

Practice Midterm

February 2014

Instructions:

1. Read all questions carefully. If you are confused ask me!
2. You should have 6 pages including this page. Make sure you have the right number of pages.
3. Make sure your name is clearly printed on the first page and your initials are on all subsequent pages.
4. If necessary you may use the back of pages.
5. Box your answers when appropriate.

Name: _____

UO ID: _____

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

[10 pts] 1. Let X_1, X_2, \dots be a sequence of i.i.d. random variables with

$$E[X_i] = \mu \neq 0 \quad \text{and} \quad E[|X_i|] < \infty.$$

Show that

$$M_n = \frac{1}{\mu^n} \prod_{i=1}^n X_i$$

is a martingale. (Note: there are two things you have to verify).

Solution: First,

$$E[|M_n|] = E\left[\frac{1}{|\mu^n|} \prod_{i=1}^n |X_i|\right] = \frac{1}{|\mu^n|} E[|X_i|^n] < \infty.$$

Second,

$$\begin{aligned} E[M_n | M_{n-1} = m_{n-1}, \dots, M_1 = m_1] &= E\left[\frac{X_n}{\mu} M_{n-1} \mid M_{n-1} = m_{n-1}, \dots, M_1 = m_1\right] \\ &= \frac{m_{n-1}}{\mu} E[X_n] \\ &= m_{n-1}. \end{aligned}$$

[10 pts] 2. Consider a chessboard with a lonely white king making random moves (a king can only move one square in any direction per turn). That is, at each move the king determines which squares he could possibly move to and then moves to one of these with equal probability.

(a) Is the corresponding Markov chain irreducible? Why or why not?

Solution: Yes. The king can get from any square to any other square through a sequence of moves.

(b) Is it aperiodic? Why or why not?

Solution: Yes. The king can return to the square it started in 2 moves, or in three moves, and hence since the greatest common divisor of 2 and 3 is 1, we have aperiodicity.

- (c) Suppose the king is replaced with a bishop (which can only move diagonally). Is the corresponding Markov chain irreducible? aperiodic? Explain.

Solution: The chain is reducible, since a bishop starting on a white square will not be able to get to a black square, and vice-versa. The chain is aperiodic, by the same argument as for the king.

[10 pts] 3. Suppose S_n is a simple random walk starting at 0. That is, $S_n = \xi_1 + \cdots + \xi_n$ where the ξ_i are independent and take the values ± 1 with equal probability. Let T be the first time (S_n) reaches c or $-c$.

(a) Show that $S_n^2 - n$ is a Martingale with respect to S_n .

Solution:

$$\begin{aligned}
 E[S_n^2 - n | S_{n-1} = s_{n-1}, \dots, S_1 = s_1] & \\
 &= E[(S_{n-1} + \xi_n)^2 - n | S_{n-1} = s_{n-1}, \dots, S_0 = s_0] \\
 &= E[(s_{n-1} + \xi_n)^2 - n] \\
 &= E[s_{n-1}^2 - (n-1) + 2s_{n-1}\xi_n + \xi_n^2 - 1] \\
 &= s_{n-1}^2 - (n-1) + 2s_{n-1}E[\xi_n] + E[\xi_n^2] - 1 \\
 &= s_{n-1}^2 - (n-1).
 \end{aligned}$$

(b) Use this to show $E[T] = c^2$. (You may use the stopping theorem without checking that the hypotheses to do so have been satisfied).

Solution:

$$0 = E[S_T^2 - T] = c^2 - E[T].$$

- [10 pts] 4. One of Markov's own applications of Markov chains was a 1913 study of how often a vowel is followed by another vowel or a consonant is followed by another consonant in Russian text. For English, the probability that a vowel is followed by another vowel is .12, the probability that a vowel is followed by a consonant is .88, the probability that a consonant is followed by a vowel is .54 and the probability that a consonant is followed by another consonant is .46.
- (a) Use this information to determine the expected proportion of English letters that are vowels.

Solution: The stationary distribution of

$$P = \begin{bmatrix} .12 & .88 \\ .54 & .46 \end{bmatrix}$$

is (approximately) (.38, .62). Thus, if words were created by selecting letters randomly (uniformly) we would expect that about 38% of letters would be vowels given the data.

- (b) How do you explain the discrepancy between your answer and the true proportion (which is between 19 and 23 percent depending on whether you consider 'Y' a vowel).

Solution: The hypothesis that words were formed by randomly, uniformly choosing letters must be incorrect.