Partition Functions in 2-D Electrostatics

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1 Moment Functions

In [1] Jeff Vaaler and Shey-Jey Chern introduced two families of analytic functions to study the range of values of Mahler measure on $\mathbb{C}[x]$ and $\mathbb{R}[x]$. These analytic functions together with geometry of numbers techniques to give asymptotic estimates for the number of polynomials in $\mathbb{Z}[x]$ of degree N with Mahler measure bounded by T as $T \to \infty$. Here we give another interpretation of these analytic functions in terms of two-dimensional electrostatics.

Given positive integer N we view $\mu : \mathbb{C}^{N+1} \to [0, \infty)$ as Mahler measure on the coefficient vectors of polynomials of degree N. That is,

$$\mu(\mathbf{a}) = M(a_1 x^N + a_2 x^{N-1} + \cdots + a_{N+1}),$$

where M is (multiplicative) Mahler measure. We also define $\widetilde{\mu} : \mathbb{C}^N \to (0, \infty)$ to be Mahler measure restricted to the vectors of non-leading coefficients of monic polynomials of degree N. The degree N real and complex *moment functions* of μ are given by

$$F_N(\mu; s) = \int_{\mathbb{R}^N} \widetilde{\mu}(\mathbf{b})^{-s}\, d\lambda_N(\mathbf{b}) \quad \text{and} \quad H_N(\mu; s) = \int_{\mathbb{C}^N} \widetilde{\mu}(\mathbf{b})^{-2s}\, d\lambda_{2N}(\mathbf{b}),$$

where λ_N and λ_{2N} are Lebesgue measure on $\mathbb{R}^N$ and $\mathbb{C}^N$ respectively.

Theorem 1.1 (Chern, Vaaler). *As $T \to \infty$,*

$$\#\left\{\mathbf{a} \in \mathbb{Z}^{N+1} : \mu(\mathbf{a}) \leq T\right\} = \frac{F_N(\mu; N+1)}{N+1} T^{N+1} + O(T^N).$$

Chern and Vaaler evaluated $H_N(\mu; s)$ and $F_N(\mu; s)$ to find

Theorem 1.2 (S.-J. Chern, J. Vaaler). *$F_N(\mu; s)$ and $H_N(\mu; s)$ analytically continue to rational functions of s. In particular,*

$$H_N(\mu; s) = \frac{\pi^N}{N!} \prod_{n=1}^{N} \frac{s}{s-n}.$$

And, if M is the integer part of $(N-1)/2$ then

$$F_N(\mu; s) = \mathfrak{C}_N \prod_{m=0}^{M} \frac{s}{s-(N-2m)} \qquad \textit{where} \qquad \mathfrak{C}_N = 2^N \prod_{m=1}^{M} \left(\frac{2m}{2m+1}\right)^{N-2m}$$

2 2-D Electrostatics

Turning to 2-D electrostatics we consider a physical system consisting of an infinite conducting cylinder and a collection of infinite wires parallel to the cylinder. By slicing our system we may identify the cylinder with the

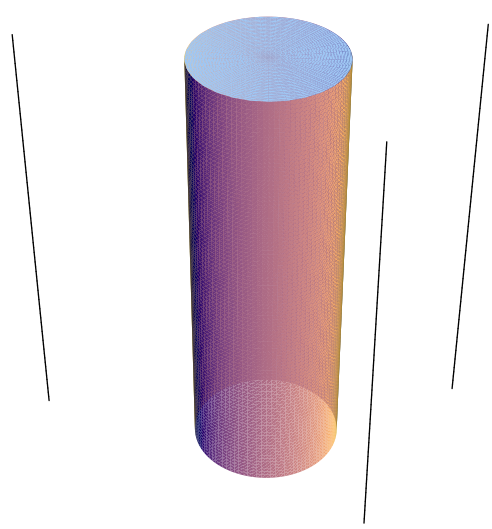

Figure 1: A model of our physical system.

closed unit disk D, and the wires $\alpha_1, \alpha_2, \ldots \alpha_N \in \mathbb{C}$. We may generalize this

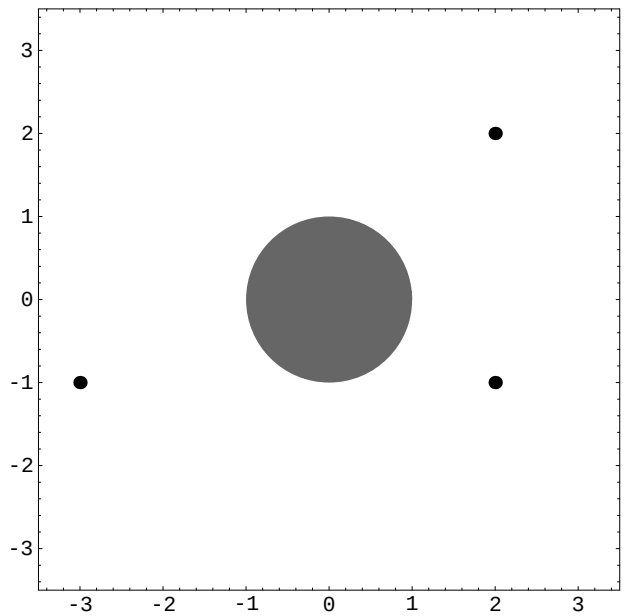

Figure 2: I
dentifying our system with a subset of $\mathbb{C}$.

situation by replacing D with any compact connected subset of $K \subseteq \mathbb{C}$.

In the sliced system we place a charge distribution on D, and view the α as N particles which are free to move in the complex plane. Moreover we assume that the temperature of the entire system is fixed. Such a system is an example of a *log-potential* Coulomb system. Additionally we will assume that each of the particles have equal charge normalized to 1 – that is we have a *one-component* system. Moreover, to keep the particles from repelling to ∞ we will assume that the total charge on D, $-\sigma$, is greater than N.

3 Interaction Energy

The electrostatic energy of the particle-particle interaction is given by

$$U_1(\boldsymbol{\alpha}) = -\sum_{1 \le j < k \le N} \log|\alpha_k - \alpha_j| = -\frac{1}{2}\log|\operatorname{disc} f|,$$

where f is the monic polynomial with roots $\alpha_1, \alpha_2, \ldots, \alpha_N$. Identifying the charge distribution on D with a probability measure ν the background-background energy is given by $\sigma^2 I$, where

$$I = -\frac{1}{2}\int_D \int_D \log|z-w|\, d\nu(z)\, d\nu(w).$$

I is, of course, a function of ν. In isolation, the charged cylinder equilibrizes to a measure which minimizes I. The measure which minimizes I is Lebesgue measure on the unit circle, and the background-background interaction energy for this measure is 0. In the general situation, where D is replaced with K, the measure which minimizes I is unique and supported on the boundary of K.

At equilibrium, the background-particle interaction energy is given by

$$U_2(\boldsymbol{\alpha}) = \sigma \int_0^1 \log|f(e^{2\pi i\theta})|\, d\theta = \sigma M(f) \qquad \text{where} \qquad f(z) = \prod_{n=1}^{N}(x-\alpha_n).$$

The total energy of the system determined by $\boldsymbol{\alpha}$ is thus

$$U_{\text{pot}}(\boldsymbol{\alpha}) = U_1(\boldsymbol{\alpha}) + U_2(\boldsymbol{\alpha}).$$

The temperature of the total system is fixed, not the energy. Configurations with high energy are less probable than those with low energy. In fact, the probability that the system is in the state specified by $\alpha_1, \alpha_2, \ldots, \alpha_N$ is given by the *Boltzmann factor*:

$$\frac{e^{-\beta U_{\text{pot}}(\boldsymbol{\alpha})}}{Z} = \frac{|\operatorname{disc}(f)|^{\beta/2} M(f)^{-\beta\sigma}}{Z}.$$

Z is a normalizing constant, and β is proportional to the inverse temperature.

The normalizing 'constant' is known as the *partition function* and is given by

$$\begin{aligned} Z = Z(\sigma,\beta) &= \frac{1}{N!}\int_{\mathbb{C}^N} e^{-\beta U_{\text{pot}}(\boldsymbol{\alpha})}\, d\lambda_{2N}(\boldsymbol{\alpha}) \\ &= \frac{1}{N!}\int_{\mathbb{C}^N} \prod_{n=1}^{N} \max\{1, |\alpha_n|\}^{-\beta s} \prod_{1<j\le k<N} |\alpha_k - \alpha_j|^{\beta/2}\, d\lambda_{2N}(\boldsymbol{\alpha}). \end{aligned}$$

4 Partition Functions and Moment Functions

Returning to the complex moment function of μ,

$$H_N(\mu; s) = \int_{\mathbb{C}^N} \mu(\mathbf{b})^{-2s}\, d\lambda_{2N}(\mathbf{b}).$$

Instead of integrating over coefficient vectors of monic polynomials we may integrate over root vectors. That is, we define $E : \mathbb{C}^N \to \mathbb{C}^N$ to be the change of variables given by $E(\boldsymbol{\alpha}) = \mathbf{b}$ where $\mathbf{b}$ is the vector of non-leading coefficients of the monic polynomial whose vector of roots in given by $\boldsymbol{\alpha}$. Using this change of variables we have

$$H_N(\mu; s) = \frac{1}{N!} \int_{\mathbb{C}^N} \prod_{n=1}^{N} \max\{1, |\alpha_n|\}^{-2s}\, |\operatorname{Jac} E(\boldsymbol{\alpha})|^2\, d\lambda_{2N}(\boldsymbol{\alpha}).$$

The Jacobian of E is given by

$$\operatorname{Jac} E(\boldsymbol{\alpha}) = \prod_{1<j\le k<N} (\alpha_k - \alpha_j),$$

that is $\operatorname{Jac} E(\boldsymbol{\alpha})$ is the determinant of the Vandermonde matrix in the coordinates of $\boldsymbol{\alpha}$.

Theorem 4.1. *The partition function of the one component log-potential Coulomb system comprised of N particles in the presence of the charged unit disk at equilibrium with charge $-\sigma > N$ and temperature specified by $\beta = 2$ is given by*

$$Z(\sigma, 2) = H_N(\mu; \sigma),$$

and the probability that this system will be in the configuration specified by monic polynomial f is given by

$$\frac{|\operatorname{disc} f| M(f)^{-2\sigma}}{H_N(\mu; \sigma)}.$$

5 A Physical Interpretation of Lehmer's Conjecture

We imagine that our physical system is quantized so that the only allowable states are those which correspond to monic polynomials with integer coefficients. For $\sigma = 1$, the background-particle interaction of the system corresponding to monic polynomial f is given by $M(f)$. The lowest background-particle energy states correspond to cyclotomic polynomials. The lowest excited ground states correspond to monic polynomials with integer coefficients and small Mahler measure. Thus, in this hypothetical situation, Lehmer's conjecture reduces to the question of whether the lowest excited background-particle energy of the quantized system (independent of the number of particles) is bounded away from 0.

References

[1] Shey-Jey Chern and Jeffrey D. Vaaler. The distribution of values of Mahler's measure. *J. Reine Angew. Math.*, 540:1–47, 2001.

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