25. **Transformations of tensor fields**

a) Consider a covariant rank-\(n\) tensor field \(t_{i_1...i_n}(x)\) and find its transformation law under normal coordinate transformations that is analogous to §5.1 def.1; i.e., find how \(\tilde{t}_{i_1...i_n}(\tilde{x})\) is related to \(t_{i_1...i_n}(x)\).

b) Convince yourself that your result is consistent with the transformation properties of (i) a covector \(x_i\) (the case \(n = 1\)), and (ii) the covariant components of the metric tensor \(g_{ij}\).

(4 points)

26. **Curl and divergence**

Show that the curl and the divergence of a vector field transform as a pseudovector field and a scalar field, respectively.

(3 points)

27. **Tensor products, and tensor traces**

Prove Propositions 1 and 2 from ch. 1 §5.3.

(4 points)

28. **Lindhard function**

Consider the function \(f : \mathbb{C} \to \mathbb{C}\) (which plays an important role in the theory of many-electron systems) defined by

\[
f(z) = \log \left( \frac{z - 1}{z + 1} \right)
\]

The *spectrum* \(f'' : \mathbb{R} \to \mathbb{R}\) and the *reactive part* \(f' : \mathbb{R} \to \mathbb{R}\) of \(f\) are defined by

\[
f''(\omega) := \frac{1}{2i} \left[ f(\omega + i0) - f(\omega - i0) \right], \quad f'(\omega) := \frac{1}{2} \left[ f(\omega + i0) + f(\omega - i0) \right]
\]

a) Show that \(f'\) and \(f''\) are indeed real-valued functions.

b) Determine \(f''\) and \(f'\) explicitly, and plot them for \(-3 < \omega < 3\).

c) Show that

\[
\int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{f''(\omega)}{\omega - z} = f(z)
\]

(5 points)