37. **Potentials in Coulomb gauge**

Consider the potentials $\varphi$ and $\mathbf{A}$ in the Coulomb gauge, i.e., the field equations from ch.4 §1.2 proposition 2. Show explicitly that the resulting asymptotic electric and magnetic fields are the same as those calculated in the Lorenz gauge in ch.4 §3.

*hint:* Show that the scalar potential does not contribute to the electric field, and show that the asymptotic vector potential now reads

$$\mathbf{A}(\mathbf{x}, t) = -\hat{x} \times \left[ \hat{x} \times \frac{1}{r c} \int d\mathbf{y} \, j(\mathbf{y}, t_r) \right]$$

instead of the expression derived in ch. 4 §3.1. Then calculate the fields.

(8 points)

38. **Radiation from cyclotron motion**

Consider a point mass $m$ with charge $e$ that moves in a plane perpendicular to a homogeneous magnetic field $\mathbf{B}$. Assume nonrelativistic motion, $v \ll c$

a) Find the power radiated by the particle.

b) Show that the energy of the particle decreases with time according to $E(t) = E_0 e^{-t/\tau}$, and determine the timescale $\tau$.

c) Find $\tau$ in seconds for an electron in a magnetic field of 1 Tesla.

(4 points)

39. **Radiating harmonic oscillator**

Consider particle with charge $e$ and mass $m$ in a one-dimensional harmonic potential. Let the frequency of the harmonic oscillator by $\omega_0$.

a) Find the power radiated by the particle, averaged over one oscillation period, as a function of the energy $E$ of the oscillator.

*hint:* Remember the virial theorem, which for a harmonic potential says $\bar{V} = \bar{T} = E/2$, with $V$, $T$, and $E$ the potential, kinetic, and total energy, respectively, of the particle, and the bar denoting a time average.

b) Show that the energy of the oscillator again decreases exponentially, $E(t) = E_0 e^{-t/\tau}$.

c) Determine $\tau$ in seconds for $e$ and $m$ the electron charge and mass, respectively, and $\omega_0 = 10^{15}$ sec$^{-1}$ (a typical atomic frequency).

(4 points)

41. **Absence of dipole radiation**

Show that a system of particles that all have to the same ratio of charge to mass and are not subject to any external forces cannot emit either electric or magnetic dipole radiation.

(3 points)
\[ \mathbf{\nabla} \cdot \mathbf{A} = \frac{q_0}{\varepsilon} \quad (\text{i}) \]

\[ \mathbf{\nabla} \times \mathbf{A} = -\mu \frac{\partial \mathbf{A}}{\partial t} \quad (\text{ii}) \]

with the condition \( \nabla \cdot \mathbf{A} = 0 \).

(i) is solved by Poisson's formula

\[ \psi(x,t) = \frac{1}{4\pi} \int \frac{\psi_0(r,t)}{|x - r|} \, dr \]

\[ \Rightarrow \text{Asymptotically} \quad \psi(x,t) = \frac{1}{\varepsilon} \int \psi_0(r,t) \, dr + O(1/r^2) \quad (\text{e}) \]

\[ \Rightarrow \nabla \psi = O(1/r^2) \]

\[ \Rightarrow \text{The scalar potential cannot contribute to the asymptotic electric field} \mathbf{E}, \quad \text{while decay as} \ 1/r. \]

\[ \Rightarrow \text{For the asymptotic fields,} \mathbf{\nabla} \cdot \mathbf{A} = 0 \]

\[ \mathbf{E}(x,t) = -\frac{1}{\varepsilon} \frac{\partial}{\partial t} \mathbf{A}(x,t) + O(1/r^2) \]

\[ \mathbf{B}(x,t) = \mathbf{\nabla} \times \mathbf{A}(x,t) \]

Now rewrite (ii) for \( \mathbf{A} \). For the wave \( \mathbf{F} \) and \( \mathbf{\nabla} \psi \),

do a spherical Fourier Transforms \( \mathbf{F} \).

\[ -\mathbf{k} \cdot \psi(x,t) = -q_0 \psi(\mathbf{k},t) \]

The charge conservation implies \( \partial_t \psi(\mathbf{k},t) = -\mathbf{k} \cdot \mathbf{J}(\mathbf{k},t) \)

\[ \Rightarrow \partial_t \psi(\mathbf{k},t) = i\mathbf{k} \cdot \mathbf{J}(\mathbf{k},t) \]

\[ \Rightarrow \partial_t \psi(\mathbf{k},t) = \frac{q_0}{\varepsilon} i\mathbf{k} \cdot \mathbf{J}(\mathbf{k},t) \]
\[
\begin{align*}
\text{In this } \varphi (x) \Rightarrow \\
\left( \frac{1}{c^2} \partial_x^2 + \vec{\lambda}^2 \right) \varphi (x, t) &= \frac{\text{i} \hbar}{c} \frac{\partial}{\partial t} \varphi (x, t) - \frac{\text{i}}{\hbar} \left( (-\text{i} \hbar) \frac{\partial}{\partial x} \cdot \vec{\lambda} \cdot \varphi (x, t) \right) \\
&= \frac{\text{i} \hbar}{c} \left[ \frac{\partial}{\partial t} \varphi (x, t) - \vec{\lambda} \cdot \varphi (x, t, t) \right]
\end{align*}
\]

\[\text{For } \varphi \| \vec{x} \Rightarrow \vec{\lambda} = \vec{x}\]

\[\text{Form the back propagation yields, } \]
\[\Box \varphi (x, t) = \frac{-\text{i} \hbar}{c} \left[ \frac{\partial}{\partial t} \varphi (x, t) - \vec{x} \cdot \partial_x \varphi (x, t) \right] \]
\[= \frac{-\text{i} \hbar}{c} \vec{x} \times (\vec{x} \cdot \varphi (x, t)) \]

where this as \( \vec{x} \| \vec{x} \) yields, i.e. plane of the expression \( \vec{x} \| \vec{x} \)

\[\varphi (x, t) = -\vec{x} \times \left[ \vec{x} \times \frac{1}{\text{rc}} \int \text{d}\vec{y} \varphi (\vec{y}, t) \right] \]

Now calculate this yields. Let

\[-\vec{x} \times (\vec{x} \cdot \varphi ) = \vec{j} - \vec{x} \times (\vec{x} \cdot \varphi ) = \vec{j}_T\]

where \( \vec{j}_T = \vec{j} - \vec{x} \times (\vec{x} \cdot \varphi ) \) is the transverse part

\[= \vec{j} - \vec{j}_L \quad \text{will } \vec{j}_L \cdot \vec{x} (\vec{x} \cdot \varphi ) \quad \text{the longitudinal part} \]

But the field is pure transverse

\[\Rightarrow \vec{x} \cdot \vec{j}_L = 0\]

\[\Rightarrow \Delta \cdot \vec{j}_L = \vec{\Delta} \cdot \vec{j}_L\]
\[ \mathcal{E}(x,t) = \mathcal{D}(x,t) = \frac{1}{c^2} \mathbf{E} \times \mathbf{v} \mathcal{J}(x,t) + o(1/t^2) \]

which is the same result as in 1.5 § 2.1.

For the electric field, we have

\[ \mathcal{E}(x,t) = -\frac{1}{c^2} \mathbf{E} \times \mathbf{v} \mathcal{J}(x,t) = \frac{1}{c^2} \mathbf{E} \times \left[ \mathbf{E} \times \mathcal{J}(x,t) \right] \]

which again is the same result as in 1.5 § 2.1, and hence

\[ \mathcal{E}(x,t) = -\mathbf{E} \times \mathcal{J}(x,t) \]

Remark: A nice way to see it is the boundary part of
\[ \mathcal{E} \] is constant, so add the contribution \(-\mathbf{E} \mathcal{J}\) to \(\mathcal{E}\) in constant space, which leads to the divergence theorem: It cancels the derivative part of \(-\frac{1}{c^2} \mathbf{E} \times \mathbf{v} \mathcal{J}\) in constant space, as the proof of the proposition in 1.5 § 2.1.
b) \[ \mathcal{P} = -\frac{d\mathcal{E}}{dt} \quad \Rightarrow \quad \frac{d\mathcal{E}}{dt} = -\frac{1}{c} \mathcal{E} \quad \text{viii} \quad \frac{1}{c} = \frac{4e^2}{3c^3 m} \]

\[ \mathcal{E}(t) = \mathcal{E}(t_0) e^{-t/c} \]

c) \[ \mathcal{T} = \frac{3e^2 (\mathcal{E} \mathcal{H})}{4e^2} \quad \text{vii} \quad \frac{\mathcal{E} \mathcal{H}}{2e^2} \]

\[ \text{Length:} \quad \mathcal{T} = 4.8 \times 10^{-10} \quad \text{mm} \quad 1 \text{f} = 10^4 \text{f} \]

\[ m = 9.1 \times 10^{-6} \quad \text{g} \]

\[ \Rightarrow \quad c = \frac{\sqrt{3} \times 10^{10}}{4 \left(4.8 \times 10^{10}\right)^2 \times 10^8} \quad \approx 2.6 \text{ f} \]
a)\[ \dot{x} = -\frac{\omega_0^2}{m} x \]
\[ \ddot{x} = -\frac{\omega_0^2}{m} x \Rightarrow (\ddot{x})^2 = \frac{\omega_0^4}{m^2} x^2 = \frac{\omega_0^4}{m^2} V(x) \]
\[ \Rightarrow (\ddot{x})^2 = \frac{\omega_0^4}{m} V(x) = \frac{\omega_0^4}{m} \frac{\varepsilon}{\varepsilon_0} \quad \text{by the} \quad \text{uniform} \]

b) \[ \ddot{\rho} = -\frac{\rho E}{m} = \frac{1}{\varepsilon} E \quad \text{voltage} \quad \frac{1}{\varepsilon} = \frac{2e^2}{3c^2 \omega_0^2} \]
\[ \Rightarrow E(t) = E(t-v) e^{-t/\tau} \]

c) \[ \omega_0 = 10^{15} \text{ } \text{s}^{-1} \]
\[ \Rightarrow \tau = \frac{2e^2}{2e^2 \omega_0^2} \frac{1}{2(3\times10^{10})^2} \frac{9.1\times10^{-24}}{10^{30}} \approx 1.6 \times 10^{-7} \text{ } \text{s} \]
(41) Write a net of dr of charge \( e \) in \( \text{mass} \ m_2 \).

The electric dipole moment is

\[ \vec{d} = \sum e \vec{x}_k = \sum \frac{e}{m_k} \vec{x}_k \vec{x}_k \]

\[ \Rightarrow \vec{d} = \text{wst} \sum \frac{e}{m_k} \vec{x}_k \vec{x}_k = \text{wst} \cdot \vec{x}(t) \]

will \( \vec{x}(t) \) the mass of mass. That \( \vec{x}(t) \) mass

uniform \( \vec{D} \rightarrow \vec{d}(t) \times \vec{x}(t) = 0 \rightarrow \text{no electric dipole moment} \)

The magnetic dipole moment is

\[ \vec{m} = \frac{1}{lc} \sum e \vec{x}_k \vec{x}_k \vec{p}_k = \frac{1}{lc} \sum \frac{e}{m_k} \vec{x}_k \vec{x}_k \vec{p}_k \]

\[ = \text{wst} \times \frac{1}{lc} \sum \frac{e}{m_k} \vec{x}_k \vec{x}_k \vec{p}_k = \text{wst} \times \frac{1}{lc} \vec{L}(t) \]

will \( \vec{L} = \sum \vec{x}_k \vec{p}_k = \sum \vec{x}_k \times \vec{m}_k \) the total angular momentum

that angular momentum is constant \( \Rightarrow \vec{L}(t) = 0 \)

\[ \Rightarrow \vec{m}(t) = 0 \rightarrow \text{no magnetic dipole moment} \]