1) Dielectric Response

As a simple model of a dielectric, consider \( N \) charged harmonic oscillators (charge \( e \), mass \( m \), resonance frequency \( \omega_0 \), damping coefficient \( \gamma \)) in a volume \( V \), for an oscillator density \( n = N/V \). The oscillators are driven by an external electric field \( E(t) \), so the equation of motion for each oscillator is

\[
m(\ddot{x} + \omega_0^2 x + \gamma \dot{x}) = eE(t)
\]

a) Show that the Fourier transform of the resulting polarization \( P \) is

\[
P(\omega) = \chi(\omega) E(\omega)
\]

with a susceptibility

\[
\chi(\omega) = \frac{\omega_p^2/4\pi}{-\omega^2 + \omega_0^2 - i\gamma\omega}
\]

where \( \omega_p = \sqrt{4\pi ne^2/m} \) is the plasma frequency.

b) Fourier backtransform to find the time-dependent susceptibility \( \chi(t) \). Convince yourself that the response is properly retarded, i.e., that \( \chi(t) = 0 \) for \( t < 0 \).

c) What is the relation between the polarization \( P(t) \) at time \( t \) and the field \( E(t') \) at time \( t' \)? What is the length \( T \) of the time interval during which the source field \( E \) substantially influences the response of the system?

(13 points)

2) Linear accelerator

As a simple model for a linear accelerator, consider a charged point particle that travels with constant velocity \( v_0 \) for times \( t < 0 \), then gets accelerated with constant acceleration \( a \) for times \( 0 \leq t \leq T \), and travels with constant velocity \( v_1 = v_0 + aT \) again for times \( t > T \).

a) Calculate, sketch, and discuss the radiated energy per frequency,

\[
dU/d\omega = \frac{2e^2}{3\pi c^3} |\dot{\psi}(\omega)|^2
\]

How does the spectrum change qualitatively as a function of \( T \)?

b) Calculate the total radiated power \( P \), and show that you recover the Larmor formula

\[
P = \frac{2e^2}{3c^3} a^2
\]

for a uniformly accelerated point particle.

hint: \[
\int_0^{\infty} dx \frac{\sin^2 x}{x^2} = \frac{\pi}{2}
\]

(11 points)