2. Complex-valued functions of complex arguments

Consider the field $C$ of complex numbers, $z = x + iy = a + bi$, $(z, x, y, a, b, i \in \mathbb{R})$ is constructed in $\mathbb{C}$.

2.1 Complex functions

Def. 1: (a) Let $f: C \rightarrow C$ be a mapping in the ring of $\mathbb{C}$. The
or cell $f$ is a (high-valued) complex-valued function of
a complex argument.

(b) Generalize the concept of a mapping
and let one proceed in the
ring $\mathbb{C}$.

Example: (1) $f(z) = z^2$ is a high-valued function.

(2) $f(z) = e^z$ is a high-valued function.

(3) $f(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is a high-valued function.

(4) $\frac{dz}{dx} e^{iy} = e^{iy} (i(\pi + 2\pi n), n \in \mathbb{Z})$

$\log z = \log (r e^{i(\pi + 2\pi n)}) = \log r + i \theta$

is a $\mathbb{Z}$-valued function.

Def. 2: A multivalued function $f(z)$ on a non-ordinary region $R$ is said to have
a branch cut when it is not continuous.

A cut is cut for $z \in C$.
Example: \( f(t) = e^{-i\pi t} \) right-branch by choosing the cut along the imaginary axis. The \( i^{1/4} \) uniquely, etc.

(6) \( f(t) = \ln t \) can be made right-branch by choosing the same branch cut.

Remark: (1) The branch cut is a property of the individual function, not of the complex plane in general.

(2) For a given function, the choice of the branch cut is not unique. For instance, choosing the cut in \( \ln z \) along the positive real axis corresponds to \( \theta \in [0, 2\pi) \).

(3) For functions of the form \( f(z) \) the branch cut will start at a "branch point" to determine by \( f(z_0) = 0 \), return then at the origin.

Example: (7) \( f(z) = \ln(2z - 1) \)

(8) \( f(z) = \ln \left( \frac{z-1}{z+1} \right) = \ln(z-1) - \ln(z+1) \)

The branch cut must end and start for \( [-x_0, -x_1] \)

Def. 2: For a two-valued function, one can continue the function across the cut onto a second sheet, so that the function takes on the other possible value. The two sheets will form the \( \infty \)-plane 2-folded from the Riemann surface for the function. In analogy, a fundamental works for a \( \infty \)-function.
\begin{align*}
\text{In[11]} &= \text{Plot3D} \left[ \text{Im} \left[ \sqrt{x+1} y \right], \{x, -2, 2\}, \{y, -2, 2\}, \text{PlotPoints} \to 30 \right] \\
\text{Out[11]} &=
\end{align*}

\begin{align*}
\text{In[14]} &= \text{ParametricPlot3D} \left[ \{r \cos[\phi], r \sin[\phi], \sqrt{r} \sin[\phi/2]\}, \{r, 0, 1\}, \{\phi, 0, 4\pi\}, \text{PlotPoints} \to \{20, 60\} \right] \\
\text{Out[14]} &=
\end{align*}
Example: \( f(t) = t^4 \)

Continue this on either sheet past the cut strip on both to the other sheet:

\[ t = e^{\pm i\pi}, e^{-3\pi i/4} \]

Proof:

**Def 1:** \( f(t) \) is called **continuous** in the point \( t_0 \in C \) if \( f(t_0) \) exists and \( \lim_{t \to t_0} f(t) = f(t_0) \).

**Def 2:** \( f(t) \) is called **differentiable** in \( t_0 \) if \( f(t) \) has a derivative \( df/dt \) at \( t_0 \), if the limit

\[ \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} = \frac{df}{dt} |_{t_0} \]

exists.

Remark:

1. There are obvious prerequisites of the continuity except for real functions.
2. The limits must exist no matter how \( t \) approaches to \( t_0 \) in the complex plane!

\( \Rightarrow \) Then \( f(t) \) will strongly resemble the complex one for real-valued functions of one real variable.

**Def 3:** Let \( D \subset \mathbb{C} \) be a region in \( \mathbb{C} \) and \( f: D \to \mathbb{C} \) a function. \( f \) is called **analytic** on \( D \) if it is differentiable in all points \( t \in D \).
**Theorem:**

\[ f'(z, t^n) + i f''(z, t^n) \text{ is analytic in } D \text{ iff } \]

\[
\frac{\partial f'}{\partial t} = \frac{\partial f'}{\partial z} \quad \text{and} \quad \frac{\partial f''}{\partial t} = -\frac{\partial f}{\partial z}, \quad t \in \mathbb{C}
\]

**Proof:**

Consider the differential quotient \( \frac{df}{dt} = \frac{f'(z, t^n) + i f''(z, t^n) - f'(z_0, t_0^n)}{t - t_0} \).

Note: \( \frac{df}{dt} \) to exist, the limit must exist for \( t \to t_0 \).

Formally, \( \frac{df}{dt} \) must exist at \( t \to t_0 \) along each level line of \( f \) at the singularity \( z \). \( f \) is differentiable at \( t_0 \) if the partial derivatives \( \frac{\partial f'}{\partial t}, \frac{\partial f'}{\partial z}, \frac{\partial f''}{\partial t}, \frac{\partial f''}{\partial z} \) exist.

Now approach to along the direction parallel to the red axis, i.e., for fixed \( t_0 \),

\[
\lim_{t \to t_0} \frac{df}{dt} = \frac{\partial f'}{\partial t} + i \frac{\partial f''}{\partial z} \quad (5)
\]

And now for fixed \( t_0 \), \( \frac{df}{dt} \) at \( t_0 \).

Thus, if \( f \) is differentiable, then \( (z) = (t_0) \) and hence

\[
\frac{\partial f'}{\partial t} = \frac{\partial f'}{\partial z} \quad \text{and} \quad \frac{\partial f''}{\partial t} = -\frac{\partial f}{\partial z}
\]

Now we need to prove the converse. We have

\[
f(t) - f(t_0) = f'(z', t^n) + i f''(z', t^n) - f'(z_0, t_0^n) - i f''(z_0, t_0^n)
\]

That from Taylor's theorem on \( f \), for \( t \to t_0 \),

\[
f'(z', t^n) - f'(z_0, t_0^n) \to \frac{\partial f'}{\partial z} \mid_{t_0} (z' - z_0^n) + \frac{\partial f'}{\partial t} \mid_{t_0} (t^n - t_0^n)
\]

And

\[
f''(z', t^n) - f''(z_0, t_0^n) \to \frac{\partial f''}{\partial z} \mid_{t_0} (z' - z_0^n) + \frac{\partial f''}{\partial t} \mid_{t_0} (t^n - t_0^n)
\]
\[
\frac{f(z) - f(z_0)}{z - z_0} \rightarrow \frac{1}{z - z_0} \left[ \frac{df'}{dz} \bigg|_{z_0} (z - t_0) + \frac{df''}{dz} \bigg|_{t_0} (t - z_0) + i \frac{df'}{dz} \bigg|_{z_0} (z - t_0) + i \frac{df''}{dz} \bigg|_{t_0} (t - z_0) \right]
\]

\[
= \frac{1}{z - z_0} \left[ \frac{df'}{dz} \bigg|_{z_0} (z - t_0 + it - it_0) + \frac{df''}{dz} \bigg|_{t_0} (t - z_0 - it + it_0) \right]
\]

\[
= \frac{df'}{dz} \bigg|_{z_0} \frac{z - t_0}{z - z_0} - i \frac{df''}{dz} \bigg|_{z_0} \frac{z - t_0}{z - z_0} + \frac{df'}{dz} \bigg|_{t_0} + i \frac{df''}{dz} \bigg|_{t_0}
\]

The rhs exists, so then for \( \frac{df}{dz} \big|_{z_0} \) exists and it is independent of how the limit is taken.

Example: (1) \( f(z) = \frac{z^2}{2} = \frac{(z^2 - \overline{z}^2)}{2} + i \overline{z}^2 \frac{z}{2} \)

\( \frac{df'}{dz} = 2z = 2 \overline{z} \frac{z}{2} = \frac{df''}{dz} \)

\( \frac{df'}{dz} = -2 \overline{z} = -\frac{df''}{dz} \)

\( \Rightarrow f(z) \) is analytic on \( C \).

(2) \( f(z) = \frac{1}{z} = \frac{1}{z^2 + z \overline{z}} - i \frac{1}{z^2 + z \overline{z}} \) \( \Rightarrow f' = \frac{1}{z^2} \), \( f'' = -\frac{2}{z^4} \)

\( \frac{df'}{dz} = \frac{1}{z^2} - \frac{2}{z^4} \frac{z}{2} \)

\( \frac{df''}{dz} = -\frac{1}{z^4} + \frac{2}{z^6} \)

\( \frac{df'}{dz} = \frac{1}{z^2} - \frac{2}{z^4} \frac{z}{2} \)

\( \frac{df''}{dz} = -\frac{1}{z^4} + \frac{2}{z^6} \)

\( \Rightarrow f(z) \) is analytic on \( C \setminus \{0\} \).

(3) \( f(z) = \frac{1}{z(z - z_0)^2} \), \( \Re f \) is analytic on \( C \setminus \{z_0\} \).

Wolfe: \( f \colon \mathbb{R} \to \mathbb{C} \) is analytic, then \( f' + f \) satisfies Laplace's differential equation \( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \).

\( \phi = x^2 + y^2 \)

Proof: For \( z \to z_0^n \) \( \Rightarrow \frac{\partial f'}{\partial z} = \frac{\partial f''}{\partial z} = \frac{\partial f'}{\partial z} \)

\( \Rightarrow \phi = x^2 + y^2 \).

Remark: As we have assumed that the second derivatives exist. One can doubt whether we do, as it does all higher derivatives! This is another indication of how