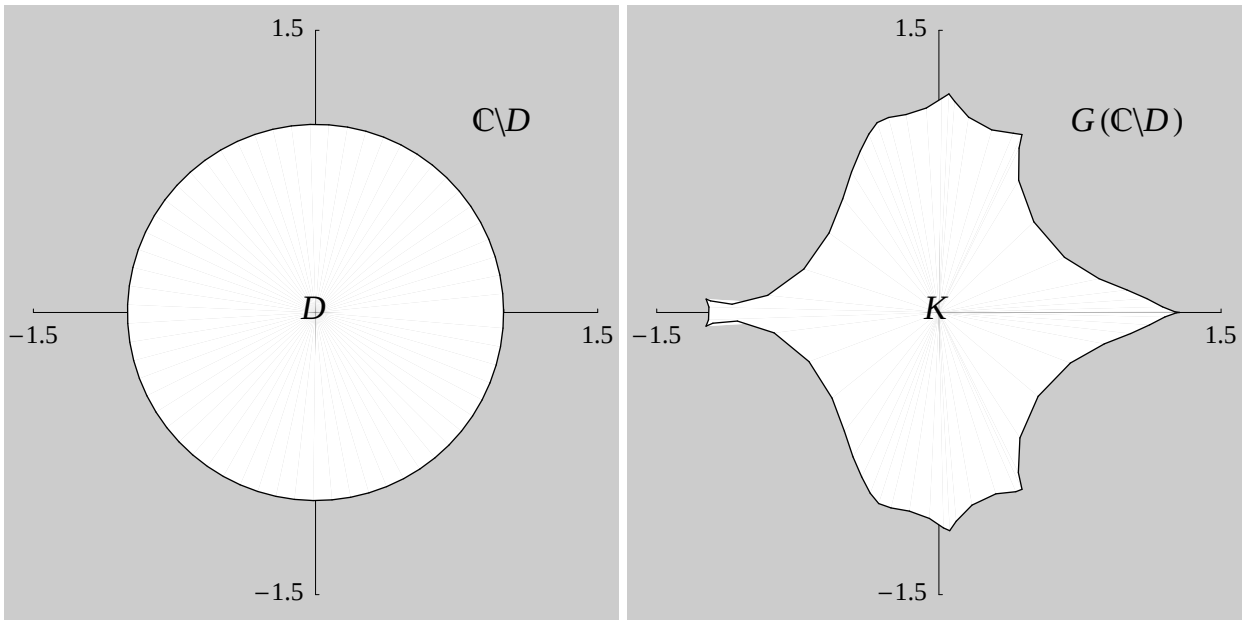

Figure 1: K corresponding to $G = x + .2x^{-3} + .1x^{-5} - .05x^{-9} + .03x^{-12}$

Corollary 3.2. *The reciprocal Mahler measure satisfies:*

$$\rho = P_I.$$

If μ_t is the pullback of Mahler measure through $x + t/x$, where $t \in (0,1)$, then

$$\mu_t = P_{E_t},$$

where E_t is the elliptical region given by

$$E_t = \left\{ x + iy : \frac{x^2}{(1+t)^2} + \frac{y^2}{(1-t)^2} \leq 1 \right\}.$$

Theorem 3.1 is a generalization of Jensen's formula, since it equates a pullback of Mahler measure (initially defined in terms of its behavior on roots of polynomials) with a multiplicative distance functions defined via an integral. We may formulate multivariable analogs of these multiplicative distance functions.

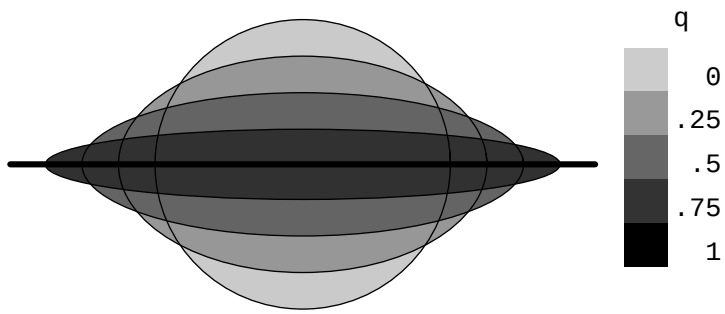

Figure 2: E_q for a few values of q

Question 3. It seems likely, perhaps easily verifiable, that if $F(x_1, \ldots, x_L) \in \mathbb{C}[x_1, \ldots, x_L]$ then $\rho(F) = \mu \circ F(x_1 + 1/x_1, \ldots, x_L + 1/x_L)$. There are examples of such F for which $\rho(F)$ has been explicitly computed. Is there any relationship between $\rho(F)$ and $\mu(F)$ from a special values perspective? Does the underlying compact set I appear in any tangible way in the formulas for $\rho(F)$?

4 Moment Functions

Returning to the general situation, we view Φ as the restriction of a multiplicative distance function to the coefficient vectors of polynomials of degree N. That is, we identify the set of polynomials of degree N with $\mathbb{C}^{N+1}$ and view Φ as a function $\mathbb{C}^{N+1} \to [0, \infty)$. As such we may identify the set of monic polynomials of degree N with $\mathbb{C}^N$ and define the function $\widetilde{\Phi} : \mathbb{C}^N \to [0, \infty)$ to be the *monic restriction* of Φ. The degree N *moment functions* of Φ are then defined to be

$$H_N(\Phi; s) = \int_{\mathbb{C}^N} \widetilde{\Phi}(\mathbf{b})^{-2s}\, d\lambda_{2N}(\mathbf{b}) \quad \text{and} \quad F_N(\Phi; s) = \int_{\mathbb{R}^N} \widetilde{\Phi}(\mathbf{b})^{-2s}\, d\lambda_N(\mathbf{b}),$$

where s is a complex variable and the λ's are Lebesgue measure on $\mathbb{C}^N$ and $\mathbb{R}^N$. The integrals defining $H_N(\Phi; s)$ and $F_N(\Phi; s)$ converge to analytic functions of s in the half plane $\Re(s) > N$.

These functions were introduced for $\Phi = \mu$ by S-J. Chern and J. Vaaler to study the range of values of Mahler measure. In particular, they demon-

strated the following Theorem in the case $\Phi = \mu$.

Theorem 4.1. *Let Φ be a multiplicative distance functions. Then, as $T \to \infty$,*

$$\#\left\{\mathbf{a} \in \mathbb{Z}^{N+1} : \Phi(\mathbf{a}) \leq T\right\} = \frac{F_N(\Phi; N+1)}{N+1} T^{N+1} + O(T^N).$$

Chern and Vaaler evaluated $H_N(\mu; s)$ and $F_N(\mu; s)$ to find

Theorem 4.2 (S.-J. Chern, J. Vaaler). *$F_N(\mu; s)$ and $H_N(\mu; s)$ analytically continue to rational functions of s. In particular,*

$$H_N(\mu; s) = \frac{\pi^N}{N!} \prod_{n=1}^{N} \frac{s}{s-n}.$$

And, if M is the integer part of $(N-1)/2$ then

$$F_N(\mu; s) = \mathcal{C}_N \prod_{m=0}^{M} \frac{s}{s-(N-2m)} \qquad \textit{where} \qquad \mathcal{C}_N = 2^N \prod_{m=1}^{M} \left(\frac{2m}{2m+1}\right)^{N-2m}.$$

In my thesis I evaluated $H_N(\rho; s)$ and $F_N(\rho; s)$,

Theorem 4.3 (S-). *$F_N(\rho; s)$ and $H_N(\rho; s)$ analytically continue to rational functions of s. In particular,*

$$H_N(\rho; s) = 2^N \pi^N \prod_{n=1}^{N} \frac{s}{s^2 - n^2}.$$

And, if J is the integer part of $(N-1)/2$ then

$$F_N(\rho; s) = v_N \prod_{j=0}^{J} \frac{s^2}{s^2 - (N-2j)^2}, \qquad \textit{where} \qquad v_N = \frac{2^N}{N!} \prod_{n=1}^{N} \left(\frac{2n}{2n-1}\right)^{N+1-n}.$$

The fact that the moment functions of Mahler measure are rational functions of s with poles at integers was surprising, but might have been accidental. But the fact that the moment functions of ρ demonstrate similar structure suggests that there is a phenomenon to be explored.

Several questions suggest themselves

Question 4. [1]

1. Why are the moment functions of μ and ρ rational functions of s? $\star$
2. Why do they have poles only at integers? $\star$
3. Why are those poles simple and what is their significance?
4. What is the significance of the high multiplicity zero at the origin?
5. Why do they have rational coefficients? $\star$
6. What is the significance of $\mathcal{C}_N$ and v_N? $\star$
7. Why do the moment functions have a "simple" product representation? $\star\star$

And another question:

Question 5. Which multiplicative distance functions have moment functions which are

1. rational functions of s?
2. rational functions with poles at integers?
3. rational functions with coefficients in $\mathbb{Q}$ (or $\pi^N\mathbb{Q}$)?

Conjecture 4.4. *If $p(x) \in \mathbb{R}[x]$ is a monic polynomial of degree N, and $G(x) = p(x)/x^M$ is such that $G'(x)$ does not vanish. Then the degree N moment functions of $G^*\mu$ have an analytic continuation to*

1. *rational functions of s with poles at integers between $-MN$ and N.*
2. *Moreover, the poles at positive integers are simple.*
3. *Moreover, if $p(x) \in \mathbb{Q}[x]$, then the coefficients are rational (or in $\pi^N\mathbb{Q}$).*

The moment functions for the pullbacks of Mahler measure through $x+t/x$ have also been computed. And, while the formulas for the moment functions of μ_t are more involved than those of Mahler measure and the reciprocal Mahler measure, they share many of the qualitative properties.

[1] $\star$ indicates that there is some progress in answering this question, $\star\star$ will be explained in the next section

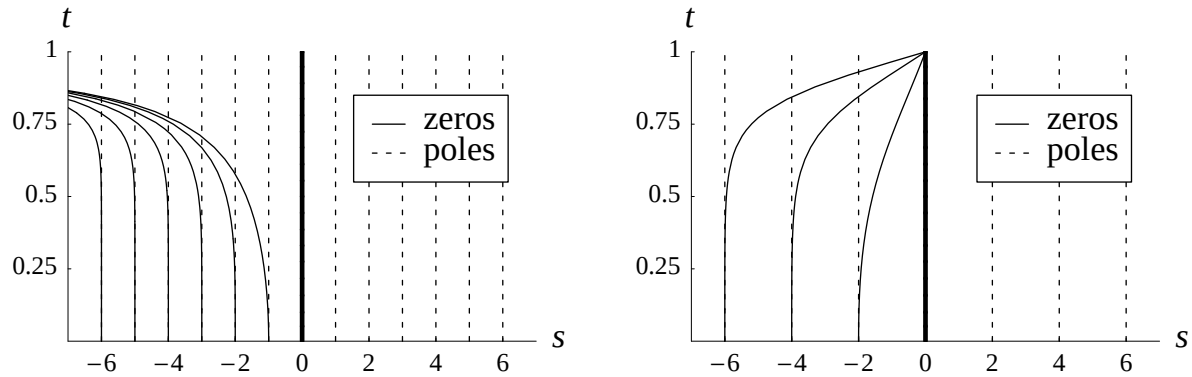

Figure 3: The location of the zeros and poles of $H_6(\mu_t; s)$ and $F_6(\mu_t; s)$

5 The Structure of Moment Functions

Next, we give structure theorems for moment functions, explaining why the examples of moment functions we have seen so far have a simple product formulation.

Definition 5.1. Suppose $\mathbf{P} = \{P_1, P_2, \ldots, P_N\}$ is a collection of monic polynomials in $\mathbb{C}[\gamma]$ such that $\deg P_n = n - 1$, then we call $\mathbf{P}$ a *complete* family of N polynomials.

For each s with $\Re(s) > N$ we define the complex inner product $\langle \cdot | \cdot \rangle$ given by

$$\langle P|Q\rangle = \int_{\mathbb{C}} \phi(\gamma)^{-2s} P(\gamma)\overline{Q(\gamma)}\, d\lambda_2(\gamma).$$

If $\mathbf{P}$ is any complete family of N polynomials, then the inner product of any two polynomials in $\mathbf{P}$ is defined.

Theorem 5.1. *Let $\Re(s) > N$, and let $\mathbf{P}$ be any complete set of N polynomials. Then,*

$$H_N(\Phi; s) = \det W_{\mathbf{P}},$$

where $W_{\mathbf{P}}$ is the $N \times N$ matrix whose j, k entry is given by $W_{\mathbf{P}}[j, k] = \langle P_j | P_k \rangle$.

The matrix $W_{\mathbf{P}}$ is an example of a Gram matrix, and its determinant is a Grammian.

Corollary 5.2. *Let $\Re(s) > N$ and let $\mathbf{Q}$ be the complete family of monic polynomials specified by*

$$\langle Q_j | Q_k \rangle = \mathfrak{N}_s(Q_k)\, \delta_{kj} \qquad \textit{where} \qquad \mathfrak{N}_s(Q_k) = \langle Q_k | Q_k \rangle.$$

Then,

$$H_N(\Phi; s) = \prod_{n=1}^{N} \mathfrak{N}_s(Q_k).$$

To give a structure theorem for the real moment function, define two skew-symmetric complex inner products by

$$\langle P, Q \rangle_{\mathbb{R}} = \int_{\mathbb{R}^2} \phi(x)^{-s} \phi(y)^{-s} P(x) Q(y) \operatorname{sgn}(y - x) dx\, dy,$$

and

$$\langle P, Q \rangle_{\mathbb{C}} = -2i \int_{\mathbb{C}} \phi(\beta)^{-s} \phi(\overline{\beta})^{-s} P(\overline{\beta}) Q(\beta) \operatorname{sgn} \Im(\beta)\, d\lambda_2(\beta).$$

We then define the skew-symmetric complex inner product $\langle \cdot, \cdot \rangle$ by

$$\langle P, Q \rangle = \langle P, Q \rangle_{\mathbb{R}} + \langle P, Q \rangle_{\mathbb{C}}.$$

Theorem 5.3. *Let $\Re(s) > N$ and let J be the integer part of $N + 1$. If $\mathbf{P}$ is any complete set of N polynomials, then*

$$F_N(\Phi; s) = \operatorname{Pf} U_{\mathbf{P}},$$

where $U_{\mathbf{P}}$ is the $2J \times 2J$ antisymmetric matrix whose j, k entry is given by

$$U_{\mathbf{P}}[j, k] = \begin{cases} \langle P_j, P_k \rangle & \textit{if } j, k \leq N, \\ \operatorname{sgn}(k - j) \displaystyle\int_{\mathbb{R}} \phi(x)^{-s}\, P_{\min\{j,k\}}(x)\, dx & \textit{otherwise.} \end{cases}$$

The matrix $U_{\mathbf{P}}$ is the skew-symmetric analog of a Gram matrix, and we may refer to its Pfaffian as a *Pframmian*.

Corollary 5.4. *Suppose $N = 2J$, $\Re(s) > N$ and let $\mathbf{Q}$ be any complete family of monic polynomials specified by*

$$\langle Q_{2k-1}, Q_{2j}\rangle = -\langle Q_{2j}, Q_{2k-1}\rangle = \delta_{kj}\mathfrak{M}_s(Q_j) \quad \textit{and} \quad \langle Q_{2j}, Q_{2k}\rangle = \langle Q_{2j-1}, Q_{2k-1}\rangle = 0,$$

Then,

$$F_N(\Phi; s) = \prod_{j=1}^{J} \mathfrak{M}_s(Q_j).$$

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