5. **Energy-momentum tensor**

Consider the electromagnetic field in the absence of matter.

a) Show that the tensor field

\[ H^\nu_\mu(x) = (\partial_\mu A_\alpha(x)) \frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\alpha(x))} - \delta^\nu_\mu \mathcal{L} \]

obeys the continuity equation

\[ \partial_\nu H^\nu_\mu(x) = 0 \quad (\ast) \]

*note:* Notice that \( H^\nu_\mu(x) \) is a generalization of Jacobi’s integral in Classical Mechanics.

b) Show that \((\ast)\) also holds for

\[ \tilde{T}^\nu_\mu(x) = H^\nu_\mu(x) + \partial_\alpha \psi^\nu_\mu(x) \]

where \( \psi^\nu_\mu(x) \) is any tensor field that is antisymmetric in the second and third indices, \( \psi^\nu_\mu(x) = -\psi^\alpha_\nu(x) \).

c) Show that \( \psi^\nu_\mu(x) \) can be chosen such that \( \tilde{T}^\nu_\mu(x) = T^\nu_\mu(x) \), which provides an alternative proof that \( T^\nu_\mu(x) \) obeys \((\ast)\).

(5 points)

6. **Energy-momentum conservation in the presence of matter**

Prove the corollary of ch. 1 §2.3: In the presence of matter, the energy-momentum tensor obeys the continuity equation

\[ \partial_\nu T^\nu_\mu(x) = -\frac{1}{c} F^\nu_\mu(x) J_\nu(x) \]

(2 points)

7. **Energy-momentum tensor for a massive scalar field**

Consider the massive scalar field from Problem 3:

\[ \mathcal{L} = \frac{1}{2} (\partial_\mu \varphi) (\partial^\mu \varphi) - \frac{m^2}{2} \varphi^2 \]

and the tensor field \( H^\nu_\mu \) defined analogously to Problem 5:

\[ H^\nu_\mu = (\partial_\mu \varphi) \frac{\partial \mathcal{L}}{\partial (\partial_\nu \varphi)} - \delta^\nu_\mu \mathcal{L} \]

Determine \( H^\nu_\mu \) explicitly and show that

\[ \partial_\nu H^\nu_\mu = 0 \]

*hint:* Use the Euler-Lagrange equation determined in Problem 3a).

(3 points)

.../over
8. **Coulomb gauge**

Consider the 4-vector potential $A^\mu(x) = (\varphi(x), A(x))$. Show that one can always find a gauge transformation such that

$$\nabla \cdot A(x) = 0$$

This choice is called *Coulomb gauge*. 

(2 points)
\[ \begin{align*}
\text{a) } & \quad \partial_v \delta_{\mu} \nu L = \partial_\mu L = \frac{\partial L}{\partial A_\mu} \partial_\alpha A_\mu + \frac{\partial L}{\partial (\partial_\alpha A_\mu)} \partial_\nu (\partial_\alpha A_\mu) \\
& \quad = \partial_\nu \frac{\partial L}{\partial (\partial_\alpha A_\mu)} \partial_\alpha A_\mu \\
& \implies 0 = \partial_\nu \left( \frac{\partial L}{\partial (\partial_\alpha A_\mu)} \partial_\alpha A_\mu + \delta_{\mu} \nu L \right) = \partial_\nu \eta_{\mu} \nu
\end{align*} \]

\[ \begin{align*}
\text{b) } & \quad \partial_\nu \partial_\mu \gamma_{\nu} \kappa \mu = -\partial_\nu \partial_\mu \gamma_{\nu} \kappa \mu = -\partial_\nu \partial_\mu \gamma_{\nu} \kappa \mu = -\partial_\nu \partial_\mu \gamma_{\nu} \kappa \mu \\
& \implies \partial_\nu \gamma_{\nu} \kappa \mu = 0 \\
& \implies \partial_\nu \gamma_{\mu} = 0
\end{align*} \]

\[ \begin{align*}
\text{c) } & \quad \text{Covariant } \gamma_{\mu} \kappa \nu = \frac{\gamma_{\mu} \kappa \nu}{c^2} A^\gamma F^{\kappa \nu} = -\frac{i}{c^2} A^\gamma F^{\kappa \nu} - \gamma_{\mu} \kappa \nu
\end{align*} \]

\[ \begin{align*}
& \implies \partial_\nu \gamma_{\mu} = 0 \\
& \implies \gamma_{\mu} \kappa \nu = \gamma_{\mu} \kappa \nu + \partial_\lambda \gamma_{\mu} \kappa \nu = (\partial_\lambda A_\mu) \frac{\partial L}{\partial (\partial_\lambda A_\mu)} F^{\kappa \nu} + \frac{i}{c^2} \partial_\lambda A^\gamma F^{\kappa \nu} \\
& \quad + \frac{i}{2c^2} (\partial_\lambda A^\gamma - \partial_\gamma A^\lambda) F^{\kappa \nu} + \frac{i}{16c^2} \partial_\mu \gamma_{\kappa \nu} A^\lambda F^{\gamma \lambda} \\
& \quad + \frac{i}{16c^2} (\partial_\mu A_{\gamma \lambda} - \partial_\gamma A_{\mu \lambda}) F^{\kappa \nu} + \frac{i}{16c^2} \partial_\lambda \gamma_{\kappa \nu} A^\gamma F^{\mu \lambda} \\
& \quad + \frac{i}{16c^2} \partial_\mu \gamma_{\kappa \nu} F^{\mu \lambda} + \frac{i}{16c^2} \partial_\kappa \gamma_{\nu} F^{\mu \lambda} = \gamma_{\mu} \kappa \nu
\end{align*} \]
6. In the proof of the proposition $\phi$, we find

The only difference is that now the EM fields

$$\phi_x = \frac{\partial}{\partial x} f \phi$$

$$\Rightarrow \phi_x = \frac{1}{c} \left[-(\partial \phi_x^k) F_x^k - F_x^k \partial_x \phi_x + \frac{1}{4} \phi_x^2 F_x^k F_x^k \right]$$

$$\Rightarrow \phi_x = \frac{1}{c} \left[-(\partial \phi_x^k) F_x^k + \frac{1}{2} (\partial_x F_x^k) F_x^k \right]$$

$$= \frac{1}{c} \phi_x^k F_x^k$$

$$= 0 \quad \text{by (2.3)}$$
\[ \frac{\partial}{\partial t} \nabla^\mu \varphi = (\partial_t \varphi) \left( \frac{\partial \varphi}{\partial x^\mu} \right) - \delta_\mu^\nu \frac{\partial}{\partial x^\nu} \varphi \]

\[ = (\partial_t \varphi) (\partial^\mu \varphi) - \delta_\mu^\nu \frac{\partial}{\partial x^\nu} \left( \partial_t \varphi \right) \left( \partial^\nu \varphi \right) + \delta_\mu^\nu \frac{\partial}{\partial x^\nu} \varphi \]

\[ \Rightarrow \partial_\nu \nabla^\nu = (\partial_\nu \partial_t \varphi) (\partial^\nu \varphi) + (\partial_t \varphi) (\partial_\nu \partial^\nu \varphi) - (\partial_\nu \partial_\xi \varphi) (\partial^\nu \varphi) + m^2 \varphi \partial_\nu \varphi \]

\[ = (\partial_\nu \partial_t \varphi) (\partial^\nu \varphi) - (\partial_\nu \partial_\xi \varphi) (\partial^\nu \varphi) + (\partial_t \varphi) (\partial_\nu \partial^\nu \varphi + m^2 \varphi) \]

\[ (\partial_t \varphi) (\partial_\nu \partial^\nu + m^2) \varphi = 0 \quad \text{by Klein-Gordon eq.} \quad \text{Robinson (Ia)} \]
8.1) Gauge Loop: \[ A^\tau \rightarrow A^\tau - \partial \tau \chi \]
\[ \rightarrow \bar{A} \rightarrow \bar{A} - \bar{\nabla} \chi \]
\[ \rightarrow \bar{\nabla} \cdot \bar{A} \rightarrow \bar{\nabla} \cdot \bar{A} - \bar{\nabla}^2 \chi \]

Now choose \( \chi \) as any solution of the Poisson eq.
\[ \bar{\nabla}^2 \chi(x) = \bar{\nabla} \cdot \bar{A}(x) \]

In the transformed \( \bar{A} \) has the property
\[ \bar{\nabla} \cdot \bar{A}(x) = \bar{\nabla} \cdot \bar{A}(x) - \bar{\nabla}^2 \chi(x) = 0 \]