25. **Field due to distant charges**

Consider the electric field generated by a charge density $\rho(y)$ that vanishes inside a sphere with radius $r_0$: $\rho(y) = 0$ for $|y| \leq r_0$. Show that

a) If $\rho$ is invariant under parity operations, $\rho(-y) = \rho(y)$, then the electric field at the origin vanishes.

b) If $\rho(y)$ is invariant under rotations about the $z$-axis through multiples of an angle $\alpha$ with $|\alpha| < \pi$, then the field-gradient tensor at the origin has the form $\varphi_{ij}(x = 0) = \begin{pmatrix} \varphi & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & -2\varphi \end{pmatrix}$

c) If $\rho(y)$ has cubic symmetry, i.e., if $\rho(y)$ is invariant under rotations through $\pi/2$ about any of the three axes $x$, $y$, and $z$, then the field-gradient tensor at the origin vanishes.

(6 points)

26. **Electrostatic interaction**

Consider the following classical model for a nuclear quadrupole moment in a crystal lattice: A rectangular parallelepiped (height $A$, length and width $B$) carries a charge $e$ at each of its eight corners. At the center of the parallelepiped is a homogeneously charged spheroid (charge $Q$, semi-axes $a$ and $b$). The symmetry axis of the spheroid forms an angle $\theta$ with the $A$-axis of the parallelepiped. The center of the spheroid is fixed, but the angle $\theta$ can vary. Let $A \gg a$, $B \gg b$.

a) Calculate the electrostatic interaction energy $U$ of this system to quadrupolar order. Show that $U$ can be expressed in terms of $e$, the lattice constants $A$ and $B$, and the quadrupole moment $Q_{33}$ of the spheroid in the coordinate system of the lattice.

b) Calculate the quadrupole moment $Q'_{33}$ of the spheroid in its principal-axes system, and then calculate $Q_{33}$ by transforming into the lattice system. Express $U$ as a function of the angle $\theta$.

*hint:* In general, lining up the principal-axes systems would require three Euler angles. However, due to the symmetries of the problem $Q'_{33}$ and $Q_{33}$ in the present case are related by only one angle, viz., $\theta$.

c) Find the equilibrium positions of the spheroid. Make sure to distinguish the cases of prolate and oblate spheroids ($a > b$ and $a < b$, respectively), as well as between the cases $A > B$ and $A < B$.

(15 points)

.../over
27. Electric charges in an external field

Consider a static electric charge distribution $\rho(x)$ subject to a static potential $\varphi(x)$. Consider the force $F_{el}$ on the charge distribution and show that $F_{el} = -\nabla U$, with $U$ the electrostatic energy calculated in ch.3 §3.6. In particular, convince yourself that the dipole term in the multipole expansion of $U$ gives the correct potential energy for an electric dipole moment $d$ in an electric field $E$.

(3 points)
\[ \phi(x) - \int d^2 \frac{\delta(x)}{|x-k|} = \phi(x=0) + x \cdot \nabla \phi \bigg|_{x=0} + \frac{i}{2} x_i x_j \frac{\partial^2 \phi}{\partial x_i \partial x_j} \bigg|_{x=0} + \ldots \]

\[ = \phi_0 - x \cdot \vec{e} + \frac{i}{2} x_i x_j \phi_{ij} + \ldots \]

\[ = \phi_0 + \phi_0(x) + \phi_2(x) + \ldots \]

a) \[ \delta(x) = \delta(-x) \Rightarrow \phi(-x) = \int d^2 \frac{\delta(x)}{|x-k|} = \int d^2 \frac{\delta(x)}{|x-k|} = \phi(x) \]

\[ \Rightarrow \text{All } n \text{ odd } i \text{ vanish, i.e. perturbation } \vec{e}^2 = 0 \]

b) \[ \phi_{ij} \text{ is not symmetric } \Rightarrow \text{ normal condition } \text{ and let } \phi_{ij} \text{ is diagonal} \]

\[ \phi(x) \text{ obeys Laplace's eq. } \forall r < r_0 \]

\[ \Rightarrow \frac{\delta}{\delta x_i} = 0 \]

\[ \Rightarrow \phi_{ij} \text{ has the form } \phi_{ij} = \begin{pmatrix} \phi_+ + \phi_- & 0 & 0 \\ 0 & \phi_+ - \phi_- & 0 \\ 0 & 0 & -2\phi_+ \end{pmatrix} \]

\[ \text{where } \phi_+ = \frac{i}{\xi} (\phi_{xx} - \phi_{yy}) \]

\[ \Rightarrow \phi_0(x) = \frac{i}{\xi} x^2 \frac{\partial^2}{\partial x^2} \phi_+ \phi_- \]

\[ + \frac{i}{\xi} \frac{\partial^2}{\partial x^2} \phi_+ \phi_- \]

\[ + \frac{i}{\xi} \frac{\partial^2}{\partial x^2} (-2\phi_+) \]

\[ = \frac{i}{\xi} \left[ (1 - \frac{1}{2} w^2) \phi_+ \phi_- \right] \]

Rotational invariance of \( \phi(x) \) implies rotational invariance of \( \phi(x) \), i.e. perturbation of \( \phi_0(x) \)
\[ \psi_c \left( r, \alpha, \varphi + \delta \right) = \frac{1}{2} \ \frac{e^{i}}{\mathcal{L}} \ \left[ (1 - \cos^2 \varphi) \ \psi_+ + \cos^2 \varphi \ \psi_- \right] \]

\[ = \psi_c \left( r, \alpha, \varphi \right) \]

\[ \psi_+ \ \cos^2 \varphi + \psi_- \ \sin^2 \varphi = \psi_+ \cos^2 \varphi \]

\[ \Rightarrow \ \psi_- = 0 \]

1) Certain family \[ \mathcal{F} \] invariant under rotations about any of the three axes \( x, y, z \).

b) \[ \Rightarrow \ \psi_+ = 0 \] due to invarian under rotation about \( t \) and about \( x \), \( y \), \( z \) \[ \Rightarrow x' = 0 \]

The invariance of \[ \mathcal{F} \] implies invariance of \[ \psi(x) \]

\[ \Rightarrow \ \psi_c \left( r, \alpha + \frac{\pi}{2}, \varphi \right) = \frac{1}{2} \ \frac{e^{i}}{\mathcal{L}} \ \left[ (1 - \cos^2 \varphi) \ \psi_+ + \cos^2 \varphi \ \psi_- \right] \]

\[ = \frac{1}{2} \ \frac{e^{i}}{\mathcal{L}} \ \psi_+ \left[ (1 - \cos^2 \varphi \cos^2 \varphi) \right] \]

\[ \Rightarrow \ \psi_+ \cos^2 \varphi \cos^2 \varphi = \psi_+ \cos^2 \varphi \]

\[ \Rightarrow \ \psi_+ = 0 \]

\[ \psi \left( x + \hat{e}_1 \right) = \psi \left( x \right) \]

\[ \Rightarrow \ \psi_+ = 0 \]

\[ \Rightarrow \ \psi_+ = 0 \]


26. a) Consider the potential due to the 2 charges:

\[ \phi(x) = e \frac{\delta}{\delta x} \frac{1}{|x - \frac{x_1}{2}|} \quad \text{where} \quad \phi_0 = e \left( \frac{1}{|x - \frac{x_1}{2}|} \right) \]

We have:

\[ \psi_0 = \psi(x = 0) = e \frac{\delta}{\delta x} \frac{1}{|x - \frac{x_1}{2}|} \]

\[ E = -\nabla \phi(x = 0) = e \frac{1}{|x - \frac{x_1}{2}|} \]

\[ \psi_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} \psi \bigg|_{x = 0} = \begin{pmatrix} \psi_0 & 0 & 0 \\ 0 & \psi_0 & 0 \\ 0 & 0 & -2\psi_0 \end{pmatrix} \]

By symmetry, we find:

\[ \psi = \psi_{xx} = \frac{\partial^2}{\partial x^2} \left( e \frac{\delta}{\delta x} \frac{1}{|x - \frac{x_1}{2}|} \right) = e \left( \frac{1}{(r_1)^2} - \frac{1}{(r_0)^2} \right) \]

Again, \( r_0 = \frac{1}{A^{1/4}} \rightarrow r_1^{(2)} = \frac{1}{2} r_0 \)

\[ \rightarrow \quad \psi_0 = \frac{16e}{r_0} \]

\[ \psi = e \left( \frac{1}{(r_0)^2} - \frac{1}{(r_1)^2} \right) \]

\[ = e \frac{1}{r_0^2} \left( \frac{1}{(r_0^2 - A^{1/4})} \right) = \frac{2e}{r_0^2} \left( \frac{1}{r_0^2 - A^{1/4}} - \frac{1}{A^{1/4}} \right) = \frac{2e}{r_0^2} \left( \frac{1}{r_0^2 - A^{1/4}} \right) \]

\[ u = \psi_0 Q_1 + \frac{1}{2} \left( \psi_{xx} + \psi_{xx} - 2\psi \right) \]

\[ = \psi_0 Q + \frac{1}{2} \psi \left( Q_{xx} + Q_{xx} - 2Q_{xx} \right) \]

\[ \sum_{i} Q_{xx} = 0 \quad \rightarrow \quad \psi_0 Q + \psi Q_{xx} \]

\[ \sum_{i} Q_{xx} = 0 \quad \rightarrow \quad \psi_0 Q - \psi Q_{xx} \]
mark: Here $Q_{ij}$ is the quadrupole moment of the spherical
in the lattice coordinate axes!

\[ Q_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2q \end{pmatrix} \]

Transform to the lattice axis by means of rotation
matrices 

\[ a_{ij} = \sum \delta_{ik} \delta_{lj} \]

\[ a_{ij} = \sum \delta_{ik} \delta_{lj} \]

\[ a_{ij} = \sum \delta_{ik} \delta_{lj} \]

Now $a_{ij}$ is in orthogonal base \[ \sum a_{ij} = 1 \]

ed a must define the $x'$-axis will the t-axis \[ A_{xx} = \frac{3}{4} \]

\[ A_{xx} = \frac{3}{4} \]

Finally, Problem 20 will \[ q = \frac{a}{10} (\Delta^2 - \bar{a}) \]

\[ U = \psi_0 A - \frac{2e}{\hbar c} \left( \psi_0 A \right) \frac{1}{2} \left( 1 - 2\psi_0 A \right) \]

\[ U = \psi_0 A + \frac{2e}{\hbar c} \left( A^2 - \psi_0 A \right)^2 \left( 1 - \psi_0 A \right) \left( 1 - \psi_0 A \right) \]
(b) \( J^2 \) is minimized for \( \zeta = 0 \) \( \Rightarrow \lambda = 1/2 \) 

maximized for \( \zeta = \pm \xi \) 
\( \Rightarrow \lambda = 0, \xi \)

let \( \lambda > 0 \) \( \Rightarrow \) \( \lambda \) is minimized for
\( \lambda = \frac{\xi}{2} \) if \( (\lambda^2 - \lambda^3)(e^2 - e^3) > 0 \)
\( \lambda = 0 \) if \( (\lambda^2 - \lambda^3)(e^2 - e^3) < 0 \)

prolate spheroid \( (\lambda > 0) \)

oblate \( (\lambda < 0) \): flips the two cones

\( \lambda < 0 \): Flips the two cones again.
27. ) \( d^2 g_x = 3 \pi \varepsilon_0 \psi(x) \) 

\( \nabla \psi(x) = \sum \varepsilon_x \delta(x - \bar{x}_x) \) 

\[ \mathbf{E} = -\varepsilon_x \nabla \psi(\bar{x}) = -\int d\bar{x} \psi(\bar{x}) \nabla \psi(\bar{x}) = + \int d\bar{x} \bar{E}(\bar{x}) \bar{E}(\bar{x}) \]

Now expand \( i = d^2 \bar{E} \):

\[ \mathbf{E} = -\int d\bar{x} \psi(\bar{x}) \left[ \frac{\partial \psi}{\partial x} \bigg|_{x=0} + \frac{\partial \psi}{\partial y} \bigg|_{x=0} + \cdots \right] \]

\[ = -Q \frac{\partial \psi}{\partial x} \bigg|_{x=0} - \int d\bar{x} \psi(\bar{x}) \sum \delta(x - \bar{x}_x) \bar{E} \frac{\partial \psi}{\partial x} \bigg|_{x=0} \]

\[ \frac{\partial \psi}{\partial x} = -Q \frac{\partial \psi}{\partial x} \bigg|_{x=0} + \int d\bar{x} \psi(\bar{x}) \bar{E} \frac{\partial \psi}{\partial x} \bigg|_{x=0} \]

\[ \therefore \frac{\partial }{\partial x} \left[ \bar{E} \cdot d \right] = -Q \frac{\partial \psi}{\partial x} \bigg|_{x=0} \]

\[ \Rightarrow \mathbf{E} = -\nabla \psi \bigg|_{x=0} \] 

\( \text{will } \psi(x) = Q \psi(\bar{x}) - \bar{E} \cdot \bar{E}(\bar{x}) \) 

\( \text{if } x \bar{E} \text{ is put to zero after calculating the product} \)

In particular, the dipole has - the electrostatic energy 

\[ U_{dipole} = -\bar{E} \cdot d \]

which describes the potential energy of a fixed electric dipole in an electric field \( \bar{E} \).