Mathematical Methods for Scientists

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September 27, 2019
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## Acknowledgments, and Disclaimer

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Acknowledgments

These notes are based on a one-quarter course taught various times at the University of Oregon. This course is the first quarter of a year-long sequence, with the following two quarters covering electromagnetism; the selection of material reflects this. The lectures were typeset by Wenqian Sun based on handwritten notes. The \LaTeX{} code is based on a template written for a different course by Joshua Frye, with help from Rebecca Tumblin and Brandon Schloemann.

Disclaimer

These notes are a work in progress. If you notice any mistakes, whether it’s trivial typos or conceptual problems, please send email to dbelitz@uoregon.edu.
Chapter 1

Algebraic Structures

NOTATION

\( \in \), belongs to
\( \notin \), does not belong to
\( \Rightarrow \), implies
\( \land \), and
\( \lor \), or
\( :\equiv \), is defined to be
\( \equiv \), identically equals
\( \exists \), there exists
\( \exists! \), there exists exactly one
\( \forall \), for all
\( \square \), the end of a proof
\( \iff \), if and only if
\( \cong \), is isomorphic to
1 Sets and Mappings

1.1 Sets

Let us consider a collection of well-defined, distinct objects that can be either real or imagined, such as coins, cars, numbers, letters, or pieces of chalk.

**Definition 1.**

(a) A set \( M \) is defined by any property that each of the objects does or does not possess. If \( m \) is an object that has the property, then we say “\( m \) is an element of \( M \)” or “\( m \) is in \( M \)” and write \( m \in M \). Otherwise, we write \( m \notin M \).

(b) The set containing no elements is called empty set or null set and denoted by \( \emptyset \).

**Example 1.** All pieces of blue chalk in the classroom form a set \( M_{bc} \).

If a set \( M \) has elements \( m_1, m_2, \ldots \), then we write \( M = \{m_1, m_2, \ldots\} \). If \( p \) is the property that determines \( M \), then we write \( M = \{m; m \text{ has the property } p\} \).

**Example 2.** Some common number sets are

- the set of natural numbers denoted by \( \mathbb{N} = \{1, 2, 3, \ldots\} \),
- the set of integers denoted by \( \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \),
- the set of rationals denoted by \( \mathbb{Q} = \{p/q; p, q \in \mathbb{Z} \land q \neq 0\} \)
- the set of real numbers denoted by \( \mathbb{R} \),
- the set of complex numbers denoted by \( \mathbb{C} \), etc.

**Remark 1.** It is assumed that the reader has an intuitive understanding of these number sets. For a formal definition of \( \mathbb{N} \), see *Introduction to Mathematical Philosophy* by Bertrand Russell (1993); for \( \mathbb{R} \), see *Algebra* by van der Waerden (1991), which is listed on the class website.

**Remark 2.** If the objects themselves are sets, problems may result that we will ignore. See Problem 1, Russell’s Paradox.

**Definition 2.** Let \( A \) and \( B \) be sets.

(a) \( A \) is called a **subset** of \( B \) (\( A \subseteq B \)) if \( a \in A \) implies \( a \in B \) (\( a \in A \implies a \in B \)).

(b) \( A \) and \( B \) are **equal** (\( A = B \)) if \( A \subseteq B \land B \subseteq A \).

(c) \( A \) is called a **proper subset** of \( B \) (\( A \subset B \)) if \( A \subseteq B \land A \neq B \).

(d) \( \emptyset \) is a subset of any set.

**Remark 3.** The relation \( A \subseteq B \) can be illustrated by a **Venn diagram**, as Fig.1.1.1 shows.

![Venn Diagram](image-url)
Remark 4. The relation $\subseteq$ is transitive, i.e., $\forall A, B, C, A \subseteq B \land B \subseteq C \implies A \subseteq C$. Fig. 1.1.2 depicts the transitive property.

Definition 3. Let $A$ and $B$ be sets. Let us define the following relations,
(a) the union of $A$ and $B$ by $A \cup B := \{x; x \in A \lor x \in B\}$,
(b) the intersection of $A$ and $B$ by $A \cap B := \{x; x \in A \land x \in B\}$ and
(c) the difference between $A$ and $B$ or the complement of $B$ in $A$ by $A \setminus B := \{x; x \in A \land x \notin B\}$.

Remark 5. These relations can also be illustrated by Venn diagrams, as Fig. 1.1.3 shows.

Definition 4. Let $A$ and $B$ be sets. If $A \cap B = \emptyset$, then we say that $A$ and $B$ are disjoint.

Definition 5. The Cartesian product of two sets $A$ and $B$, denoted by $A \times B$, is the set of all possible ordered pairs, where the first components of the ordered pairs are from $A$, and the second from $B$. We write $A \times B = \{(a, b); a \in A \land b \in B\}$.

Example 3. $\mathbb{R} \times \mathbb{R} \equiv \mathbb{R}^2$ is an algebraic representation of the Cartesian plane.
1.2 Mappings

Definition 1. Let $X, Y$ be sets.

(a) Let $\varphi$ be a prescription that associates with every $x \in X$ one and only one $y = \varphi(x) \in Y$. Then $\varphi$ is called a mapping from $X$ to $Y$, and we write $\varphi : X \to Y$.

(b) $y = \varphi(x)$ is called the image of $x$ under $\varphi$, and $x$ is called a pre-image of $y$. We write $x \xrightarrow{\varphi} y$ or $\varphi : x \to y$.

(c) If every $y \in Y$ has at least one pre-image in $X$, then $\varphi$ is called a surjective mapping. We write $Y = \varphi(X)$ and say that $\varphi$ maps $X$ onto $Y$.

(d) If every image $y \in Y$ has one and only one pre-image in $X$, then $\varphi$ is called an injective or one-to-one mapping.

(e) A mapping that is both injective and surjective is called a bijective mapping.

(f) Let $X$ be a set and let $\varphi$ be a bijective mapping from $\mathbb{N}$ to $X$. Then $X$ is called a countable set.

Example 1. $\mathbb{Z}$ and $\mathbb{Q}$ are countable sets. $\mathbb{R}$ is not countable.
Remark 1. No pre-image can have more than one image. Every \( x \in X \) must be a pre-image of some \( y \in Y \).

![Fig. 1.2.3. A non-mapping.](image)

Remark 2. An image can have multiple pre-images (See \( x_1 \) and \( x_3 \) in Fig. 1.2.1).

**Example 2.** Let \( X = Y = \mathbb{R} \). \( x \mapsto \sqrt{x} \) is not a mapping. But, if we choose \( X = \{ x; x \in \mathbb{R} \land x \geq 0 \} \) and \( Y = \mathbb{R} \), \( x \mapsto \sqrt{x} \) will be a mapping.

Remark 3. If \( \varphi : X \to Y \) is bijective, then there exists exactly one mapping \( \varphi^{-1} : Y \to X \) such that \( \varphi : x \to y \) implies that \( \varphi^{-1} : y \to x \). \( \varphi^{-1} \) is called the **inverse** of \( \varphi \). (This is plausible, but requires a proof, which we skip for now.)

**Definition 2.** Let \( X \) and \( Y \) be two identical sets. Let \( x \) be an arbitrary element of \( X \). The mapping \( \varphi : x \to x \) is called the **identity mapping** of \( X \) denoted by \( I_X \) or \( \text{id}_X \).

Remark 4. It is obvious that \( \text{id}_X \) is bijective, and \( \text{id}_X^{-1} = \text{id}_Y = \text{id}_X \).

Remark 5. If \( X \) and \( Y \) are number sets, then mappings \( f : X \to Y \) are called **functions**, and we write \( y = f(x) \). For functions, we sometimes relax the rule that no pre-image can have more than one image; functions violating the rule are called **multivalued functions**.

**Definition 3.** Let \( X \) be a set. Let \( I \) be another set called **index set**. We say that the images \( x_i \) of an arbitrary mapping \( \varphi : i \in I \to x_i \in X \) are a system of elements of \( X \) that is **labelled** or **indexed** by \( I \).

Remark 6. We often choose \( I = \mathbb{N} \). However, this is not necessary; in general, \( I \) does not even have to be countable.

**Example 3.** Counting is an example of indexing objects with \( I = \mathbb{N} \).

**Example 4.** Take rotations \( \rho \) in Cartesian plane as another example. We can label each \( \rho \) with the corresponding angle of rotation \( \alpha \). In other words, we use an uncountable set \( I = [0, 2\pi) \) to label rotations: \( \varphi : \alpha \in I \to \rho_\alpha \).

Remark 7. We can use \( I \) to index sets. This allows us to generalize our previous concepts of union and intersection: for more than two sets, the union of these sets (labelled by \( I \)) can be defined by

\[
\bigcup_{i \in I} X_i := \{ x; \exists i \in I : x \in X_i \},
\]
and the intersection by
\[
\bigcap_{i \in I} X_i := \{x; \forall i \in I, x \in X_i\}.
\]

**Fig. 1.2.4.** Union and intersection of three sets.

**Remark 8.** We can also generalize the concept of Cartesian product with the help of \(I\), e.g., \(\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \equiv \mathbb{R}^n := \{(x_1, x_2, \ldots, x_n); \forall i \in [1, n] \cap \mathbb{N}, x_i \in \mathbb{R}\} \).

**Definition 4.** Let \(X, Y, Z\) be sets. Let \(f : X \to Y\) and \(g : Y \to Z\) be mappings. The relation that connects each \(x \in X\) to some \(g(f(x)) \in Z\) defines another mapping \(g \circ f : X \to Z\) called the **composition** of \(f\) and \(g\). We say “\(g\) after \(f\)”, or “\(g\) follows \(f\)". 

**Fig. 1.2.5.** The composition of \(f\) and \(g\).
Proposition 1. Let $X_1, X_2, X_3, X_4$ be sets. Let $f_1 : X_1 \to X_2$, $f_2 : X_2 \to X_3$, $f_3 : X_3 \to X_4$ be mappings. Then $f_3 \circ (f_2 \circ f_1) = (f_3 \circ f_2) \circ f_1 \equiv f_3 \circ f_2 \circ f_1$.

Proof. Let $x$ be an arbitrary element of $X_1$. It follows that $(f_3 \circ (f_2 \circ f_1))(x) \equiv f_3((f_2 \circ f_1)(x)) \equiv f_3(f_2(f_1(x)))$. We also have $((f_3 \circ f_2) \circ f_1)(x) \equiv (f_3 \circ f_2)(f_1(x)) \equiv f_3(f_2(f_1(x)))$. The equality $f_3 \circ (f_2 \circ f_1) = (f_3 \circ f_2) \circ f_1$ is thus established: for all $x \in X_1$, both of these two mappings map $x$ to $f_3(f_2(f_1(x))) \in X_4$. We say that the operation $\circ$ is associative and write $f_3 \circ f_2 \circ f_1$. \hfill $\square$

Remark 9. In general, the operation $\circ$ is not commutative, i.e., $f_2 \circ f_1 \neq f_1 \circ f_2$.

Example 5. Let us consider two real functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x + 1$ and $g(x) = x^2$, respectively. It is easy to see that $g \circ f \neq f \circ g$: for an arbitrary $x \in \mathbb{R}$, $(g \circ f)(x) \equiv g(f(x)) = (x + 1)^2 \neq x^2 + 1 = f(g(x)) \equiv (f \circ g)(x)$.

1.3 Ordered Sets

Definition 1. Let $X$ be a set. An order on $X$ can be any relation $x \sim y$ between components of ordered pairs $(x, y) \in X \times X$ that possesses the following properties: $\forall x, y, z \in X$,

1. $x \sim x$; (reflexivity)
2. $(x \sim y \land y \sim x) \implies x = y$;
3. $(x \sim y \land y \sim z) \implies x \sim z$. (transitivity)

What is more, we say that the order is linear if $\forall (x, y) \in X \times X, x \sim y \lor y \sim x$.

Example 1. Let $m, n \in \mathbb{N}$. The relation that $m$ divides $n$ ($m|n$) is an order on $\mathbb{N}$. But, it is not linear, since $2|3$ and $3 \not| 2$.

Example 2. Let us consider the set of all people and the relation that one begets the other. This is not an order, since reflexivity is not satisfied.

Example 3. The ordinary $\leq$ on $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ is a linear order.

Remark 1. We often use $\leq$ to denote orders.

Definition 2.

(a) Let $Y$ be a set and $\leq$ an order on $Y$. Let $X \subseteq Y$. Let $b \in X$ have the property: $\forall y \in Y, y \leq b$ ($b \leq y$). Such a $b$ is termed an upper (a lower) bound of $Y$, and we say that $Y$ is bounded above (below) by $b$.

(b) Let $B$ be the set of upper (lower) bounds of $Y$. Let $b_0 \in B$ have the property: $\forall b \in B, b_0 \leq b$ ($b \leq b_0$). We call $b_0$ the least upper bound (greatest lower bound) of $Y$. It is common practice to refer to the least upper bound (greatest lower bound) as supremum (infimum), and we write $b_0 = \sup Y$ ($b_0 = \inf Y$).

Remark 2. The supremum or infimum of a set is not necessarily contained in the set.
Example 4. Let \( X = [0, 1) \subseteq \mathbb{R} \). We have \( \sup X = 1 \notin X \), and \( \inf X = 0 \in X \).

1.4 Natural Numbers and the Principle of Mathematical Induction

Definition 1. An axiom is a statement that is regarded as being established and always serves as a premise for further arguments. For instance, we are able to define \( \mathbb{N} \) with Peano axioms:

1. The number 1 \( \in \mathbb{N} \);
2. For all \( n \in \mathbb{N} \), there exists a unique successor \( n^+ \in \mathbb{N} \);
3. For all \( n \in \mathbb{N} \), \( n^+ \neq 1 \), i.e., 1 never serves as a successor;
4. If \( m^+ = n^+ \), then \( m = n \), i.e., every natural number is the unique successor of one and only one number except for 1;
5. Let \( M \subseteq \mathbb{N} \). If \( M \) satisfies
   (a) \( 1 \in M \) and
   (b) \( \forall m \in M, \exists! m^+ \in M \),
then \( M = \mathbb{N} \).

Remark 1. The successor \( n^+ \) is usually denoted by \( n + 1 \). We write \( 1^+ = 1 + 1 = 2 \), \( 2^+ = 3 \), \( 3^+ = 4 \), etc.

Remark 2. Axiom 5 is termed the principle of mathematical induction. If
(a) a statement \( S \) is true for \( n = 1 \); (base case)
(b) \( S \) is valid for some \( k \in \mathbb{N} \) (inductive hypothesis) implies that \( S \) holds for \( k + 1 \), (step case)
then \( S \) is true for all \( n \in \mathbb{N} \).

Proposition 1. Let us use mathematical induction to show that for all \( n \in \mathbb{N} \),

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.
\]

Proof. The base case is obviously true:

\[
\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}.
\]

For the step case, let us assume that for some \( k \in \mathbb{N} \),

\[
\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.
\]

Now, we add \( k + 1 \) to both sides of the equality:

\[
\sum_{i=1}^{k+1} i = (k + 1) + \sum_{i=1}^{k} i = (k + 1) + \frac{k(k + 1)}{2} = \frac{(k + 1)((k + 1) + 1)}{2}.
\]

Thus, we have shown that the step case is also true. Hence, the statement is true for all \( n \in \mathbb{N} \) by the principle of mathematical induction. \( \square \)

Remark 3. Principle of mathematical induction still applies for statements whose base cases start from some natural number \( n_0 > 1 \). This is due to the fact that there exist bijective functions from \( \{n_0, n_0 + 1, n_0 + 2, \ldots\} \) to \( \mathbb{N} \).
CHAPTER 1. ALGEBRAIC STRUCTURES

2 Groups

2.1 Definition of a Group

Definition 1. The Cartesian product of two sets $A$ and $B$, denoted by $A \times B$, is the set of all possible ordered pairs, where the first components of the ordered pairs are from $A$, and the second from $B$. We write $A \times B = \{(a, b); a \in A \land b \in B\}$.

Definition 2. Let $G$ be a non-empty set. Let $\varphi : G \times G \to G$, a mapping that relates every ordered pair $(a, b) \in G \times G$ to an element of $G$, denoted by $a \vartriangleright b$. If $\vartriangleright$ possesses the following properties: $\forall a, b, c \in G$,

i. $a \vartriangleright b \in G$; (closure)

ii. $(a \vartriangleright b) \vartriangleright c = a \vartriangleright (b \vartriangleright c) \equiv a \vartriangleright b \vartriangleright c$; (associative laws)

iii. $\exists e \in G : e \vartriangleright a = a$; (existence of a neutral element)

iv. $\exists a^{-1} \in G : a^{-1} \vartriangleright a = e$, (existence of inverses)

we will call $G$ a group under the operation $\vartriangleright$ and write $(G, \vartriangleright)$. What is more, if $\vartriangleright$ has another property:

v. $a \vartriangleright b = b \vartriangleright a$, (commutative laws)

we will call $G$ an abelian group and $\vartriangleright$ a commutative operation.

Remark 1. “$\vartriangleright$” is used here to denote the mapping, i.e., $\varphi((a, b)) \equiv a \vartriangleright b$. This should not be confused with the logical operator “or”, which connects two statements.

Remark 2. The notations $a \vartriangleright b \equiv a \cdot b \equiv a * b \equiv ab$ and $e \equiv 1$ are more commonly used when the groups we consider are groups under multiplication, or simply multiplication.

Remark 3. For abelian groups, “$\vartriangleright$” is often written as “$+$” and termed addition. In this case, $e$ is denoted by 0, and $a^{-1}$ by $-a$. One usually writes $a - a = 0$ instead of $a + (-a) = 0$. We name groups with such an operation groups under addition, or simply addition.

Example 1.  
(1) $(\mathbb{Z}, +)$ with the ordinary addition is an abelian group whose neutral element is the number 0.
(2) $(\mathbb{R}, +)$ is another abelian group.

Proposition 1. $\mathbb{R} \setminus \{0\}$ is an abelian group under ordinary multiplication. Its neutral element is the number 1.

Proof. To complete the proof, we will simply check that the group satisfies all the five properties: $\forall a, b, c \in \mathbb{R} \setminus \{0\}$,

(i) $ab \in \mathbb{R} \setminus \{0\}$;

(ii) $(ab)c = a(bc)$;

(iii) $1a = a$;

(iv) $\exists a^{-1} = \frac{1}{a} \in \mathbb{R} \setminus \{0\}: a^{-1}a = \frac{1}{a}a = 1$;

(v) $ab = ba$.\qed
CHAPTER 1. ALGEBRAIC STRUCTURES

2.2 Operation Rules

**Proposition 2.** Let \((G, \lor)\) be an arbitrary group. For all \(a \in G\), \(a^{-1} \lor a = a \lor a^{-1} = e\), and \((a^{-1})^{-1} = a\).

What is more, the neutral element \(e\) is unique and satisfies \(e \lor a = a \lor e = a\).

*Proof.* We know that \(\forall a \in G, \exists a^{-1} \in G : a^{-1} \lor a = e\). We would like to show that \(a \lor a^{-1} = e\). This equality can be established in the following way. Let us first write \(a^{-1} = e \lor a^{-1} = (a^{-1} \lor a) \lor a^{-1} = a^{-1} \lor (a \lor a^{-1})\). It is true that \(\forall a^{-1} \in G, \exists (a^{-1})^{-1} \in G : (a^{-1})^{-1} \lor a^{-1} = e\). This implies that \(e = (a^{-1})^{-1} \lor a^{-1} = (a^{-1})^{-1} \lor (a^{-1} \lor (a \lor a^{-1})) = ((a^{-1})^{-1} \lor a^{-1}) \lor (a \lor a^{-1}) = e \lor (a \lor a^{-1}) = a \lor a^{-1}\). It directly follows that \((a^{-1})^{-1} = a\), since \(a \lor a^{-1} = (a^{-1})^{-1} \lor a^{-1} = e\).

Now, it becomes trivial to show that \(e \lor a = a \lor e\), since \(e \lor a = (a \lor a^{-1}) \lor a = a \lor (a^{-1} \lor a) = a \lor e\). To show uniqueness of the neutral element, let us assume that there exist two neutral elements \(e_1\) and \(e_2\). However, they must be identical, since \(e_1 = e_2 \lor e_1 = e_1 \lor e_2 = e_2\). □

**Example 2.** The set \(\{a, e\}\) with an operation \(\lor\) defined by \(e \lor e = e\), \(e \lor a = a \lor e = a\), and \(a \lor a = e\) forms an abelian group.

**Remark 4.** For finite groups, the operation scheme can be represented by a table. For instance, for the group referred in **Example 2**, we have

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<th>(e)</th>
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<td>(e)</td>
<td>(a)</td>
</tr>
<tr>
<td>(e)</td>
<td>(a)</td>
<td>(e)</td>
</tr>
</tbody>
</table>

2.2 Operation Rules

**Proposition 1.** Let \((G, \lor)\) be a group. For all \(a, b \in G\), \((a \lor b)^{-1} = b^{-1} \lor a^{-1}\).

*Proof.* We know that \(a \lor b \in G\). To complete the proof, we simply write \((a \lor b)^{-1} \lor (a \lor b) = e = b^{-1} \lor b = b^{-1} \lor (e \lor b) = b^{-1} \lor ((a^{-1} \lor a) \lor b) = (b^{-1} \lor a^{-1}) \lor (a \lor b)\). □

**Definition 1.**

(a) Let \((G, \lor)\) be a multiplicative group. We write the element that is composite of \(n \in \mathbb{N}\) elements in \(G\) as

\[
a_1 \lor a_2 \lor \cdots \lor a_{n-1} \lor a_n \equiv a_1 a_2 \cdots a_{n-1} a_n =: \prod_{\alpha=1}^{n} a_{\alpha}
\]

and define recursively

\[
\prod_{\alpha=1}^{n+1} a_{\alpha} := \left( \prod_{\alpha=1}^{n} a_{\alpha} \right) a_{n+1}.
\]

We call the element the **product of factors** \(a_1, a_2, \ldots, a_{n-1}, a_n\).

(b) The product of \(n\) identical factors

\[
\prod_{\alpha=1}^{n} a := a^n
\]

is termed the **\(n\)-th power** of \(a\).
Proposition 2. Let \((G, \vee)\) be a multiplicative group. We have
\[
\left( \prod_{\alpha=1}^{m} a_{\alpha} \right) \left( \prod_{\beta=1}^{n} a_{m+\beta} \right) = \prod_{\rho=1}^{m+n} a_{\rho}.
\]

Proof. We are going to complete the proof by applying mathematical induction. First, we will check that for \(n = 1\), the statement is true. It is obvious that
\[
\left( \prod_{\alpha=1}^{m} a_{\alpha} \right) \left( \prod_{\beta=1}^{1} a_{m+\beta} \right) \equiv \left( \prod_{\alpha=1}^{m} a_{\alpha} \right) a_{m+1} = \prod_{\rho=1}^{m+1} a_{\rho}.
\]

Then, supposing that the statement holds for some \(k \in \mathbb{N}\), we want to show that it is still valid for \(n = k + 1\). For \(n = k\), we have
\[
\left( \prod_{\alpha=1}^{m} a_{\alpha} \right) \left( \prod_{\beta=1}^{k} a_{m+\beta} \right) = \prod_{\rho=1}^{m+k} a_{\rho}.
\]

Now, we multiply both sides of the equation by \(a_{m+k+1}\). The left-hand side of the equation becomes
\[
\left( \prod_{\alpha=1}^{m} a_{\alpha} \right) \left( \prod_{\beta=1}^{k} a_{m+\beta} \right) a_{m+k+1} = \left( \prod_{\alpha=1}^{m} a_{\alpha} \right) \left( \prod_{\beta=1}^{k} a_{m+\beta} \right) a_{m+k+1} = \left( \prod_{\alpha=1}^{m} a_{\alpha} \right) \left( \prod_{\beta=1}^{k+1} a_{m+\beta} \right).
\]

The right-hand side of the equation becomes
\[
\prod_{\rho=1}^{m+k} a_{\rho} a_{m+k+1} = \prod_{\rho=1}^{m+k+1} a_{\rho}.
\]

Thus, we have shown that the statement is true for \(n = k + 1\):
\[
\left( \prod_{\alpha=1}^{m} a_{\alpha} \right) \left( \prod_{\beta=1}^{k+1} a_{m+\beta} \right) = \prod_{\rho=1}^{m+k+1} a_{\rho}.
\]

Hence, the statement is true for all \(n \in \mathbb{N}\) by the principle of mathematical induction.

Corollary 1. Let \((G, \vee)\) be a multiplicative group and \(a \in G\) be an arbitrary element in the group. We have
(a) \(a^{m} a^{n} = a^{m+n}\);
(b) \((a^{m})^{n} = a^{mn}\).

Proof. See Problem 8.

Definition 2. The zeroth power of \(a\) is defined by \(a^{0} := e\), and the negative powers of \(a\) by \(a^{-n} := (a^{-1})^{n}\).

Remark 1. The latter definition complies with Corollary 1, (b).
Remark 2. For additive groups, we write
\[ a_1 \lor a_2 \lor \cdots \lor a_{n-1} \lor a_n \equiv a_1 + a_2 + \cdots + a_{n-1} + a_n =: \sum_{\alpha=1}^{n} a_\alpha \]
and name the composite element the \textit{sum} of the \(a_\alpha\)'s. A sum of identical elements is a \textit{multiple} of that element:
\[ \sum_{\alpha=1}^{n} a =: \text{na}. \]

Proposition 2 and its corollaries still hold with \(\prod\) replaced by \(\sum\):
\[ \left( \sum_{\alpha=1}^{m} a_\alpha \right) + \left( \sum_{\beta=1}^{k} a_{m+\beta} \right) = \sum_{\rho=1}^{m+k} a_\rho; \]
\[ ma + na = (m + n)a, \]
and
\[ mna = nma. \]
The proofs are left as exercises to the reader (See Problem 9).

2.3 Permutations

**Definition 1.** Let \(M\) be a finite set and \(P : M \to M\) be a bijective mapping. We call \(P\) a \textit{permutation} of \(M\).

**Remark 1.** If \(M\) is finite with \(n \in \mathbb{N}\) elements, then \(M\) and the set \(\{1, 2, \ldots, n-1, n\} \equiv \{i\}_{i=1}^{n}\) share the same cardinality. We are able to characterize every permutation \(P\) of \(M\) with its action on \(\{i\}_{i=1}^{n}\):
\[ E = \begin{pmatrix} 1, 2, 3, \ldots, n \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1, 2, 3, \ldots, n \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1, 2, 3, \ldots, n \end{pmatrix}, \quad \text{etc.} \]

**Definition 2.** If it takes an even number of (adjacent) pairwise exchanges to convert a permutation \(P\) into \(E\), then we say that \(P\) is an \textit{even permutation} and write \(\text{sgn} P = 1\). Otherwise, if it takes an odd number of exchanges, then we say that \(P\) is an \textit{odd permutation} and write \(\text{sgn} P = -1\).

**Example 1.** For permutations listed in Remark 1, we have \(\text{sgn} E = 1, \text{sgn} P_1 = -1, \text{and} \text{sgn} P_2 = -1\). The permutation
\[ \begin{pmatrix} 1, 2, 3, \ldots, n \end{pmatrix} \]
\[ \begin{pmatrix} 3, 1, 2, \ldots, n \end{pmatrix} \]
is even.
Proposition 1. All the permutations of a finite set $M$ with $n \in \mathbb{N}$ elements form a group under composition $\circ$ termed the symmetric group $S_n$.

Proof. To complete the proof, we need to check that the set of all the permutations satisfies the four group axioms: $\forall P_1, P_2 \in S_n$,

(i) $(P_1 : M \to M) \land (P_2 : M \to M) \implies P_1 \circ P_2 : M \to M$;

(ii) associative laws are satisfied because of Proposition 1 in 1.2;

(iii) $E = (1, 2, 3, \ldots, n)$ serves as the neutral element;

(iv) existence of inverses is due to the fact that bijective mappings have inverses.

Remark 2. In general, $S_n$ is not abelian. See Problem 10.

2.4 Subgroups

Definition 1. Let $(G, \lor)$ be a group and $H \neq \emptyset \subset G$. If $H$ is also a group under $\lor$, we call it a subgroup of $G$.

Example 1. Let $e$ be the neutral element of a group $G$. $\{e\}$ is a subgroup of $G$.

Theorem 1. $H$ is a subgroup of $(G, \lor)$ if and only if for all $a, b \in H$, $a \lor b^{-1} \in H$.

Proof. First, it is trivial that $H$ is a subgroup implies that for all $a, b \in H$, $a \lor b^{-1} \in H$. It is more instructive to complete the proof by contrapositive. That is, we would like to show the statement that there exists some $a, b \in H$ such that $a \lor b^{-1} \notin H$ implies that $H$ is not a subgroup. Suppose that such $a$ and $b$ exist. We know that if $H$ is a subgroup, then $a \lor b^{-1} \in H$. It directly follows that $H$ cannot be a subgroup. Proof by contrapositive can be very useful in some cases (although the reader might find that in the current example, it seems somewhat unnecessary).

Second, we want to show that for all $a, b \in H$, $a \lor b^{-1} \in H$ implies that $H$ is a subgroup. Suppose that for all $a, b \in H$, $a \lor b^{-1} \in H$. We need to check that $H$ satisfies the four group axioms. Notice that if we choose two identical elements $x = y$, then $e = x \lor x^{-1} = x \lor y^{-1} \in H$. We have thus established that the neutral element is contained in $H$. Now, if we choose the neutral element as one of the two elements in our assumption (let $a = e$), we will have $e \lor b^{-1} = b^{-1} \in H$, for all $b \in H$; that is, existence of inverses is satisfied. What is more, combining existence of inverses and the assumption, we have $\forall b \in H, \exists! b^{-1} \in H : \forall a \in H, a \lor b = a \lor (b^{-1})^{-1} \in H$; in other words, we have $\forall a, b \in H, a \lor b \in H$, the closure. At last but not least, the fact that $(G, \lor)$ is a group ensures that the operation $\lor$ is associative.
Example 2. Let us consider the following two elements of $S_3$, 

$$E = \begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1, 2, 3 \\ 1, 3, 2 \end{pmatrix}.$$ 

We want to apply Theorem 1 to check whether or not $g = \{E, P\}$ is a subgroup of $S_3$ (under composition $\circ$). First, notice that $E^{-1} = E$, and $P^{-1} = P$, since $E \circ E = E$, and $P \circ P = E$. It is straightforward to check the following:

- $E \circ E^{-1} = E \in g$;
- $E \circ P^{-1} = E \circ P = P \in g$;
- $P \circ E^{-1} = P \circ E = P \in g$;
- $P \circ P^{-1} = E \in g$.

Hence, $g = \{E, P\}$ is a subgroup of $S_3$ by Theorem 1.

2.5 Isomorphisms and Automorphisms

Definition 1.

(a) Let $(G, \lor)$ and $(H, \ast)$ be groups. Let $\varphi : G \to H$ be a bijective mapping such that for all $a, b \in G$, $\varphi(a \lor b) = \varphi(a) \ast \varphi(b)$. Such a $\varphi$ is termed an isomorphism between $G$ and $H$. We say that $G$ is isomorphic to $H$ and write $G \cong H$.

(b) Furthermore, if $G = H$, then we call $\varphi$ an automorphism on $G$. That is, an isomorphism between a group and itself is an automorphism on the group.

Remark 1. One says that $\varphi$ respects the operations.

Example 1. Let us consider a particular set of real $2 \times 2$ matrices,

$$G = \left\{ \text{real } 2 \times 2 \text{ matrices } g_\alpha \equiv \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} : \alpha \in [0, 2\pi] \right\},$$

and the set $H = \{\text{complex numbers } h_\beta \equiv e^{i\beta}; \beta \in [0, 2\pi]\}$. It is easy to show that $G$ is a group under matrix multiplication (denoted by $\cdot$), and $H$ is a group under multiplication of complex numbers (denoted by $\ast$). The proofs are left to the reader. Let us define a mapping $\varphi : G \to H$ by the relation $\varphi(g_\alpha) := h_\alpha$. It is obvious that $\varphi$ is bijective (by definition). We would like to check that $\varphi$ is an isomorphism between $G$ and $H$. First, notice that for all $g_\alpha, g_\beta \in G$,

$$g_\alpha \cdot g_\beta = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \cdot \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha + \beta) & \sin(\alpha + \beta) \\ -\sin(\alpha + \beta) & \cos(\alpha + \beta) \end{pmatrix} = g_{\alpha + \beta}.$$ 

Accordingly, we have $\varphi(g_\alpha \cdot g_\beta) = \varphi(g_{\alpha + \beta}) = h_{\alpha + \beta} = e^{i\alpha + \beta}$. Hence, we have shown that $G \cong H$.

Remark 2. People often call $G$ the special orthogonal group or rotation group of degree two, denoted $SO(2)$. $H$ is often called the unitary group of degree one, denoted $U(1)$. 


3 Fields

3.1 Bilinear Mappings

**Definition 1.** Let $A, B, C$ be additive groups with neutral elements $0_A, 0_B, 0_C$, respectively. Let $\varphi : A \times B \to C$ be a mapping defined by the relation $\varphi((a, b)) \equiv \varphi(a, b) \equiv a \cdot b \in C$. If $\varphi$ satisfies distributive laws: $\forall a_1, a_2, a_3 \in A \land b_1, b_2, b_3 \in B$,

i. $$(a_1 + a_2) \cdot b_1 = a_1 \cdot b_1 + a_2 \cdot b_1;$$

ii. $$a_1 \cdot (b_1 + b_2) = a_1 \cdot b_1 + a_1 \cdot b_2,$$

then we call $\varphi$ a **bilinear mapping**.

**Remark 1.** In i, the $+$ on the left-hand side (LHS) of the equality is the addition on $A$. In ii, the $+$ on the LHS is the addition on $B$. In both i and ii, the $+$ on the right-hand side (RHS) is the addition on $C$.

**Remark 2.** Usually called multiplication, $\cdot$ is referred to as an exterior operation, since it connects elements from two different groups. On the other hand, $+$’s are interior operations because of the closure.

**Proposition 1.** Consider $A, B, C$ and $\varphi$ in **Definition 1**. The following statements are true: $\forall a \in A \land b \in B$,

1. $0_A \cdot b = a \cdot 0_B = 0_C$;
2. $-a \cdot b = a \cdot (-b) = -(a \cdot b)$;
3. $-a \cdot (-b) = a \cdot b$.

**Proof.** For this proof, we will just write symbolic expressions.

1. $0_A = 0_A + 0_A \implies 0_A \cdot b = (0_A + 0_A) \cdot b = 0_A \cdot b + 0_A \cdot b$
   $0_C = 0_A \cdot b + (-0_A \cdot b) \equiv 0_A \cdot b - 0_A \cdot b$
   $\therefore 0_C = 0_A \cdot b + 0_A \cdot b - 0_A \cdot b = 0_A \cdot b$

2. $0_B = 0_B + 0_B \implies a \cdot 0_B = a \cdot (0_B + 0_B) = a \cdot 0_B + a \cdot 0_B$
   $0_C = a \cdot 0_B - a \cdot 0_B$
   $\therefore 0_C = a \cdot 0_B + a \cdot 0_B - a \cdot 0_B = a \cdot 0_B$

3. $0_C = 0_A \cdot b = (a + (-a)) \cdot b = a \cdot b + (-a) \cdot b \implies -a \cdot b = -(a \cdot b)$
   $0_C = a \cdot 0_B = a \cdot (b + (-b)) = a \cdot b + a \cdot (-b) \implies a \cdot (-b) = -(a \cdot b)$
   $0_C = 0_C \implies -a \cdot b = a \cdot (-b) = -(a \cdot b)$

3.2 Fields

**Definition 1.** Let $(K, +)$ be an additive group with neutral element $0$. Let $\cdot : K \times K \to K$ be an associative bilinear multiplication. If $K \setminus \{0\}$ is a group under $\cdot$, then we call $K$ a **field**.

**Example 1.** $\mathbb{R}$ under ordinary addition and multiplication is a commutative field. So is $\mathbb{Q}$. $\mathbb{Z}$ is not a field, since not every element has an inverse.
3.3 The Field of Complex Numbers

**Theorem 1.** We are able to construct a commutative field \( \mathbb{C} \) termed the field of complex numbers with the following properties:

1. \( \mathbb{R} \subset \mathbb{C} \);
2. \( \exists ! i \in \mathbb{C} : i^2 = -1 \);
3. \( \mathbb{C} = \{ z = z_1 + iz_2 : z_1, z_2 \in \mathbb{R} \} \), i.e., every element \( z \in \mathbb{C} \) can be uniquely written as \( z = z_1 + iz_2 \equiv z' + iz'' \) for some \( z_1, z_2 \in \mathbb{R} \).

**Remark 1.** \( z_1 (z') \) and \( z_2 (z'') \) are called the real and imaginary parts of a complex number \( z \), respectively. Note that they are both real numbers. We name \( z' - iz'' =: z^* \equiv \bar{z} \) the complex conjugate of \( z = z' + iz'' \).

**Proof.** Let us consider the Cartesian product \( \mathbb{R}^2 \equiv \mathbb{R} \times \mathbb{R} \). We would like to first establish that \( \mathbb{R}^2 \) is a field under certain addition and multiplication; thereafter, to complete the proof, we simply show that \( \mathbb{C} \cong \mathbb{R}^2 \).

First, let us turn \( \mathbb{R}^2 \) into an additive group by defining a proper addition: \( \forall a, b \in \mathbb{R}^2, a + b := (a_1 + b_1, a_2 + b_2) \). It is easy to check that \( (\mathbb{R}^2, +) \) is a group with neutral element \((0, 0)\). Second, we need to define a proper multiplication on \( \mathbb{R}^2 \): \( \forall a, b \in \mathbb{R}^2, a \cdot b \equiv ab := (a_1b_1 - a_2b_2, a_1b_2 + a_2b_1) \). Notice that the multiplication is both commutative and distributive: \( \forall a, b, c \in \mathbb{R}^2, ab = (a_1b_1 - a_2b_2, a_1b_2 + a_2b_1) = (b_1a_1 - b_2a_2, b_1a_2 + b_2a_1) = ba; \)

\[
(b + c)a = a(b + c) = (a_1(b_1 + c_1), a_2(b_2 + c_2)) = (a_1(b_1 + c_1) = a_2(b_2 + c_2)) \]

\[
= (a_1b_1 + a_2b_2, a_1c_2 + a_2c_1) = (a_1b_1 - a_2b_2, a_1c_2 + a_2c_1) = ab + ac.
\]

Similarly, one can also check that associative laws are satisfied: \( \forall a, b, c \in \mathbb{R}^2, \)

\[
a(bc) = a(b_1c_1 - b_2c_2, b_1c_2 + b_2c_1) = \frac{a_1}{a_1^2 + a_2^2} \cdot \frac{a_2}{a_1^2 + a_2^2}\]

\[
= \frac{a_1^2 + a_2^2}{a_1^2 + a_2^2}, \frac{a_1a_2 + a_2a_1}{a_1^2 + a_2^2} = (a_1b_1 - a_2b_2, a_1c_2 + a_2c_1) = (a_1b_1 - a_2b_2, a_1c_2 + a_2c_1) = (ab)c.
\]

We have thus shown that \( \cdot \) is an associative bilinear multiplication. Now, we want to show that \( (\mathbb{R}^2 \setminus \{(0,0)\}, \cdot) \) is a group. The fact that \( \mathbb{R}^2 \) under ordinary addition and multiplication is a field ensures that the closure is satisfied. The multiplicative identity is \((1,0)\), because \( \forall a \in \mathbb{R}^2, a \cdot (1,0) = (1,0) \cdot a = (a_1, a_2) = a \). We have already proven that \( \cdot \) is associative. To show that existence of (multiplicative) inverses holds, let \( a \neq (0,0) \). It follows that \( a_1^2 + a_2^2 \neq 0 \). Notice that \( \forall a \in \mathbb{R}^2, \)

\[
a^{-1} = \frac{a_1}{a_1^2 + a_2^2} - \frac{a_2}{a_1^2 + a_2^2} \]

Hence, we have shown that \( \mathbb{R}^2 \) is a field under designated addition and multiplication.

Notice the curious fact that \((0,1)^2 = (0,1) \cdot (0,1) = (-1,0)\) is very similar to \( i^2 = -1 \). Now, we can define \( \mathbb{C} \) by means of an isomorphism \( \varphi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C} \) given by the relations: \( \forall z_1, z_2 \in \mathbb{R}, \)

\[
\varphi((0,1)) = i \in \mathbb{C};
\]

\[
\varphi((z_1, z_2)) = z_1 + iz_2 \in \mathbb{C}.
\]
Remark 2. The isomorphism can be graphically represented by the complex plane.

![Fig. 3.3.1. The complex plane.](image)

**Proposition 1.** The set of complex numbers \( \{ z = e^{i\alpha}; \alpha \in [0, 2\pi] \} \) forms a circle centered at the origin \( 0 + i0 \) with radius 1 in the complex plane. **Euler’s formula** reads

\[
e^{i\alpha} = \cos \alpha + i \sin \alpha.
\]

**Proof.** Recall the Maclaurin series of \( e^x \): for \( |x| < \infty \),

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.
\]

It directly follows that

\[
e^{i\alpha} = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!}
\]

\[
= \sum_{m=0}^{\infty} \frac{(i\alpha)^{2m}}{(2m)!} + \sum_{n=0}^{\infty} \frac{(i\alpha)^{2n+1}}{(2n+1)!}
\]

\[
= \sum_{m=0}^{\infty} \frac{(-1)^m \alpha^{2m}}{(2m)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!}
\]

\[
= \cos \alpha + i \sin \alpha.
\]

We also know that for each \( \alpha \in [0, 2\pi] \), \( e^{i\alpha} = z_1 + iz_2 \) with some \( z_1, z_2 \in \mathbb{R} \). Accordingly, we have \( z_1^2 + z_2^2 = \cos^2 \alpha + \sin^2 \alpha = 1 \), which describes a circle centered at the origin with radius 1 in the complex plane. \( \square \)

**Corollary 1.** Let \( z \in \mathbb{C} \). There exist some \( r \in [0, \infty) \) and \( \phi \in [-\pi, \pi) \) such that \( z = re^{i\phi} \).

**Proof.** From **Proposition 1** it directly follows that \( z = z_1 + iz_2 = r \cos \phi + ir \sin \phi = re^{i\phi} \) with \( r = \sqrt{z_1^2 + z_2^2} \) and \( \phi = \arctan \left( \frac{z_2}{z_1} \right) \). \( \square \)

**Remark 3.** \( r \) is termed the **modulus** or **absolute value** of \( z \) and denoted by \( r = |z| \). \( \phi \) is called the **argument** of \( z \); one writes \( \phi = \arg z \).
Remark 4. \( \phi \in [-\pi, \pi) \) is merely a particular choice. In general, \( \phi \) can be defined on any interval of length \( 2\pi \).

Remark 5. Let \( z = re^{i\phi} \in \mathbb{C} \) for some \( r \in [0, \infty) \) and \( \phi \in \mathbb{R} \) mod \( 2\pi \). Notice that \( \forall n \in \mathbb{Z}, e^{i2n\pi} = 1 \implies z = re^{i\phi} = re^{i(\phi+2n\pi)} \). That is, a complex number has multiple arguments.

**Definition 1.** Let \( z = re^{i\phi} \in \mathbb{C} \) for some \( r \in [0, \infty) \) and \( \phi \in \mathbb{R} \) mod \( 2\pi \). Real powers of \( z \) are defined by \( z^x := r^xe^{ix\phi} \) for all \( x \in \mathbb{R} \).

Remark 6. The definition is consistent with Corollary 1, (b) in 2.2, since \( z^x = (re^{i\phi})^x = r^xe^{ix\phi} \). Note the difference that the corollary only holds for \( n \in \mathbb{N} \).

Remark 7. For \( x \notin \mathbb{N} \), \( z^x \) is not unique. In particular, when \( x = \frac{1}{n} \) (\( n \in \mathbb{N} \)), \( z^x \) has \( n \) different values called \( n \)-th roots of \( z \).

**Example 1.** Let us compute second roots of \( i \). Let us first write \( i = e^{i\frac{\pi}{2}} = e^{i\left(\frac{\pi}{2} + 2\pi k\right)} \). The second roots are \((i^{\frac{1}{2}})_0 = e^{i\frac{\pi}{4}}\) and \((i^{\frac{1}{2}})_1 = e^{i\frac{1}{2}\left(\frac{\pi}{2} + 2\pi\right)} = e^{i\frac{5\pi}{4}}\).

![Fig. 3.3.2. The second roots of i in the complex plane.](image)
4 Vector Spaces and Tensor Spaces

4.1 Vector Spaces

**Definition 1.** Let \((V, +)\) be an additive group. Let \(K\) be a field. We define an *exterior multiplication* \(\varphi : K \times V \to V\) that possesses the following properties:

(i) bilinearity,

(ii) associativity that \(\forall \lambda, \mu \in K \land x \in V, (\lambda \mu) x = \lambda (\mu x)\) and

(iii) \(\forall x \in X, 1_K x = x\), where \(1_K\) is the multiplicative identity of \(K\).

We then call \(V\) a *vector space* over \(K\) or \(K\)-*vector space*. The term *linear space* over \(K\) is sometimes also used.

**Remark 1.** For the sake of simplicity, we assume that \(K\) is commutative, i.e., \(\forall \lambda, \mu \in K, \lambda \mu = \mu \lambda\).

**Remark 2.** Elements of \(V\) are termed *vectors*, and elements of \(K\) *scalars*.

**Example 1.** Four common \(\mathbb{R}\)-vector spaces are tabulated below.

<table>
<thead>
<tr>
<th>No.</th>
<th>(V)</th>
<th>The Addition on (V)</th>
<th>(K)</th>
<th>Operations on (K)</th>
<th>The Exterior Multiplication</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>(\mathbb{R})</td>
<td>the ordinary +</td>
<td>(\mathbb{R})</td>
<td>the ordinary + and ·</td>
<td>the ordinary ·</td>
</tr>
<tr>
<td>#2</td>
<td>(\mathbb{C})</td>
<td>(\forall z_1, z_2 \in \mathbb{C}, z_1 + z_2 := (z_1 + z_2') + i(z_1'' + z_2''))</td>
<td>(\mathbb{R})</td>
<td>the ordinary + and ·</td>
<td>(\forall \lambda \in \mathbb{R} \land z \in \mathbb{C}, \lambda \cdot z \equiv \lambda z := \lambda z' + i\lambda z'').</td>
</tr>
<tr>
<td>#3</td>
<td>(\mathbb{R}^2)</td>
<td>the + defined in <strong>Theorem 1 of 3.3</strong></td>
<td>(\mathbb{R})</td>
<td>the ordinary + and ·</td>
<td>(\forall \lambda \in \mathbb{R} \land (x, y) \in \mathbb{R}^2, \lambda (x, y) := (\lambda x, \lambda y)).</td>
</tr>
<tr>
<td>#4</td>
<td>(\mathbb{R}^n)</td>
<td>a componentwise + similar to the one on (\mathbb{R}^2)</td>
<td>(\mathbb{R})</td>
<td>the ordinary + and ·</td>
<td>(\forall \lambda \in \mathbb{R} \land \lambda \equiv (x_1, \ldots, x_n) \in \mathbb{R}^n, \lambda x := (\lambda x_1, \ldots, \lambda x_n)).</td>
</tr>
</tbody>
</table>

In general, we are able to construct a vector space from any arbitrary field. Let \(K\) be a field. Let us first define a proper addition on \(K^n\) to turn it into an additive group: \(\forall k \equiv (k_1, \ldots, k_n), l \equiv (l_1, \ldots, l_n) \in K^n, k + l := (k_1 + l_1, \ldots, k_n + l_n)\). It is easy to check that \((K^n, +)\) is a group with neutral element \((0_{K_1}, \ldots, 0_{K_n})\), where \(0_{K}\) is the additive identity of \(K\). \(K^n\) will be further promoted to a \(K\)-vector space, if we define the exterior multiplication by \(\forall k \in K \land l \in K^n, kl := (kl_1, \ldots, kl_n)\).

4.2 Basis Sets

**Definition 1.** Let \(V\) be a \(K\)-vector space. If there exist a finite number of vectors \(p_1, p_2, \ldots, p_n \in V\) such that \(\forall x \in V, \exists \lambda_1, \lambda_2, \ldots, \lambda_n \in K : x = \sum_{i=1}^{n} \lambda_i p_i\), then we say that the set \(\{p_i\}_{i=1}^{n}\) spans \(V\), which is a *finite-dimensional* vector space.

**Example 1.** Let us consider \(\mathbb{R}^2\) as an \(\mathbb{R}\)-vector space. The set \(\{(1, 0), (0, 2)\}\) spans \(\mathbb{R}^2\); so does \(\{(1, 0), (0, 1), (1, 1)\}\).
Definition 2. Let \( \{p_i\}_{i=1}^n \) span a \( K \)-vector space, \( V \). We say that the vectors \( p_1, p_2, \ldots, p_n \) are \textit{linearly dependent}, if one of them can be written as a linear combination of the others. Otherwise, we say that these vectors are \textit{linearly independent}.

Example 2. In \( \mathbb{R}^2 \), \((1,0), (0,1) \) and \((1,1)\) are linearly dependent.

Definition 3. A \textit{basis} is a set of linearly independent vectors that spans a vector space. We call these vectors \textit{basis vectors} and denote them with \( e_1, e_2, \ldots, e_n \). If there are \( n \) basis vectors in a basis, then we say that the corresponding vector space is \textit{n-dimensional}.

Example 3. \( \{(1,0), (0,2)\}, \{(1,0), (0,1)\}, \{(1,0), (1,1)\} \) and \( \{(0,1), (1,1)\} \) are all bases of \( \mathbb{R}^2 \).

Proposition 1. Let \( V \) be a \( K \)-vector space with neutral element \( \vartheta \). Let \( p_1, p_2, \ldots, p_n \) be linearly independent vectors. We then have
\[
\sum_{i=1}^n \lambda_i p_i = \vartheta \implies \forall i \in \{i\}_{i=1}^n, \lambda_i = 0_K,
\]
where \( 0_K \) is the additive identity of \( K \).

Proof. Let us complete the proof by contradiction. Let us assume that there exists a \( \lambda_c \neq 0_K \) such that \( \sum_{i=1}^n \lambda_i p_i = \vartheta \) still holds. It directly follows that
\[
p_c = -\frac{1}{\lambda_c} \sum_{i \neq c} \lambda_i p_i = \sum_{i \neq c} \left( -\frac{\lambda_i}{\lambda_c} \right) p_i.
\]
Hence, we arrive at a contradiction, since \( p_1, p_2, \ldots, p_n \) are linearly independent. \( \square \)

Proposition 2. Let \( V \) be a \( K \)-vector space with neutral element \( \vartheta \). Let \( \{e_i\}_{i=1}^n \) be a basis of \( V \). Any arbitrary vector \( x \in V \) can be written as
\[
x = \sum_{i=1}^n \lambda_i e_i,
\]
where \( \{\lambda_i\}_{i=1}^n \subseteq K \) is a unique set of scalars.

Remark 1. We refer to the formula
\[
x = \sum_{i=1}^n \lambda_i e_i
\]
as expanding \( x \) in the basis \( \{e_i\}_{i=1}^n \). We say that the set of scalars \( \{\lambda_i\}_{i=1}^n \) is a \textit{representation} of \( x \).
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Proof. The fact that \( \{e_i\}_{i=1}^{n} \) spans \( V \) implies that there exist \( \lambda_1, \lambda_2, \ldots, \lambda_n \in K \) such that for all \( x \in V \),
\[
x = \sum_{i=1}^{n} \lambda_i e_i.
\]
Let us now show that \( \{\lambda_i\}_{i=1}^{n} \) is unique. Suppose that \( x \) can also be written as
\[
x = \sum_{i=1}^{n} \alpha_i e_i,
\]
where \( \alpha_1, \alpha_2, \ldots, \alpha_n \in K \). It directly follows that
\[
\vartheta = x - x = \sum_{i=1}^{n} \lambda_i e_i - \sum_{j=1}^{n} \alpha_j e_j = \sum_{i=1}^{n} (\lambda_i - \alpha_i) e_i.
\]
Proposition 1 further implies that \( \forall i, \lambda_i = \alpha_i \).

Remark 2. We often use the notation \( \lambda_i \equiv x^i \) and call \( \lambda_i \)'s the components or coordinates of the vector \( x \) in the basis \( \{e_i\}_{i=1}^{n} \). We write
\[
x = \sum_{i=1}^{n} \lambda_i e_i \equiv \sum_{i=1}^{n} x^i e_i \equiv x^i e_i.
\]
Thereinto, the latter is called the Einstein summation convention.

Example 4. \( \{e_1 = (1,0,0,\ldots,0), e_2 = (0,1,0,\ldots,0), \ldots, e_n = (0,0,0,\ldots,1)\} \) is termed the standard basis of \( \mathbb{R}^n \).

Remark 3. Proposition 2 indicates that there is a one-to-one correspondence between any vector \( x \in V \) and the \( n \)-tuple of its components. One can further show that all \( n \)-dimensional \( K \)-vector spaces are isomorphic to \( K^n \).

4.3 Tensor Spaces

Definition 1. Let \( V \) be a \( K \)-vector space.
(a) A mapping \( f : V \to K \) is called a linear form if \( \forall x, y \in V \wedge \lambda \in K \),
(i) \( f(x + y) = f(x) + f(y) \);
(ii) \( f(\lambda x) = \lambda f(x) \).
(b) A mapping \( g : V \times V \to K \) is termed a bilinear form if \( \forall x, y, z \in V \wedge \lambda \in K \),
(i) \( g(x + y, z) = g(x, z) + g(y, z) \);
(ii) \( g(x, y + z) = g(x, y) + g(x, z) \);
(iii) \( g(\lambda x, y) = \lambda g(x, y) = g(x, \lambda y) \).

Remark 1. Note the relation between bilinear forms and bilinear mappings (See 3.1).

Definition 2. Let \( V \) be a \( K \)-vector space. Let \( \{e_i\}_{i=1}^{n} \) be a basis of \( V \). Let \( f : V \times V \to K \) be a bilinear form. The scalars \( t_{ij} := f(e_i, e_j) \) are called the components or coordinates of \( f \) in the basis \( \{e_i\}_{i=1}^{n} \).
Proposition 1. A bilinear form can be completely characterized by its components.

Proof. Let $V$ be a $K$-vector space. Let $\{e_i\}_{i=1}^n$ be a basis of $V$. Let $f : V \times V \to K$ be a bilinear form. Let $x, y \in V$. We have

$$x = \sum_{i=1}^n x^i e_i \equiv x^i e_i, \quad \text{and} \quad y = \sum_{j=1}^n y^j e_j \equiv y^j e_j.$$ 

It follows that

$$f(x, y) = f \left( \sum_{i=1}^n x^i e_i, y \right) \equiv f(x^i, y)$$

$$= \sum_{i=1}^n x^i f(e_i, y) \equiv x^i f(e_i, y)$$

$$= \sum_{i=1}^n \sum_{j=1}^n x^i y^j f(e_i, e_j) \equiv x^i y^j f(e_i, e_j) \equiv x^i y^j t_{ij}.$$ 

Hence, after we obtain all the components $t_{ij}$, the bilinear form is completely determined, because we are able to compute $f(x, y)$ for any arbitrary $x$ and $y$. The reader ought to appreciate the Einstein summation convention from now on, unless he or she really relishes $\sum$. For his convenience, the slothful typist will employ the Einstein summation convention in the rest of this note. □

Definition 3. Let $\{e_i\}_{i=1}^n$ be a basis of a $K$-vector space $V$. Let $f : V \times V \to K$ be a bilinear form. The scalars $t_{ij} = f(e_i, e_j)$ are also called the components or coordinates of a rank-2 tensor $t$ in the basis $\{e_i\}_{i=1}^n$. A rank-2 tensor is equivalent to a bilinear form. In general, a high-rank tensor is corresponding to a multilinear form.

Theorem 1. Let $K$ be a field. The set of all rank-2 tensors is an $n^2$-dimensional $K$-vector space.

Proof. See Problem 13. □

Example 1. Let us consider $\mathbb{R}^3$ with the standard basis $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$. Recall that $\mathbb{R}^3$ is an $\mathbb{R}$-vector space. The well-known Levi-Civita tensor or completely antisymmetric tensor of rank 3 is the tensor corresponding to the trilinear form $\varepsilon : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ with components $\varepsilon(e_i, e_j, e_k) = \varepsilon_{ijk}$, where the Levi-Civita symbol $\varepsilon_{ijk}$ is given by

$$\varepsilon_{ijk} = \text{sgn} \binom{i, j, k}{1, 2, 3}.$$ 

Remark 2. Notice that in Definition 3, we singled out the phrase “in the basis $\{e_i\}_{i=1}^n$”. We would like to accentuate the fact that components of a tensor depend on the basis chosen. For instance, components of the Levi-Civita tensor in an arbitrary basis $\{\tilde{e}_i\}_{i=1}^n$ are generally not given by the Levi-Civita symbol, i.e., $\varepsilon(\tilde{e}_i, \tilde{e}_j, \tilde{e}_k) \neq \varepsilon(e_i, e_j, e_k) = \varepsilon_{ijk}$. Also, for now, note that a symbol is merely a token, i.e., the Levi-Civita symbol is conveniently introduced to express components of the Levi-Civita tensor in the standard basis.
Example 2. Now, let us consider $\mathbb{R}^n$ with the standard basis $\{e_i\}_{i=1}^n$. The rank-2 tensor corresponding to the bilinear form $\delta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with components

$$\delta(e_i, e_j) = \delta_{ij} := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise}, \end{cases} \quad (4.3.1)$$

is termed (Euclidean) Kronecker delta.

4.4 Dual Spaces

Let $V$ be an $n$-dimensional $K$-vector space. Let $\{e_i\}_{i=1}^n$ be a basis of $V$. Let $f : V \to K$ be a linear form. For all $x \in V$, we have $x = x^i e_i$, where $x^i \in K$. It directly follows that

$$f(x) = f(x^i e_i) = x^i f(e_i) = x^i u_i,$$

where $u_i := f(e_i) \in K$.

Remark 1. Like bilinear forms, every linear form can also be completely characterized by $u_i$ (See Proposition 1 of 4.3). Moreover, the set of all linear forms is a $K$-vector space, denoted $V^*$, that is isomorphic to $K^n$ and hence $V$ (See Theorem 1 of 4.3 and Remark 3 of 4.2). Because of these similarities, we can say that a linear form is equivalent to a rank-1 tensor.

Definition 1. Let $V$ be an $n$-dimensional $K$-vector space. The set of all linear forms from $V$ to $K$ is also a $K$-vector space, termed the dual vector space to $V$; we often use $V^*$ to denote the dual vector space. Elements of $V^*$ are called covectors. There exists a one-to-one correspondence between covectors and vectors; the latter are elements of $V$.

Remark 2. As elements of the dual vector space, covectors can be regarded as rank-1 tensors.

Definition 2. The scalar $f(x)$ is termed the dot product or scalar product of the vector $x$ and covector $u$ corresponding to the linear form $f$. We write $x \cdot u := x^i u_i$.

Remark 3. Covectors are also named covariant vectors, in which case vectors are referred to as contravariant vectors.

Fig. 4.4.1. A summary of various isomorphisms.
Remark 4. Because $V^* \cong V$, there is no need to distinguish between them. We can define the \textbf{covariant components} of a vector $y$ to be the components of its corresponding covector $u$ under isomorphism, i.e., $y_i := u_i$. Components of $y$ itself, $y^i$ are called its \textbf{contravariant components}. As a result, we can write the scalar product as $x \cdot u = x^i y_i$. We will revisit these concepts in 4.8.

Remark 5. Now, we are able to expand a vector $x$ in two different ways: $x^i e_i = x = x^i e_i^t$. The set of covectors $\{e^i\}_{i=1}^n$ is a basis of $V^*$ that is corresponding to $\{e_i\}_{i=1}^n$. We call it a \textbf{cobasis}.

Remark 6. We can further define the scalar product

$$e^i \cdot e_j = \delta^i_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise}. \end{cases} \quad (4.4.1)$$

Notice the subtle difference between Eqs. (4.3.1) and (4.4.1): the latter holds for any arbitrary basis and its corresponding cobasis, while the former is only valid for the standard basis.

Remark 7. In quantum mechanics, we often use $|x\rangle$ and $\langle y|$ to denote vectors and covectors, respectively. The scalar product is written as $\langle y| x\rangle = y_i x^i$.

\textbf{Definition 3.}

(a) A bilinear form $f : V^* \times V^* \to K$ is equivalent to a \textbf{contravariant tensor of rank 2} whose components are given by $t^{ij} := f(e_i, e_j)$. In general, a multilinear form from a Cartesian product of $V^*$’s to $K$ is corresponding to a high-rank contravariant tensor.

(b) As its name suggests, a \textbf{mixed tensor of rank 2} is corresponding to a bilinear form $f : V \times V^* \to K$ (or $f : V^* \times V \to K$). A high-rank mixed tensor can be defined in a similar way, e.g., $f : V^* \times V \times V^* \to K$ is equivalent to a rank-3 mixed tensor with components $t^{ij}_k := f(e^i, e_j, e^k)$.

\textbf{Example 1.} $\delta^i_j$ in Remark 6 is a mixed tensor of rank 2.

Remark 8. Like covectors, vectors can be regarded as rank-1 contravariant tensors.

Remark 9. Like vectors, covectors also have both contravariant and covariant components.

Remark 10. The covariant components of a cobasis vector $e^i$ are given by $\delta^i_j$, since $\delta^i_j e^j = e^i = (e^i)_j e^j$. Similarly, $\delta_i^j$ captures the contravariant components of a basis vector $e_i$, because $\delta_i^j e_j = e_i = (e_i)_j e_j$. We have thus established $\delta^i_j = \delta_i^j$. Because there is no need to distinguish between them, we can just employ the symbol $\delta^i_j$ instead.

\textbf{Definition 4.} Let $x$ and $y$ be two contravariant vectors. The \textbf{tensor product} of $x$ and $y$ yields a contravariant tensor whose components equal the product of the two vectors’ components, i.e., $t^{ij} := x^i y^j$. We write $t = x \otimes y$.

Remark 11. Although $V \cong V^*$, we do not yet know the isomorphism explicitly. Nevertheless, Euclidean space is an exception: in this space, there is no need to distinguish between being contravariant and covariant.

Remark 12. We have frequently used “contravariant” and “covariant” in this section. The reader probably gets a little bit bewildered. In fact, these two terms only appear in mainly two aspects, namely categories of tensors and components of vectors. We often characterize a tensor by its components: if a tensor’s components are only labelled by upper (lower) indices, then the tensor is a contravariant (covariant) tensor. The same rule applies to vectors, since they can be regarded as rank-1 tensors. However, as we previously discussed, vectors have both contravariant and covariant components. This indicates that a vector can be both contravariant and covariant.
4.5 Metric Spaces

**Definition 1.** Let $M$ be a set. Let $\rho : M \times M \to \mathbb{R}$ be a mapping. If $\rho$ possesses the following properties:

1. $\forall x, y, z \in M, \quad \rho(x, y) \geq 0 \land (\rho(x, y) = 0 \iff x = y);$ (positive semidefiniteness)
2. $\rho(x, y) = \rho(y, x);$ (symmetry)
3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z),$ (triangle inequality)

we will call $M$ a **metric space** with the **metric** $\rho$.

**Remark 1.** A set with a designated metric defines a metric space.

**Example 1.** Let $M = \mathbb{R}$. We can define a metric on $M$: $\forall x, y \in M,$

$$\rho(x, y) = |x - y| := \begin{cases} x - y, & \text{if } x \geq y, \\ y - x, & \text{otherwise}. \end{cases}$$

The reader ought to verify that such a $\rho$ satisfies the three properties in **Definition 1** (See Problem 17).

**Definition 2.** Let $M$ be a metric space with metric $\rho$. Let $(a_n)_{n \in \mathbb{N}} \subseteq M$ be an infinite sequence. We say that $L \in M$ is the **limit** of the sequence (or the sequence **converges** to $L$), if $\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n > N, \rho(a_n, L) < \epsilon$. We write $\lim_{n \to \infty} a_n = L, a_n \to L$ or $\lim_{n \to \infty} \rho(a_n, L) = 0$.

**Proposition 1.** A sequence has at most one limit.

**Proof.** See Problem 18.

**Definition 3.** Let $M$ be a metric space with metric $\rho$. An infinite sequence $(a_n)_{n \in \mathbb{N}} \subseteq M$ is called a **Cauchy sequence** if it satisfies the **Cauchy condition**:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall m, n > N, \rho(a_m, a_n) < \epsilon.$$ 

**Remark 2.** For Cauchy sequences, we write $\lim_{m, n \to \infty} \rho(a_m, a_n) = 0$, or simply $\rho(a_m, a_n) \to 0$. 
**Proposition 2.** Every convergent sequence is a Cauchy sequence.

*Proof.* See Problem 18.

**Remark 3.** The converse of Proposition 2 is not true in a general metric space.

**Example 2.** Let $M = \mathbb{Q}$ with the metric defined in Example 1 and $a_n = (1 + \frac{1}{n})^n$. $a_n$ is divergent, since $\lim_{n \to \infty} a_n = e \notin \mathbb{Q}$.

**Proposition 3.** Let $M = \mathbb{R}$ with the metric defined in Example 1. Every Cauchy sequence in $M$ is convergent.

*Sketch of Proof.*

(i) Firstly, we need to show that every Cauchy sequence is bounded. This follows from the Cauchy condition.

(ii) Secondly, we establish that a Cauchy sequence is convergent if and only if it has a convergent subsequence.

(iii) Finally, we employ *Bolzano-Weierstrass theorem* that every bounded sequence in $\mathbb{R}$ has a convergent subsequence to complete the proof.

**Definition 4.** A metric space $M$ is called *complete* if every Cauchy sequence in $M$ converges.

**Remark 4.** In Example 2, we saw that $\mathbb{Q}$ is not complete.

**Proposition 4.** An incomplete metric space $M$ can be completed by adding an appropriate set. The completion of $M$ is unique up to isomorphism.

We omit the proof, because it is beyond the level of this course.

**Example 3.** $\mathbb{R}$ is a completion of $\mathbb{Q}$.

### 4.6 Banach Spaces

**Definition 1.** Let $B$ be an $\mathbb{R}$-vector space (or a $\mathbb{C}$-vector space) with null vector $\vartheta$. Let $\|\cdot\| : B \to \mathbb{R}$ be a mapping. If $\|\cdot\|$ possesses the following properties: $\forall x, y \in B \land a \in \mathbb{R}$ (or $\mathbb{C}$),

1. $\|x\| \geq 0 \land (\|x\| = 0 \iff x = \vartheta)$; (positive semidefiniteness)
2. $\|x + y\| \leq \|x\| + \|y\|$; (triangle inequality)
3. $\|ax\| = |a| \|x\|$. (linearity)

we will call $\|\cdot\|$ a *norm* on $B$.

The image of a vector $x \in B$, $\|x\|$ is called its *norm*. The norm of the difference of two vectors $x, y \in B$, $\|x - y\| = d(x, y)$ is called the *distance* between $x$ and $y$.

**Remark 1.** $d : B \times B \to \mathbb{R}$ defines a metric on $B$. The reader ought to verify that $d$ satisfies the three properties in **Definition 1** of 4.5. There exist other metrics.
Remark 2. Let \( x \in B \). Notice that \( \| x \| = \| x - \vartheta \| = d(x, \vartheta) \).

Remark 3. As we saw, a normed vector space is a metric space. However, for an arbitrary set, the existence of a metric does not imply that a norm also exists.

Remark 4. Every linear space over \( \mathbb{R} \) (or \( \mathbb{C} \)) with a norm is a metric space.

Definition 2. A linear space over \( \mathbb{R} \) (or \( \mathbb{C} \)) with a norm that is complete is called a **Banach space**, or simply **B-space**.

Example 1. As a vector space, \( \mathbb{R} \) with the norm given by \( \| x \| := |x| \) (\( x \in \mathbb{R} \)) is a Banach space. Similarly, \( \mathbb{C} \) with \( \| z \| := |z| = \sqrt{z_1^2 + z_2^2} \) (\( z = z_1 + iz_2 \in \mathbb{C} \)) is also a B-space.

Definition 3. Let \( B \) be a Banach space over \( \mathbb{C} \). Let \( \ell : B \to \mathbb{C} \) be a linear form (See Definition 1, (a) of 4.3). Let \( x \in B \). The norm of \( \ell \) is defined as

\[
\| \ell \| := \sup_{\| x \| = 1} \{|\ell(x)|\}.
\]

Remark 5. The vector space of linear forms on \( B \), \( B^* \) is the dual vector space to \( B \) (See Definition 1 of 4.4).

Proposition 1. The norm of linear forms defines a norm on \( B^* \).

Proof. See Problem 19.

Theorem 1. \( B^* \) is complete and hence a B-space.

We omit the proof, because it is beyond the level of this course. The interested reader might want to consult *A Course of Higher Mathematics, Vol. 5* by V. I. Smirnov.

4.7 Hilbert Spaces

Definition 1. Let \( H \) be a linear space over \( \mathbb{C} \) with null vector \( \vartheta \). Let \( (\cdot, \cdot) : H \times H \to \mathbb{C} \) be a mapping that possesses the following properties: \( \forall x, y, z \in H \land \lambda \in \mathbb{C} \),

(i) \((x, y) = (y, x)^*\); (symmetry)

(ii) \((x, x) \geq 0 \land ((x, x) = 0 \iff x = \vartheta)\); (positive semidefiniteness)

(iii) \((x + y, z) = (x, z) + (y, z)\);

(iv) \((\lambda x, y) = \lambda^* (x, y)\).

The norm of a vector \( x \in H \) is defined by \( \| x \| := \sqrt{(x, x)} \).

Remark 1. The mapping \((\cdot, \cdot)\) is usually called a **scalar product** on \( H \).

Remark 2. Note the subtle difference between the scalar product and bilinear forms (See Definition 1, (b) of 4.3).
Lemma 1. For \( x, y \in H \),
\[ |(x, y)|^2 \leq (x, x)(y, y). \]
The inequality is named **Cauchy-Schwarz inequality** (also known as **Bunyakovsky inequality**).

**Proof.** Obviously, the inequality holds when either \( x \) or \( y \) is the null vector. Let \( x, y \neq \vartheta \). As a result, \( (x, x), (y, y) > 0 \). Let us define
\[ z := x - \frac{(y, x)}{(y, y)} y. \]

Notice that
\[ (z, y) = \left( x - \frac{(y, x)}{(y, y)} y, y \right) = (x, y) - \frac{(y, x)^*}{(y, y)^*} (y, y) = (x, y) - (y, x)^* = 0. \]
Now, let us compute \( (x, x) \):
\[ (x, x) = \left( z + \frac{(y, x)}{(y, y)} y, z + \frac{(y, x)}{(y, y)} y \right) \]
\[ = (z, z) + \left( \frac{(y, x)}{(y, y)} z, \frac{(y, x)}{(y, y)} z \right) + \left( \frac{(y, x)}{(y, y)} y, \frac{(y, x)}{(y, y)} y \right) \]
\[ = (z, z) + \frac{(y, x)}{(y, y)} (z, y) + \frac{(y, x)^*}{(y, y)^*} (z, y)^* + \frac{|(y, x)|^2}{(y, y)^*} (y, y) \]
\[ = (z, z) + \frac{|(y, x)|^2}{(y, y)}. \]
Because of positive semidefiniteness, we have
\[ (x, x) \geq \frac{|(y, x)|^2}{(y, y)}, \]
which implies the Cauchy-Schwarz inequality.

Remark 3. A similar inequality exists for any scalar product defined on an arbitrary linear space. \((\cdot, \cdot)\) on \( H \) is a particular instance.

**Proposition 1.** The norm in **Definition 1** is indeed a norm (See **Definition 1 of 4.6**).

**Proof.** See Problem 20.

**Definition 2.** Let \( x, y \in H \). A metric on \( H \) can be defined by
\[ \rho(x, y) := \|x - y\| = \sqrt{(x - y, x - y)}. \]

**Proposition 2.** The metric in **Definition 2** satisfies the three properties in **Definition 1 of 4.5**.

The proof is straightforward and thus left as an exercise to the reader.
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Definition 3. A complete $H$ is called a **Hilbert space**, or simply an **H-space**.

Remark 4. Every H-space is a B-space.

Definition 4. Let $y \in H$ be given. We can define a linear form $\ell : H \to \mathbb{C}$ by $\ell(x) := (y, x)$ for all $x \in H$.

Proposition 3. The linear form in Definition 4 is indeed a linear form (See Definition 1 of 4.3).

Proof. See Problem 20.

Proposition 4. Every linear form on $H$ can be expressed in the form of $\ell(x)$, i.e., $\forall \ell : H \to \mathbb{C}, \exists y \in H : \forall x \in H, \ell(x) = (y, x)$.

We omit the proof, because it is beyond the level of this course.

Corollary 1. Like B-spaces, the vector space of linear forms on $H$, $H^*$ is the dual vector space to $H$ (See Definition 1 of 4.4). $H^*$ is isomorphic to $H$. What is more, $H^*$ itself is an H-space.

We omit the proof, because it is beyond the level of this course.

Definition 5. Let $\ell \in H^*$ with the corresponding vector $y \in H$. We can define a mapping $\langle \cdot | \cdot \rangle : H^* \times H \to \mathbb{C}$ by $\langle \ell | x \rangle := \ell(x) = (y, x)$ for all $x \in H$.

Remark 5. For each $\ell \in H^*$, there exists a unique $y \in H$ such that $\langle \ell | x \rangle = \ell(x) = (y, x)$.

Remark 6. Because $H^* \cong H$, there is no need to distinguish between them. We sloppily write $(y | x) := \langle \ell | x \rangle = (y, x)$. Note that $(y)$ is a linear functional, which takes a vector and returns a complex number.

Remark 7. In quantum mechanics, states of a system are represented by elements of a Hilbert space.

4.8 Generalized Metrics and Minkowski Spaces

4.8.1 Scalar Products

Definition 1. Let $V$ be an $n$-dimensional $\mathbb{R}$-vector space. Let $\{e_i\}_{i=1}^n$ be a basis. Let $g : V \times V \to \mathbb{R}$ be a symmetric bilinear form, i.e., $\forall x, y \in V, g(x, y) = g(y, x)$. $g$ corresponds to a symmetric rank-2 tensor whose components are given by $g_{ij} = g(e_i, e_j) = g(e_j, e_i) = g_{ji}$. Let $g$ have an inverse $g^{-1}$ with components $(g^{-1})_{ij} = g^{ij}$, where $g_{ij}g^{jk} = \delta^k_i$.

Let $x, y \in V$. We call the real number $g(x, y) \equiv x \cdot y \equiv xy = x^ig_{ij}y^j$ the **generalized scalar product** of $x$ and $y$. $g$ is termed the **generalized metric**, or equivalently the **metric tensor**.

Remark 1. The metric in Definition 1 is not the same as the one defined in 4.5. For instance, positive semidefiniteness can be violated, i.e., $\exists x, y \in V : g(x, y) < 0$; it is even possible that $g(x, x) < 0$.

Remark 2. Recall that the $\mathbb{R}$-vector space $V$ is isomorphic to $\mathbb{R}^n$ (See Remark 3 of 4.2). Therefore, in the rest of this section, we will just consider $\mathbb{R}^n$ with a metric $g$ instead of a general $\mathbb{R}$-vector space.
Definition 2. An adjoint basis (or a cobasis) \( \{e^i\}_{i=1}^n \) is a set of cobasis vectors \( e^i := g^{ij}e_j \).

Remark 3. Such defined \( e^i \)'s are vectors in \( V \), while cobasis vectors in \( 4.4 \) are elements of \( V^* \). However, because \( V \cong V^* \), we can obscure the difference here by defining cobasis vectors in \( V \).

Remark 4. The relation between \( e_i \) and \( e^j \) is given by
\[
e^i = \delta^k_i e_k = (g_{ij}g^{jk})e_k = g_{ij}(g^{jk}e_k) = g_{ij}e^j.
\]

Definition 3. Let \( x \in V \) be given. Coordinates of \( x \) in a basis \( \{e_i\}_{i=1}^n \), \( x_i \) are called contravariant. Coordinates of \( x \) in the cobasis \( \{e^i\}_{i=1}^n \), \( x^i \) are called covariant.

\[\begin{array}{c}
\text{contravariant components of } x \\
x = x^i e^i & \longrightarrow & \text{basis vectors (contravariant)} \\
= x^i e^i & \longrightarrow & \text{cobasis vectors (covariant)} \\
\end{array}\]

Fig. 4.4.2. Contravariant and covariant components of a vector \( x \).

Remark 5. So far, all the definitions in this section are consistent with the ones in \( 4.4 \). However, we have now specified a relation between bases and cobases (See Remark 11 of \( 4.4 \)).

Proposition 1. Let \( x \in V \). The contravariant and covariant components of \( x \) are related by
\[
x_i = g_{ij}x^j, \quad \text{and} \quad x^i = g^{ij}x_j.
\]

Proof. For the first equality, we have
\[
x_i e^i = x = x^i e_i = x^j (g_{ji} e^i) = (x^j g_{ji}) e^i = (g_{ij}x^j) e^i,
\]
which implies \( x_i = g_{ij}x^j \). For the second,
\[
x^i = \delta^k_i x^k = (g^{ij} g_{jk}) x^k = g^{ij} (g_{jk} x^k) = g^{ij} x_j.
\]

Corollary 1. Let \( x, y \in V \). The scalar product of \( x \) and \( y \) can now be written as
\[
g(x, y) = x^i g_{ij} y^j = \begin{cases} x^i g_{ij} y^j = x_j y^j \\ x^i (g_{ij} y^j) = x^i y_j. \end{cases}
\]

Remark 6. The form of the scalar product in Corollary 1 is consistent with Remark 4 of \( 4.4 \).

Remark 7. According to Eq. (4.4.1),
\[
g(e^i, e_j) = g^i_j = e^i \cdot e_j = \delta^i_j \neq \delta_{ij} = g_{ik} \delta^k_j = g_{ij}.
\]
In Euclidean space, \( \delta^i_j = \delta_{ij} \). We will revisit this concept in \( 4.8.3 \).
4.8.2 Basis Transformations

Definition 4. A real \( m \times n \) matrix \( D \) is a rectangular array of real numbers in \( m \) rows and \( n \) columns:

\[
D = \begin{pmatrix}
  D_{11} & D_{12} & D_{13} & \cdots & D_{1n} \\
  D_{21} & D_{22} & D_{23} & \cdots & D_{2n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  D_{m1} & D_{m2} & D_{m3} & \cdots & D_{mn}
\end{pmatrix}.
\]

The \( D_{ij} \)'s are called matrix elements. In this course, we mainly consider real square matrices, i.e., real matrices with an equal number of rows and columns.

(i) A square matrix \( D \) is invertible if there exists another square matrix \( D^{-1} \) such that \( D_{ij}(D^{-1})_{jk} = (D^{-1})_{ij}D_{jk} = \delta^i_k \). We also write \( DD^{-1} = D^{-1}D = I_n \), where \( I_n \) is the \( n \times n \) identity matrix

\[
\begin{pmatrix}
  1 & 0 & 0 & \cdots & 0 \\
  0 & 1 & 0 & \cdots & 0 \\
  0 & 0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & 1
\end{pmatrix} \equiv \text{diag}\{1,1,\ldots,1\}. \quad \text{n 1's}
\]

(ii) The transpose of an \( m \times n \) matrix \( D \), \( D^T \) is an \( n \times m \) matrix with matrix elements \((D^T)^i_j = D_j^i\).

(iii) The product of an \( l \times m \) matrix \( A \) and an \( m \times n \) matrix \( B \) is an \( l \times n \) matrix with matrix elements \((AB)^i_j := A_i^kB^k_j\).

(iv) The determinant of an \( n \times n \) square matrix \( D \) is a number defined by

\[
\det D \equiv \begin{vmatrix}
  D_{11} & D_{12} & D_{13} & \cdots & D_{1n} \\
  D_{21} & D_{22} & D_{23} & \cdots & D_{2n} \\
  D_{31} & D_{32} & D_{33} & \cdots & D_{3n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  D_{n1} & D_{n2} & D_{n3} & \cdots & D_{nn}
\end{vmatrix} = \sum_{\pi \in S_n} \left( \text{sgn } \pi \prod_{i=1}^{n} D_{\pi(i)}^i \right),
\]

where \( S_n \) is the symmetric group (See Proposition 1 of 2.3).

Remark 8. Matrix elements are not like components of a tensor; we usually do not distinguish between upper and lower indices. For matrices, the distinction is only important when Einstein summation convention is involved. Therefore, for a matrix \( D \), we can usually write \( D_{ij} = D_{ij} = D^i_j = D^{ij} \). It is crucial to tell the difference between row and column indices though.

Example 1. Let us consider a general \( 2 \times 2 \) matrix \( D \) and compute its determinant:

\[
\det D = \begin{vmatrix}
  D_{11} & D_{12} \\
  D_{21} & D_{22}
\end{vmatrix} = \sum_{\pi \in S_2} \left( \text{sgn } \pi \prod_{i=1}^{2} D_{\pi(i)}^i \right)
\]

\[
= \text{sgn } (1,2) \prod_{i=1}^{2} D_{i(1,2)}^i + \text{sgn } (1,2) \prod_{j=1}^{2} D_{j(1,2)}^j
\]

\[
= 1 \cdot D_{11} \cdot D_{22} + (-1) \cdot D_{12} \cdot D_{21} = D_{11}D_{22} - D_{12}D_{21}.
\]
CHAPTER 1. ALGEBRAIC STRUCTURES

Proposition 2. Let $A$, $B$ and $D$ be matrices. We have

(i) $(AB)^T = B^T A^T$;
(ii) $(D^{-1})^T = (D^T)^{-1}$;
(iii) $\det(AB) = \det A \cdot \det B$;
(iv) $\det(D^{-1}) = \frac{1}{\det D}$;
(v) $\det(D^T) = \det D$.

Proof. For (i) and (ii), we will just write symbolic expressions.

Proof for (i) \[
(AB)^T = (AB)^T = (A^T B^T)^T = (B^T A^T)^T = B^T A^T.
\]

Proof for (ii) \[
D^{-1} (D^{-1})^T = (D^{-1} D)^T = I_n^T = I_n.
\]

For (iii), (iv) and (v), please consult A Course of Higher Mathematics, Vol. 3 by V. I. Smirnov.

Definition 5. Let us consider $\mathbb{R}^n$. Let $\{e_i\}_{i=1}^n$ be a basis. Let $D$ be an invertible $n \times n$ real matrix. We can define a second basis $\{\tilde{e}_i\}_{i=1}^n$ by the basis transformation $\tilde{e}_i := e_j (D^{-1})^i_j$.

Remark 9. The inverse basis transformation is given by

\[
e_i = \epsilon_k \delta^k_i = \epsilon_k \left( (D^{-1})^k_j D^i_j \right) = \left( \epsilon_k (D^{-1})^k_j \right) D^i_j = \tilde{e}_j D^i_j.
\]

Proposition 3. $\{\tilde{e}_i\}_{i=1}^n$ is indeed a basis (See Definition 3 of 4.2).

Proof. To show that $\{\tilde{e}_i\}_{i=1}^n$ is a basis, we need to establish that $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n$ are linearly independent and span the $\mathbb{R}$-vector space $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$. From Remark 9 it follows that

\[
x = x^j e_j = x^j (\tilde{e}_j D^i_j) = (D^i_j x^j) \tilde{e}_i := \tilde{x}^i \tilde{e}_i,
\]

i.e., $\{\tilde{e}_i\}_{i=1}^n$ spans $\mathbb{R}^n$. Now, let $\{\tilde{\lambda}^i\}_{i=1}^n \subset \mathbb{R}$. Let us consider the sum $S = \tilde{\lambda}^i \tilde{e}_i$. According to Definition 5, we have

\[
S = \tilde{\lambda}^i \tilde{e}_i = \tilde{\lambda}^i (D^{-1})^j_i \tilde{e}_j = \lambda^j e_j.
\]

The fact that $\{e_i\}_{i=1}^n$ is a basis implies that $S = 0 \Rightarrow \lambda^j = 0 (i \in \{i\}_{i=1}^n)$. Furthermore, because $D^{-1}$ is invertible, $\lambda^j = 0 (i \in \{i\}_{i=1}^n)$. We have thus shown that $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n$ are linearly independent.

Proposition 4. Let $x \in \mathbb{R}^n$ be a vector with components $x^i$ in the basis $\{e_i\}_{i=1}^n$. In the basis $\{\tilde{e}_i\}_{i=1}^n$, components of $x$ are given by $\tilde{x}^i = D^i_j x^j$.

Proof. See Eq. (4.8.1).

Remark 10. The inverse relation is $x^i = (D^{-1})^i_j \tilde{x}^j$.

Remark 11. Such a $D$ applied on components of a vector is termed a coordinate transformation.
Proposition 5. Let \( g_{ij} = e_i \cdot e_j \) be the metric associated with the basis \( \{ e_i \}_{i=1}^n \). Let \( D^{-1} \) be a basis transformation \( \tilde{e}_i = e_j (D^{-1})^j_i \). The metric that corresponds to the new basis \( \{ \tilde{e}_i \}_{i=1}^n \) is given by
\[
\tilde{g}_{ij} = ((D^{-1})^T)^k_i g_{k\ell} (D^{-1})^\ell_j \quad \text{(or } \tilde{g} = (D^{-1})^T g D^{-1})
\]
The inverse relation is \( g = D^T \tilde{g} D \).

Proof. For this proof, we will just write symbolic expressions.
\[
\tilde{g}_{ij} = \tilde{e}_i \cdot \tilde{e}_j = e_k (D^{-1})^k_i \cdot e_\ell (D^{-1})^\ell_j = ((D^{-1})^T)^k_i (e_k \cdot e_\ell) (D^{-1})^\ell_j = ((D^{-1})^T)^k_i g_{k\ell} (D^{-1})^\ell_j
\]
(2)
\[
\tilde{g} = (D^{-1})^T g D^{-1}
\]
\[
D^T \tilde{g} D = D^T (D^{-1})^T g D^{-1} D
\]
\[
g = D^T \tilde{g} D
\]
Proof. Corollary 2 ensures the existence of a transformation such that \( \tilde{g}_{ij} = \lambda_i \cdot \delta_{ij} \), where \( \delta_{ij} \) is the Euclidean Kronecker delta, and \( \lambda_i \neq 0 \ (i \in \{i\}_{i=1}^{n}) \). To complete the proof, we would like to first permute the order of basis vectors so that \( \lambda_1, \lambda_2, \ldots, \lambda_m > 0 \), and \( \lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_n < 0 \). Now, let us define a second transformation by the symmetric matrix
\[
(D^{-1})^i_j := \frac{1}{\sqrt{|\lambda_i|}} \delta^i_j.
\]
According to Proposition 5, the new metric is given by
\[
\tilde{g}_{ij} = ((D^{-1})^T)^k_i \tilde{g}_{kl} (D^{-1})^l_j
\]
\[
= \left( \frac{1}{\sqrt{|\lambda_i|}} \cdot \delta_k^i \right) (\lambda_k \cdot \delta_{kl}) \left( \frac{1}{\sqrt{|\lambda_i|}} \cdot \delta^l_j \right)
\]
\[
= \frac{\lambda_k}{\sqrt{|\lambda_i \lambda_r|}} \cdot \delta_i^k \delta_{kl} \delta^l_j = \frac{\lambda_i}{\sqrt{|\lambda_i \lambda_r|}} \cdot \delta_{il} \delta^l_j = \frac{\lambda_i}{\sqrt{|\lambda_i \lambda_j|}} \cdot \delta_{ij}
\]
\[
= \frac{\lambda_i}{|\lambda_i|} \cdot \delta_{ij} = \begin{cases} 
\delta_{ij}, & \text{if } i \leq m, \\
-\delta_{ij}, & \text{if } m < i \leq n,
\end{cases}
\]
as desired. \( \square \)

**Definition 6.** If the metric corresponding to a basis is of the form in Eq. (4.8.2), we will call the basis a **normal coordinate system**.

**Remark 12.** The integer \( m \) in Eq. (4.8.2) is characteristic of the vector space and remains invariant under basis transformations. This is an implication of **Sylvester’s rigidity theorem**.

**Example 2.** Let \( m = n \) in Eq. (4.8.2). We have \( g = I_n \). One can further check that \( g_{ij} = \delta_{ij} \) is compatible with **Definition 1 of 4.5**. The \( \mathbb{R} \)-vector space, \( \mathbb{R}^n \) with the metric \( I_n \) is named the **n-dimensional Euclidean space**, denoted \( E^n \). In \( E^n \), normal coordinate systems are called **Cartesian coordinate systems**. Notice that for any \( x \in E^n \),
\[
x^i = \delta_{ij} x^j = g_{ij} x^j = x_i,
\]
i.e., there is no need to distinguish between being contravariant and covariant in Euclidean space (See **Remark 11 of 4.4**).

**Example 3.** Now, let \( m = 1 \) and \( n \geq 2 \). In this case, we have \( g = \text{diag}\{1,-1,\ldots,-1\} \) that is a generalized metric. \( \mathbb{R}^n \) with the metric \( g \) is termed the **n-dimensional Minkowski space** \( M^n \). In this space, normal coordinate systems are called **inertial coordinate frames**. It is straightforward to show that for any \( x \in M^n \), its contravariant and covariant components are related by
\[
x^i = \begin{cases} 
x_i, & \text{if } i = 1, \\
-x_i, & \text{if } 1 < i \leq n.
\end{cases}
\]
Remark 13. One formulation of the special relativity is based on the postulate that classical mechanical systems can be described as collections of particles moving in the space $M^4$. Let $x \in M^4$. As physicists, we often use the notation $x = (x^0, x^1, x^2, x^3) := (ct, \mathbf{x})$, where $ct$ is the temporal component of the four-vector $x$, and $\mathbf{x}$ the spatial component. Thereinto, $c$ is a characteristic velocity, namely the speed of light in vacuum.

4.8.4 Normal Coordinate Transformations

**Definition 7.** A coordinate transformation $D$ is **normal** if it transforms one normal coordinate system into another. In other words, the metric of the form in Eq. (4.8.2) is invariant under a normal coordinate transformation, i.e.,

$$g_{ij} = \tilde{g}_{ij} = \left((D^{-1})^T\right)^k_i g_{k\ell} (D^{-1})^\ell_j \quad \text{(or } g = \tilde{g} = (D^{-1})^T g D^{-1})$$

which implies that $g = D^T g D$.

**Example 4.** Let $D$ be a normal coordinate transformation in the $n$-dimensional Euclidean space. Recall that in $E^n$, $g = \mathbb{1}_n$. According to **Definition 7**, $D$ must satisfy the relation

$$\mathbb{1}_n = D^T \mathbb{1}_n D = D^T D.$$  

We call such a $D$ **orthogonal**.

**Example 5.** In $M^n$, normal coordinate transformations are termed **Lorentz transformations**.

**Lemma 2.**

(i) The inverse of a normal coordinate transformation is also normal.

(ii) The product of two normal coordinate transformations is normal as well.

**Proof.** Let $g$ be a metric of the form in Eq. (4.8.2).

(i) See **Definition 7**.

(ii) Let $D_1$ and $D_2$ be normal coordinate transformations. By definition, we have

$$g = D_1^T g D_1, \quad \text{and } g = D_2^T g D_2.$$  

Combining the two equalities, we obtain

$$g = D_1^T (D_2^T g D_2) D_1 = (D_1^T D_2^T) g (D_2 D_1) = (D_2 D_1)^T g (D_2 D_1).$$

**Theorem 2.** All the normal coordinate transformations for a specific metric form a non-abelian group under matrix multiplication.

**Proof.** To complete the proof, we need to check that the set of all the normal coordinate transformations satisfies the four group axioms:

(i) closure is satisfied because of Lemma 2, (ii);
(ii) matrix multiplication is associative;
(iii) the identity matrix $1_n$ always serves as the multiplicative identity;
(iv) existence of inverses is due to Lemma 2, (i).

Remark 14. The group of all the normal coordinate transformations in $E^n$ is called the **orthogonal group**, denoted $O(n)$. In $M^n$, the group of all the Lorentz transformations is termed the **pseudo-orthogonal group**, denoted $O(1, n - 1)$.

**Proposition 6.** Let $g$ be a metric of the form in Eq. (4.8.2). Let $D$ be a normal coordinate transformation. We have $\det D = \pm 1$.

**Proof.** According to Definition 7, we have

$$g = D^T g D$$
$$\det g = \det (D^T g D) = \det (D^T) \cdot \det g \cdot \det D$$
$$1 = (\det D)^2$$
$$\det D = \pm 1.$$


5 Tensor Fields

5.1 Tensor Fields

**Definition 1.** Let us consider the \( \mathbb{R} \)-vector space \( \mathbb{R}^n \) with a generalized metric. Let \( D \) be a normal coordinate transformation. A **tensor field** is a mapping that assigns each \( x \in \mathbb{R}^n \) a rank-\( N \) tensor \( t^{i_1i_2\ldots i_N}(x) \), which transforms under \( D \) in the following way: \( \tilde{x} = Dx \), and

\[
\tilde{t}^{i_1i_2\ldots i_N}(\tilde{x}) = (D^{i_1}_{j_1}D^{i_2}_{j_2}\ldots D^{i_N}_{j_N}) t^{j_1j_2\ldots j_N}(x) = \left( \prod_{k=1}^{N} D^{i_k}_{j_k} \right) t^{j_1j_2\ldots j_N}(x).
\]

**Remark 1.** The field in **Definition 1** is not the same as the one defined in 3.2.

**Proposition 1.** Homogeneous tensor fields, i.e., \( \forall x \in \mathbb{R}^n, t^{i_1\ldots i_N}(x) = t^{i_1\ldots i_N} \), are consistent with **Definition 3** of 4.3.

**Proof.** Let \( f : \mathbb{R}^n \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \rightarrow \mathbb{R} \) be a multilinear form. Let \( \{e_i\}_{i=1}^n \) be a normal coordinate system. Let \( D^{-1} \) be a normal coordinate transformation \( \tilde{e}_i = e_j(D^{-1})^j_i \). For any \( x_1, \ldots, x_N \in \mathbb{R}^n \), we have

\[
f(x_1, \ldots, x_N) = f(\tilde{x}_1, \ldots, \tilde{x}_N) = \left( \tilde{x}_1 \right)_{i_1} \tilde{e}_{i_1} \cdots \left( \tilde{x}_N \right)_{i_N} \tilde{e}_{i_N} = \left( \tilde{x}_1 \right)_{i_1} \cdots \left( \tilde{x}_N \right)_{i_N} f(e_{i_1} \cdots e_{i_N}) = f(x_1, \ldots, x_N).
\]

Let \( g_{ij} \) be the metric corresponding to \( \{e_i\}_{i=1}^n \) and \( \{\tilde{e}_i\}_{i=1}^n \). Now, notice that

\[
x_j = g_{ji}x^i = g_{ji}(D^{-1})^j_k \tilde{x}^k = (gD^{-1})_{jk} \tilde{x}^k = (D^T g)_{jk} \tilde{x}^k = (D^T)_{jk} g_{ik} \tilde{x}^k = D^T_{jk} \tilde{e}_i,
\]

which further implies that

\[
(x_1)_{i_1} \cdots (x_N)_{i_N} t^{j_1\ldots j_N} = (\tilde{x}_1)_{i_1} \cdots (\tilde{x}_N)_{i_N} t^{j_1\ldots j_N}.
\]

By comparison, we conclude that

\[
\tilde{t}^{i_1\ldots i_N} = (D^T_{j_1} \cdots D^T_{j_N}) t^{j_1\ldots j_N},
\]

as desired. \( \square \)

**Remark 2.** **Proposition 1** implies that all tensors must transform in the same way as homogeneous tensor fields under a normal coordinate transformation. As physicists, we often define tensors by means of this transformation property without referring to multilinear forms.

**Remark 3.** In an \( n \)-dimensional vector space, a rank-\( N \) tensor can be regarded as a set of \( n^N \) scalars \( t^{i_1\ldots i_N} \) that are associated with a basis and possess the transformation property.

**Example 1.** A vector \( x \) is a rank-1 tensor, because for any arbitrary coordinate transformation \( D \),

\[
\tilde{x}^i = D^i_j x^j.
\]
**Example 2.** Metric tensors are indeed tensors, since for any coordinate transformation \(D\),
\[
\tilde{g}^{ij} = (\tilde{g}^{-1})_{ij} = (D g^{-1} D^T)_{ij} = \tilde{D}^i_k (g^{-1})_{kl} (D^T)_j^l = \tilde{D}^i_k D^j_\ell g^{k\ell}.
\]

**Remark 4.** Metric tensors of the form in Eq. (4.8.2) are special, since for any normal coordinate transformation, \(\tilde{g} = g\); nevertheless, they still possess the transformation property.

**Example 3.** Let us apply the criterion to check whether or not the Levi-Civita tensor is a tensor. Let \(\{e_i\}_{i=1}^n\) and \(\{e^i\}_{i=1}^n\) be a basis and its corresponding cobasis, respectively. Let \(D^{-1}\) be a coordinate transformation \(\tilde{e}_i = e_j (D^{-1})^j_i\). The relation between \(\tilde{e}^i\) and \(e^j\) is given by \(\tilde{e}^i = D^i_j e^j\), because for any \(x \in \mathbb{R}^n\),
\[
\tilde{x}_i \tilde{e}^i = x = x_j e^j = (D^j_i \tilde{x}_i) e^j = \tilde{x}_i (D^j_i e^j).
\]

Now, we are able to compute components of the Levi-Civita tensor in the new cobasis \(\{\tilde{e}^i\}_{i=1}^n\):
\[
(\tilde{\varepsilon})_{ijk} = \varepsilon(\tilde{e}^i, \tilde{e}^j, \tilde{e}^k) = \varepsilon(D^i_\ell e^\ell, D^j_m e^m, D^k_n e^n) = D^i_\ell D^j_m D^k_n \varepsilon(e^\ell, e^m, e^n) = D^i_\ell D^j_m D^k_n (\varepsilon_L)_{\ell mn},
\]
which indicates that the Levi-Civita tensor is indeed a tensor.

**Remark 5.** The Levi-Civita tensor is undoubtedly a tensor, since it corresponds to a trilinear form. On the other hand, we will later show that the Levi-Civita symbol is not a tensor.

**Definition 2.** Recall that the Levi-Civita symbol \(\varepsilon^{ijk}\) is given by
\[
\varepsilon^{ijk} = \text{sgn} \left( \frac{i,j,k}{1,2,3} \right).
\]

We assign \(\varepsilon^{ijk}\) to each normal coordinate system in \(\mathbb{R}^3\) so that \(\varepsilon^{ijk}\) is promoted to an entity that is invariant under normal coordinate transformations, i.e., \(\varepsilon^{ijk} = \varepsilon^{ijk}\).

\(\text{a}\) In fact, \(\varepsilon^{ijk} = -\varepsilon_{ijk} = -\text{sgn} \left( \frac{i,j,k}{1,2,3} \right)\) (See Example 1 of 4.3). However, we omit the difference here, because we are more interested in the transformation property of the Levi-Civita symbol.

**Remark 6.** We would like to once again accentuate the fact that components of the Levi-Civita tensor in an arbitrary cobasis \(\{e^i\}_{i=1}^n\) are generally not given by the Levi-Civita symbol, i.e., \(\varepsilon(e^i, e^j, e^k) = (\varepsilon_L)^{ijk} \neq \varepsilon^{ijk}\).

**Definition 3.** A rank-\(N\) pseudo-tensor \(t^{i_1, i_2, \ldots, i_N}\) transforms under a normal coordinate transformation \(D\) in the following way:
\[
\tilde{t}^{i_1, i_2, \ldots, i_N} = \det D \left( D^{i_1}_{j_1} D^{i_2}_{j_2} \cdots D^{i_N}_{j_N} \right) t^{j_1, j_2, \ldots, j_N} = \det D \left( \prod_{k=1}^N D^{i_k}_{j_k} \right) t^{j_1, j_2, \ldots, j_N}.
\]

**Lemma 1.** Let \(x_1, \ldots, x_N \in \mathbb{R}^n\). Let \(D\) be a normal coordinate transformation. For any antisymmetric multilinear form \(f\) on \(\mathbb{R}^n\), we have
\[
f(D^{i_1}_{j_1}(x_1)^{j_1}, \ldots, D^{i_N}_{j_N}(x_N)^{j_N}) = \det D \cdot f((x_1)^{i_1}, \ldots, (x_N)^{i_N}).
\]
Example 4. Now, we are about to show that the Levi-Civita symbol is a rank-3 pseudo-tensor. Let \( \{e^i\}_{i=1}^n \) be the standard cobasis. In \( \{e^i\}_{i=1}^n \), we have \( (\varepsilon_L)^{ijk} = \varepsilon(e^i, e^j, e^k) = \tilde{\varepsilon}^{ijk} \). Let \( D \) be a normal coordinate transformation \( \tilde{e}^i = D^i_j e^j \). According to Lemma 1, components of the Levi-Civita tensor in the new cobasis \( \{\tilde{e}^i\}_{i=1}^n \) are given by

\[
(\tilde{\varepsilon}_L)^{ijk} = \varepsilon(\tilde{e}^i, \tilde{e}^j, \tilde{e}^k) = \varepsilon(D^i_l e^l, D^j_m e^m, D^k_n e^n) = D^i_l D^j_m D^k_n \varepsilon(e^l, e^m, e^n) = D^i_l D^j_m D^k_n \delta^{\ell m n} = \det D \cdot \varepsilon(e^i, e^j, e^k) = \det D \cdot \varepsilon^{ijk}.
\]

We also have \( (\tilde{\varepsilon}_L)^{ijk} = \det D \cdot \varepsilon^{ijk} = \det D \cdot \varepsilon^{ijk} \), because \( \varepsilon^{ijk} \) is invariant under the normal coordinate transformation \( D \). Recall that \( D = \pm 1 = \frac{1}{\det D} \). Therefore,

\[
\varepsilon^{ijk} = \frac{1}{\det D} D^i_l D^j_m D^k_n \delta^{\ell m n} = \det D (D^i_l D^j_m D^k_n) \delta^{\ell m n},
\]

i.e., \( \varepsilon^{ijk} \) is a pseudo-tensor.

Remark 7. It is entertaining to show that \( D^i_l D^j_m D^k_n \delta^{\ell m n} = \det D \cdot \varepsilon^{ijk} \) in the following way:

\[
\begin{align*}
D^i_l D^j_m D^k_n \delta^{\ell m n} &= \sum_{k=1}^3 \sum_{m=1}^3 \sum_{n=1}^3 D^i_l D^j_m D^k_n \text{sgn} \left( \ell, m, n \right) \\
&= \sum_{(\ell, m, n) \in S_3} \left[ \text{sgn} \left( \ell, m, n \right) \delta^{\ell m n} \right] \\
&= \sum_{(\ell, m, n) \in S_3} \left[ \text{sgn} \left( \ell, m, n \right) \delta^{\ell m n} \right] \\
&= \det D \cdot \varepsilon^{ijk}.
\end{align*}
\]

5.2 Gradient, Curl and Divergence

The pedantic typist decides to employ upright boldface letters to represent vectors and matrices in the rest of this section. Components of a vector are denoted in the usual way. For example, \( x^i \equiv (x)^i \) are the contravariant components of a vector \( x \).

Definition 1. Let us consider the \( \mathbb{R} \)-vector space \( \mathbb{R}^n \) with a generalized metric. A **scalar field** is a mapping \( f : \mathbb{R}^n \to \mathbb{R} \) that assigns each \( x \in \mathbb{R}^n \) a scalar \( f(x) \in \mathbb{R} \). Likewise, a **vector field** \( \mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n \) assigns each \( x \) a vector \( \mathbf{v}(x) \in \mathbb{R}^n \). Let \( x \in \mathbb{R}^n \) be an arbitrary vector.

(i) The **gradient** of a scalar field \( f \), \( \nabla f \) is a vector field defined by

\[
(\nabla f)_i(x) := \frac{\partial f}{\partial x^i}(x) \equiv \partial_i f(x) \quad (i \in \{i\}_{i=1}^n).
\]

(ii) The **curl** of a vector field \( \mathbf{v} \), \( \nabla \times \mathbf{v} \) is a vector field defined by

\[
(\nabla \times \mathbf{v})^i(x) := \varepsilon^{ijk} \partial_j v_k(x) \quad (i \in \{i\}_{i=1}^n).
\]

(iii) The **divergence** of the vector field \( \mathbf{v} \), \( \nabla \cdot \mathbf{v} \) is a scalar field defined by

\[
(\nabla \cdot \mathbf{v})(x) = \partial_i v^i(x).
\]
CHAPTER 1. ALGEBRAIC STRUCTURES

Proposition 1.
For any \( x \in \mathbb{R}^n \),

(i) the gradient of a scalar field at \( x \) transforms in the same way as a covariant vector under a normal coordinate transformation;

(ii) the curl of a vector field at \( x \) transforms as a pseudo-vector.

What is more,

(iii) the divergence of a vector field is indeed a scalar field and transforms as a scalar at each \( x \in \mathbb{R}^n \).

Proof. See Problem 26.

Hint. Let \( \{e_i\}_{i=1}^n \) be a basis. Let \( D^{-1} \) be a normal coordinate transformation \( \tilde{e}_i = e_j(D^{-1})^j_i \tilde{x}^j \), which implies that

\[
(D^{-1})^i_j = \frac{\partial x^i}{\partial \tilde{x}^j}.
\]

For (i), let \( f(x) \) be a scalar field. Applying the chain rule, one can easily show that

\[
(\tilde{\nabla} \tilde{f})_i(\tilde{x}) = \frac{\partial f}{\partial \tilde{x}^i}(x) = (D^{-1})^i_j \frac{\partial f}{\partial x^j}(x).
\]

To complete the proof, one just needs to establish that any covariant vector \( y_i \) transforms under \( D^{-1} \) in the following way:

\[
\tilde{y}_i = (D^{-1})^j_i y_j.
\]

Remark 1. Let \( x \in \mathbb{R}^n \) be given. The contravariant components of the gradient of a scalar field \( f \) at \( x \) can be defined by

\[
(\nabla f)^i(x) := \frac{\partial f}{\partial x_i}(x) \equiv \partial^i f(x) \quad (i \in \{i\}_{i=1}^n).
\]

The reader ought to verify that it does transform as a contravariant vector.

5.3 Tensor Products and Tensor Traces

Definition 1. Let \( s \) and \( t \) be tensors of rank \( M \) and rank \( N \), respectively. The tensor product of \( s \) and \( t \) yields a rank-\((M + N)\) tensor \( u = s \otimes t \) whose components are given by

\[
u^{i_1 \cdots i_{M+N}} = s^{i_1 \cdots i_M} t^{i_{M+1} \cdots i_{M+N}}.
\]

Proposition 1. The tensor product of two tensors or two pseudo-tensors is a tensor. The tensor product of one tensor and one pseudo-tensor is a pseudo-tensor.

Proof. See Problem 27.
Remark. Of four-vectors. Latin indices (running from 1 to 3) are used to label only the spatial component.

Remark. Like components (vectors treated as a \( \in A \)).

Let us consider Proposition 2. Such defined \( u \) is indeed a tensor (or pseudo-tensor).

Proof. See Problem 27.

Example 1. Let \( x \in \mathbb{R}^n \) be given. The curl of a vector field \( v \) at \( x \), \( (\nabla \times v)(x) \) can be regarded as successive contractions of a rank-5 pseudo-tensor:

\[
(\nabla \times v)^i(x) = \varepsilon^{ijk} \partial_j v_k(x) = g_{jk} \varepsilon^{ijk} \partial^f v_k(x) = g_{km} g_{jk} \varepsilon^{ijk} \partial^f v^m(x).
\]

According to Proposition 2, the curl is a pseudo-vector. This is consistent with Proposition 1, (ii) of 5.2.

5.4 Minkowski Tensors

Let us consider \( M^4 \), i.e., \( \mathbb{R}^4 \) with the metric \( g = \text{diag}\{1, -1, -1, -1\} \). Let \( \{e_0, e_1, e_2, e_3\} \) be a basis. Let \( A \in M^4 \) be a four-vector with contravariant components \( A^\mu = (A^0, A^1, A^2, A^3) \equiv (A^0, A) \) and covariant components \( (A_0, A_1, A_2, A_3) = A_\mu = g_{\mu\nu} A^\nu = (A^0, -A^1, -A^2, -A^3) \equiv (A^0, -A) \). Thereinto, \( A \) can be treated as a three-vector in the 3-dimensional Euclidean space \( E^3 \subset M^4 \), which is spanned by the basis vectors \( e_1, e_2, e_3 \). Let \( F \) be the rank-2 tensor

\[
F^{\mu\nu} = \begin{pmatrix}
F^{00} & F^{01} & F^{02} & F^{03} \\
F^{10} & F^{11} & F^{12} & F^{13} \\
F^{20} & F^{21} & F^{22} & F^{23} \\
F^{30} & F^{31} & F^{32} & F^{33}
\end{pmatrix} = \begin{pmatrix}
F^{00} \\
F^{10} \\
F^{20} \\
F^{30}
\end{pmatrix} \equiv \begin{pmatrix}
F_{\text{hor}} \\
F_{\text{ver}} \\
F^{ij}
\end{pmatrix}.
\]

Like \( A, F_{\text{hor}} \) and \( F_{\text{ver}} \) can also be regarded as three-vectors; \( F^{ij} \) can be considered as a rank-2 tensor in \( E^3 \).

Example 1. In electromagnetism, the field-strength tensor is given by \( F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \), where

\[
\partial^\mu = \frac{\partial}{\partial x_\mu} = \left( \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) = \left( \frac{1}{c} \frac{\partial}{\partial t}, -\nabla \right).
\]

Remark 1. Greek indices (running from 0 to 3) are employed to label both temporal and spatial components of four-vectors. Latin indices (running from 1 to 3) are used to label only the spatial component.

Remark 2. If \( F \) is symmetric, i.e., \( F^{\mu\nu} = F^{\nu\mu} \), we have \( F_{\text{hor}} = F_{\text{ver}} \) and \( F^{ij} = F^{ji} \).

Remark 3. If \( F \) is antisymmetric, i.e., \( F^{\mu\nu} = -F^{\nu\mu} \), then \( F_{\text{hor}} = -F_{\text{ver}} \), \( F^{ij} = -F^{ji} \), and \( F^{\mu\nu} = 0 \).
Lemma 1. In $E^3$, the set of antisymmetric rank-2 tensors is isomorphic to the set of pseudo-vectors.

Proof. Let $t$ be an arbitrary antisymmetric rank-2 tensor in $E^3$. $t$ can be written as

$$t^{ij} = \begin{pmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{pmatrix} = \varepsilon^{ijk}v_k,$$

for some $v \in E^3$. According to Proposition 1 of 5.3, $v$ is a pseudo-vector. We have thus shown that in $E^3$, there exists a one-to-one correspondence between antisymmetric rank-2 tensors and pseudo-vectors. 

Corollary 1. In $M^4$, any antisymmetric rank-2 tensor is of the form

$$\begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & v_3 & -v_2 \\ -a_2 & -v_3 & 0 & v_1 \\ -a_3 & v_2 & -v_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ -a^T & t^{ij} \end{pmatrix},$$

for some three-vector $a$ and pseudo-three-vector $v$.

Remark 4. Let

$$F^{\mu\nu} = \begin{pmatrix} 0 & a \\ -a^T & t^{ij} \end{pmatrix}.$$ 

Let us first derive the mixed tensors $F^{\mu\nu}$ and $F^{\mu\nu}$:

$$F^{\mu\nu} = g_{\mu\alpha}F^{\alpha\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & a \\ -a^T & t^{ij} \end{pmatrix} = \begin{pmatrix} 0 & a \\ -a^T & -t^{ij} \end{pmatrix},$$

and

$$F^{\mu\nu} = F^{\mu\alpha}g_{\alpha\nu} = \begin{pmatrix} 0 & a \\ -a^T & t^{ij} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -a \\ -a^T & -t^{ij} \end{pmatrix}.$$ 

We can further obtain $F_{\mu\nu}$:

$$F_{\mu\nu} = g_{\mu\alpha}g_{\beta\nu}F^{\alpha\beta} = F^{\mu\beta}g_{\beta\nu} = \begin{pmatrix} 0 & a \\ a^T & t^{ij} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -a \\ -a^T & t^{ij} \end{pmatrix}.$$ 

Notice that

$$F^{\mu\nu}F_{\mu\nu} = -F^{\nu\mu}F_{\mu\nu} = -(FF)^{\nu}_{\nu} = -\text{Tr}\{FF\} = -\text{Tr}\left\{\begin{pmatrix} 0 & a \\ -a^T & t^{ij} \end{pmatrix} \begin{pmatrix} 0 & -a \\ a^T & t^{ij} \end{pmatrix}\right\} = -\text{Tr}\left\{\begin{pmatrix} aa^T & at \\ ta^T & a^2 + tt \end{pmatrix}\right\} = -\|a\|^2 - \text{Tr}\{a^T a + tt\} = 2\left(\|v\|^2 - \|a\|^2\right),$$

i.e., $F^{\mu\nu}F_{\mu\nu}$ is a scalar.
Chapter 2
Mathematical Analysis

1 Real Analysis
1.1 Differentiation and Integration
1.2 Paths and Line Integrals
1.3 Surfaces and Surface Integrals
2 Complex-Valued Functions of Complex Arguments
2.1 Complex functions
2.2 Analyticity
3 Integration in the Complex Plane
3.1 Path integrals
3.2 Laurent series
3.3 The residue theorem
3.4 Simple applications of the residue theorem
3.5 Another Application of Complex Analysis: The Airy Function Ai(x)

Let us consider the Airy equation

\[
\frac{d^2 y}{dx^2} - xy(x) = 0.
\]  

(3.5.1)

This is a second order ordinary differential equation (ODE) that appears in the studies of rainbows as well as charged particles moving in an external electric field. Inspired by Laplace transform, we seek solutions of the form

\[
y(x) = \int_a^b dt \, f(t)e^{xt},
\]

(3.5.2)

for some \(a, b \in \mathbb{R} (a \neq b)\) and a real-valued function \(f\) to be decided. To determine \(f\), we substitute Eq. (3.5.2) into Eq. (3.5.1):

\[
\frac{d^2 y}{dx^2} = \frac{d^2}{dx^2} \int_a^b dt \, f(t)e^{xt} = \int_a^b dt \, \frac{\partial^2}{\partial x^2} f(t)e^{xt} = \int_a^b dt \, t^2 f(t)e^{xt},
\]

and

\[
xy = x \int_a^b dt \, f(t)e^{xt} = \int_a^b dt \, f(t)xe^{xt} = \int_a^b dt \, f(t) \frac{\partial}{\partial t} e^{xt} = f(t)e^{xt} \bigg|_{t=a}^{t=b} - \int_a^b dt \, \frac{df}{dt} \cdot e^{xt},
\]

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which implies that
\[
\int_a^b \left( t^2 f(t) + \frac{df}{dt} \right) e^{xt} dt = f(b)e^{bx} - f(a)e^{ax}.
\] (3.5.3)

Let us consider the simplest case that both sides of Eq. (3.5.3) equal zero. The following lemma summarizes our analysis so far.

**Lemma 1.** If a real-valued function \( f \) satisfies both
\[
\frac{df}{dt} + t^2 f(t) = 0
\] (3.5.4)
and the statement that there exist some \( a, b \in \mathbb{R} \) \((a \neq b)\) such that for all \( x \in \mathbb{R} \),
\[
f(b)e^{bx} = f(a)e^{ax},
\] (3.5.5)
then the function
\[
y(x) := \int_a^b dt \, f(t)e^{xt}
\]
is a solution to the Airy equation.

Notice that Eq. (3.5.4) is nicely separable:
\[
\frac{df}{f} = -t^2 \, dt,
\]
which implies that
\[
f(t) = f(0)e^{-\frac{t^3}{3}}.
\]

Accordingly, Eq. (3.5.5) becomes
\[
\exp\left(-\frac{b^3}{3} + bx\right) = \exp\left(-\frac{a^3}{3} + ax\right).
\]

If this equality holds for all \( x \in \mathbb{R} \), its both sides must equal zero. However, we know that
\[
\lim_{t \to \infty} \exp\left(-\frac{t^3}{3} + tx\right) = 0.
\]

## 4 Fourier Transforms and Generalized Functions

### 4.1 The Fourier transform in classical analysis

### 4.2 Inverse Fourier transforms

### 4.3 Test functions

### 4.4 Generalized functions

### 4.5 The \( \delta \)-function