Chapter 5

Some Aspects of the Electrodyamics of Continuous Media

4.1 Equations for a Dielectric Medium

We define a uniform medium by 

\[ \mathbf{E} = \nabla \varphi + \mathbf{v} \times \mathbf{B} \]

where \( \mathbf{E} \) is the electric field, \( \varphi \) is the electric potential, \( \mathbf{v} \) is the velocity, and \( \mathbf{B} \) is the magnetic field. The condition that \( \mathbf{E} \) be uniform is equivalent to the condition that \( \nabla \times \mathbf{E} = 0 \).

For a dielectric medium, we have 

\[ \mathbf{E} = \epsilon \mathbf{E} \]

where \( \epsilon \) is the dielectric constant.

This leads to the equation 

\[ \nabla \cdot \mathbf{E} = \rho \]

where \( \rho \) is the charge density.

We also need to consider the electric field 

\[ \mathbf{E} = \mathbf{E}(x) \]

where \( x \) is the spatial coordinate.

For a uniform medium, we have 

\[ \mathbf{E} = \mathbf{E}(x) \]

This satisfies the equation 

\[ \nabla \cdot \mathbf{E} = \rho \]

where \( \rho \) is the charge density.

We define a uniform medium by 

\[ \mathbf{E} = \mathbf{E}(x) \]

where \( x \) is the spatial coordinate.

For a dielectric medium, we have 

\[ \mathbf{E} = \epsilon \mathbf{E}(x) \]

where \( \epsilon \) is the dielectric constant.
Remark: (5) Let $V$ be any volume that completely contains the dielectric body. In
\[ \int_V \mathbf{d} \mathbf{x} \cdot \mathbf{\overrightarrow{d}}(\mathbf{x}) = 0 \]
\[ \text{and} \quad \int_V \mathbf{d} \mathbf{x} \cdot \nabla \cdot \mathbf{P}(\mathbf{x}) + \int_V \mathbf{d} \mathbf{s} \cdot \mathbf{\overrightarrow{D}}(\mathbf{x}) = 0 \]

Definition: \[ \mathbf{\overrightarrow{S}}(\mathbf{x}) = \mathbf{E}(\mathbf{x}) + \mathbf{\overrightarrow{P}}(\mathbf{x}) \] is called the electric induction.

Theorem: The curl of the electric induction is always zero.
\[ \nabla \times \mathbf{\overrightarrow{S}}(\mathbf{x}) = 0 \quad (\mathbf{x}) \]

Proof: \[ \mathbf{\nabla} \cdot \mathbf{\overrightarrow{E}} = \mathbf{\nabla} \cdot \mathbf{\overrightarrow{P}} = \mathbf{\nabla} \cdot (\mathbf{\overrightarrow{E}} + \mathbf{\overrightarrow{P}}) = 0 \]

Remark: (6) Keep the dielectric not immersed, but come as an "external" charge density \( \rho_x(\mathbf{x}) \) that has been applied by the experimenter. Then (8) plus Maxwell's
\[ \nabla \times \mathbf{\overrightarrow{D}}(\mathbf{x}) = 4\pi \mathbf{\rho}_x(\mathbf{x}) \quad (\mathbf{x}) \]

(7) Consider the dipole moment of the dielectric:
\[ \mathbf{d}_i = \int_V \mathbf{d} \mathbf{x} \cdot \mathbf{x} \cdot \mathbf{\overrightarrow{S}}(\mathbf{x}) = \int_V \mathbf{d} \mathbf{x} \cdot \mathbf{x} \cdot \mathbf{\overrightarrow{D}}(\mathbf{x}) = \int_V \mathbf{d} \mathbf{x} \cdot \mathbf{x} \cdot \mathbf{\overrightarrow{P}}(\mathbf{x}) + \int_V \mathbf{d} \mathbf{x} \cdot \mathbf{x} \cdot \mathbf{\overrightarrow{P}}(\mathbf{x}) \]
\[ \mathbf{d}_i = -\int_V \mathbf{d} \mathbf{s} \cdot \mathbf{\overrightarrow{P}}(\mathbf{x}) + \int_V \mathbf{d} \mathbf{P}_i \quad \Rightarrow \quad \mathbf{d}_i = \int_V \mathbf{d} \mathbf{P}(\mathbf{x}) \]
\[ \mathbf{d}_i = 0 \text{ when } \mathbf{P} = 0 \text{ outside the body} \]

Remark: The potential is the dipole moment density of the dielectric.

Conversely, the third Maxwell's equation is satisfied.
\[ \nabla \times \mathbf{\overrightarrow{E}}(\mathbf{x}) = 0 \quad (\mathbf{x}) \]

Remark: (8) In order for (8) and (44) to give a complete description, we shall need a relation between \( \mathbf{\overrightarrow{D}} \) and \( \mathbf{\overrightarrow{E}} \).
mark: (9) $\vec{P}$ is the dipole moment induced by $\vec{E} \rightarrow \vec{P}$ must vanish as $\vec{E} \rightarrow 0$.

amphi: For medium $\vec{E}$, and in isotropic media, $\vec{P} \propto \vec{E}$:

$$\vec{P}(x) = X(x) \vec{E}(x)$$

with $X$ the dielectric susceptibility.

mark: (10) $\vec{E}$ is homogeneous in the medium. In a homogeneous isotropic medium it is a right hand; i.e., right hand.

it is a linear: $\vec{P} = X \vec{E}$

(11) Our ignorance about the microscopic details is hidden in $X$.

For the relationship between $\vec{E}$ and $\vec{E}$ in naphtha (see def. 1):

$$\vec{D} = \varepsilon \vec{E}$$

with $\varepsilon$ the relative dielectric constant.


1.2 Magnetostatics

Now write the first Maxwell eq. from def. 1.2:

$$\nabla \cdot \vec{H} = 0$$

Conservation of total curl-free magnetic induction $\vec{H}$ gives:

$$\nabla \times \vec{H}(x) = 0$$

And the fourth Maxwell eq. for static fields:

$$\nabla \times \vec{D}(x) = \frac{\mu_0}{c^2} \frac{d}{dx} \vec{H}(x)$$
will \( \tilde{\mathbf{j}}(\mathbf{x}) = \frac{1}{\mu_0} \int \varphi^y G(x, \mathbf{y}) \) \( \mu \)-dependent unit only.

Remark: (1) \( \mu \) is a dielectric, no unit can flow
\[ \int \tilde{\mathbf{D}} \cdot \mathbf{j}(\mathbf{x}) = 0 \]
\[ \int \tilde{\mathbf{D}} \cdot \mathbf{j}(\mathbf{x}) \]
when \( \mu \) is an arbitrary number of the dielectric \( \Rightarrow \tilde{\mathbf{D}} \cdot \mathbf{j}(\mathbf{x}) = 0 \)
\[ \mathbf{j}(\mathbf{x}) \] unit in a pen and ink:

Def. 1: Define the magnetization field \( \tilde{\mathbf{H}}(\mathbf{x}) \) by
\[ \tilde{\mathbf{H}}(\mathbf{x}) = \begin{cases} \text{within of } \mathbf{x} \mathbf{E}(\mathbf{x}) = \frac{1}{\epsilon} \mathbf{E}(\mathbf{x}) & \text{inside} \\ 0 & \text{outside} \end{cases} \]

Def. 2: Define the magnetic field \( \tilde{\mathbf{B}}(\mathbf{x}) \) by
\[ \tilde{\mathbf{B}}(\mathbf{x}) = \tilde{\mathbf{H}}(\mathbf{x}) - \mu \tilde{\mathbf{E}}(\mathbf{x}) \]

Union: The \( \mu \)-dependent force from all \( \mathbf{E} \)'s.\( \nabla \times \mathbf{B}(\mathbf{x}) = 0 \)

Proof: \( \nabla \times \mathbf{B} = \nabla \times \mathbf{E} - \mu \nabla \times \mathbf{H} = \nabla \times \mathbf{E} - \frac{\mu \nabla \times \mathbf{E}}{\epsilon} = 0 \)

Remark: (2) \( \mu \) is an arbitrary dielectric, then can be no "external units".

Example: For null \( \tilde{\mathbf{B}} \), in an isotropic medium,
\[ \tilde{\mathbf{H}}(\mathbf{x}) = \chi \mu \tilde{\mathbf{B}}(\mathbf{x}) \] \( \chi \mu \) \( \mu \)-magnetic
For the relative field $\vec{E}$ and $\vec{H}$ this implies
\[ \vec{E} = \vec{E}_0 + \nabla \phi = \vec{E}_0 + \nabla \phi \cdot \vec{n} \cdot \nabla \phi \]
\[ \vec{H} = \vec{H}_0 + \nabla \times \vec{A} \]

\[ \vec{D} = \varepsilon \vec{E} \]

\[ \vec{B} = \mu \vec{H} \]

\[ \mu = 1 + \frac{\eta}{\varepsilon} \text{ the magnetic permeability} \]

1.2 Summary of static Maxwell's Eq. in a dielectric medium

\[ \nabla \cdot \vec{D}(\vec{x}) = 0 \quad (1) \]
\[ \nabla \times \vec{E}(\vec{x}) = 0 \quad (2) \]
\[ \nabla \cdot \vec{B}(\vec{x}) = \frac{4\pi}{c} \text{ext}(\vec{x}) \quad (3) \]
\[ \nabla \times \vec{H}(\vec{x}) = 0 \quad (4) \]

Vill $\vec{E}$, $\vec{D}$ the common medium electric field and the electric induction, respectively,

$\vec{H}$ the common medium magnetic induction and the magnetic field, respectively

\[ \vec{E} = \varepsilon_0 \varepsilon \vec{E} \quad \vec{B} = \mu_0 \mu \vec{H} \quad (5) \]

With $\varepsilon_0$ the dielectric constant and $\mu_0$ the magnetic permeability.

Remark: (1) The susceptibility relations (5) in a solid are of linear relation, while $\varepsilon = \varepsilon_0 \varepsilon$, $\mu = \mu_0 \mu$. This is approximately true for small fields. For strong fields (e.g., light), $\varepsilon$ becomes $\varepsilon$-dependent.
1.4 Generalised DC dynamics: Retarded dielectric response

\[ P = \chi E \]

Remark: (1) In what follows we will ignore the spatial dependence of the fields, \( \chi \) of \( X \).

For a time-dependent field, the polarization \( P \) cannot follow the field \( E \) instantaneously. The amplitudes for §1.1 need to be generalized:

\[ P(t) = \int dt' \Theta(t-t') \chi(t-t') E(t') \]

Remark: (1) Now a function \( \chi(t) \) characterizes the medium, instead of a simple matrix \( \chi \). It is sometimes called a memory function.

(2) We shall assume that the relation between \( P \) and \( E \) is linear ("linear response")

(3) The step function \( \Theta(t) \) is conserved.
2. Introduction to the theory of causal functions

2.1 Causal functions (Sect. 19-21)

Let \( f : \mathbb{R} \to \mathbb{C} \) be a weakly increasing function in the sense of \( 610/54.2 \).

**Definition:**
\[
\text{def. 1: } f_+^\pm(t) = \Theta(\pm t) f(t) \quad \text{en called the retarded (}) \quad \text{and advanced (} - \text{) part of } f.
\]

**Remark:** (1) Plot \( 610 \to \) Fourier basis of \( f, f_+, \) and \( f_- \).

Next, we define \( f_{\pm} \) in terms of \( f(t), t \in \mathbb{R} \).

**Definition:**
\[
\text{def. 2: } \text{let } t \in \mathbb{C} \text{ and define the } \text{Laplace transform of } f(t), t \in \mathbb{R},
\]
\[
F(t) := \frac{1}{\pi i} \int_{\delta-i\infty}^{\delta+i\infty} e^{st} f(t) dt.
\]

\( F \) is called the causal function associated with \( f_+, t \) is called complex frequency.

**Remark:** (2) \( t = \omega + \delta \) for \( \delta > 0 \)

\[
\Rightarrow e^{it} = e^{i\omega} e^{i\delta} = e^{i\omega} e^{-\delta}
\]

\( \Rightarrow F(t) \) is an analytic function of \( t \) for \( \delta > 0 \)

**Example:** (1) \( f(t) = e^{-at} e^{-|t|} \) - decay

\[
i \int_{-\infty}^{\infty} e^{it-cst-y} dt = \frac{-i}{it-c\omega - y} = \frac{-i}{t-c\omega - y}
\]

\[
i \int_{-\infty}^{\infty} e^{it-cst+y} dt = \frac{-i}{it-c\omega + y} = \frac{-i}{t-c\omega + y}
\]

\[
\Rightarrow F(t) = \frac{-i}{t-c\omega + iy - y}
\]
$F(t)$ is analytic in the upper half plane, and can be analytically continued into the lower half plane when it has a pole at $t = \omega_0 - i\gamma$.

Similarly, it is analytic in the lower half plane and its analytic continuation into the upper half plane has a pole at $t = \omega_0 + i\gamma$.

$F(t)$ consists of two Riemann sheets. It has a branch cut on the real axis, and

$$F(\omega + i\delta) = \frac{-1}{\omega - \omega_0 + i\delta} = \frac{-1}{(\omega - \omega_0)^2 + \delta^2}$$

Discriminant of $\ln F$ on the real axis (if one stays on one sheet in which the function is analytic in the respective half plane).

Then the discriminant of $F(t)$ covers the real axis determining the Fourier transform of $f(t)$:

$$\text{ln } F(\omega + i\delta) = \pm i \hat{f}(\omega)$$

Proof: Let $t = x + iy$, i.e., $x, y \in \mathbb{R}$, $y \geq 0$

$$e^{i \omega t} = e^{i x t} e^{-y t} \leq e^{-y t}$$

$$f(t) \text{ real}$$

$$\Rightarrow |F(t)| = \left| \int_0^\infty e^{i \omega t} f(t) dt \right| \leq \int_0^\infty e^{-y t} f(t) dt < \infty$$

$$\Rightarrow F(\omega + i\delta) \text{ bounded and analytic}$$

$$\Rightarrow F(\omega + i\delta) \text{ is a regular branch of } f(t) \text{ in the } \omega$$

in $\mathbb{R} \pm 5.610 i \mathbb{R}$ if $\delta > 0$.4
Now let \( \hat{g}(\omega) \) be a test function in the \( L^1 \) space, \( \omega \in \mathbb{R} \), \( \mu = \text{Lebesgue} \) measure on \( 610 \times 4.4 \).

\[
\int_{-\infty}^{\infty} F(t + \epsilon) \hat{g}(t) dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i \omega t} f(t) g(t + \epsilon) dt\,d\mu
\]

\[
\left. \right|_{\epsilon = 0} \Rightarrow \int_{-\infty}^{\infty} f(t) e^{-i \omega t} g(t) \,d\mu
\]

The immediate result is:

\[
\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} F(t + \epsilon) \hat{g}(t) \,d\mu = \int_{-\infty}^{\infty} f(t) \hat{g}(t) \,d\mu
\]

\[\text{Proof for } \epsilon < 0.\]

\[
\begin{bmatrix}
\hat{f}(\omega) = \hat{f}^+(\omega) + \hat{f}^-(\omega) = -i \left[ F(\omega + i0) - F(\omega - i0) \right] \\
\therefore \hat{f}(\omega) = \hat{f}^+(\omega) - \hat{f}^-(\omega) = -i \left[ F(\omega + i0) + F(\omega - i0) \right]
\end{bmatrix}
\]

Example: \( \omega \)-independent example (5) with \( \omega \to 0 \). \( \Rightarrow f(t) = e^{-\omega t} \)

\[
-1 - i \left[ F(\omega + i0) - F(\omega - i0) \right] = \lim_{\epsilon \to 0} \frac{1}{i(\omega - \omega_0) + \epsilon + \epsilon} = \lim_{\epsilon \to 0} \frac{2\epsilon}{(\omega - \omega_0)i + \epsilon} = 2\delta(\omega - \omega_0)
\]
2.2 Spectrum and derivative part of causal functions

Definition: Let \( F(t) \) be a causal function. We define

\[
F''(ω) := \frac{1}{2i} \left[ F(ω+i0) - F(ω-i0) \right] \\
F'(ω) := \frac{1}{i} \left[ F(ω+i0) + F(ω-i0) \right]
\]

\( F'' \) is called spectrum or spectral part, or dispersive part.
\( F' \) is called derivative part of \( F(t) \).

Remark: (1) Our notation implies that \( f(t), f'(u), F(t), F'(u), F''(ω), F'(ω) \) are differentiable. One often writes \( f(t), f'(u), f(t), f''(ω), f'(ω) \) distinguished. We refer only by their arguments to the prime and double prime superscripts.

(2) \( F''(ω) \) is given by the discriminant of \( F(t) \) across the real axis; \( F'(ω) \) by the envelope of \( F(t) \) across the real axis.

(3) \( F(ω+i0) = F'(ω) \pm iF''(ω) \)

(4) \[
\hat{f}(ω) = \hat{f}_+(ω) \pm \hat{f}_-(ω) = 2F''(ω) \\
\hat{f}'(ω) = \hat{f}_+(ω) - \hat{f}_-(ω) = 2F'(ω)
\]

(5) In general, \( F'(ω) \) and \( F''(ω) \) on C-valued \( f(t) \).

Known: (c) The spectrum \( F''(ω) \) uniquely determines \( f(t), F(t) \) and \( F'(u) \).

(6) \( F'(ω) \) uniquely determines \( F(t) \) and \( F''(ω) \).

Remark: (6) This is an extremely important result. We usually follow immediately from the definition.
Proof:

\[
F''(w) \xrightarrow{\text{Fourier transform}} f(t) \\
F'(w) \xrightarrow{\text{Laplace transform}} F(s)
\]

Remark: (i) We need the \underline{continuous} \ function \ \( F''(w) \), \ i.e., \ \( F''(w) \neq \text{constant} \), in order to determine \( F'(w) \) for a given \( w \).

Wolfe's Lemma: For a given function \( F''(w) \), there exists at most one \( f(t) \) for \( F(s) \) to hold.

(i) \( F(s) \) is analytic for \( \text{all } s \) with \( \text{Im}(s) \rightarrow 0 \) and

(ii) \( F(s) \rightarrow 0 \) as \( \text{Re}(s) \rightarrow \infty \).

Proof: Let \( F_1, F_2 \) be two such \( f(t) \) and \( G_0 \), \( G_1, g_2 \). Consider \( G = F_1 - F_2 \)

\[
G(s) = F_1(s) - F_2(s) = F_1(w+\text{i}0) - F_2(w+\text{i}0) \\
- F_1(w-\text{i}0) + F_2(w-\text{i}0)
\]

\[
= 2i \left[ F_1''(w) - F_2''(w) \right] = 2i \left[ F_1''(w) - F_2''(w) \right] = 0
\]

\[
\Rightarrow G(s) \text{ is analytic for } \text{all } s \text{ with } \text{Im}(s) \rightarrow 0 \text{ and } G(s) \rightarrow 0 \text{ as } \text{Re}(s) \rightarrow \infty.
\]

A known in complex analysis

\[
\Rightarrow G(s) \text{ is a polynomial in } t
\]

\[
\Rightarrow G(t) \rightarrow 0 \text{ as } t \rightarrow \infty \Rightarrow G(t) \equiv 0
\]
2.3 Hilbert-Redfield transformations

**Definition:** Let \( f: \mathbb{R} \to \mathbb{C} \) be a measurable function and let

\[
F(t) = \int_0^\infty \frac{f(u)}{u-t} \,du
\]

exist for \( m < 0 \).

The \( F(t) \) is called the **Hilbert-Redfield transform** of \( f(u) \).

**Example:**
1. \( f(u) = \delta(u-u_0) \implies F(t) = \frac{1}{u_0-t} \)

**Remark:**
1. \( f \mapsto F \) is a linear map that maps \( \mathcal{L}^1(\mathbb{R}) : \mathbb{R} \to \mathbb{C} \)

\( F: \mathbb{C} \to \mathbb{C} \).

2. \( F(t) \) is analytic for \( m < 0 \) and has a branch cut for \( m = 0 \).

**Proposition:**
1. If \( f(u) \) is even (odd), then \( F(t) \) is odd (even).
2. If \( f(u) \to F(t) \), then \( f^k(u) \to F(t)^k \).
3. If \( f(u) \in \mathbb{R} \), then \( F(t) = F(t^*) \).
4. If \( f(u) \in i\mathbb{R} \), then \( F(t) = -F(t^*) \).

**Proof:**
1. Let \( f(u) \to f(-u) \implies F(-t) = \int_0^\infty \frac{f(u)}{u+t} \,du - \int_0^{-\infty} \frac{f(-u)}{u-t} \,du = -F(t) \)

2. \( \int_0^\infty \frac{f(u)^k}{u-t} \,du = \left( \int_0^\infty \frac{f(u)}{u-t} \,du \right)^k = F(t)^k \)

3. Let \( f(u) \in \mathbb{R} \implies F(t^*) \cdot \int_0^\infty \frac{f(u)}{u-t} \,du = \left( \int_0^\infty \frac{f(u)}{u-t} \,du \right)^* \cdot F(t)^* \)

4. Let \( f(u) \in i\mathbb{R} \implies F(t^*) \cdot \int_0^\infty \frac{-f(u)}{u-t} \,du = -\left( \int_0^\infty \frac{f(u)}{u-t} \,du \right)^* \cdot F(t)^* \)
Theorem 1. (Sokhotski–Plemelj)

If \( F(t) \) is analytic for \( |t| < \varepsilon \), then

\[
\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \left[ F(u+i\varepsilon) - F(u-i\varepsilon) \right] = f(u)
\]

\[\text{Proof: } \]

Let \( g(u) \) be a test function. Then

\[
\frac{1}{2\varepsilon} \int g(u) \left[ F(u+i\varepsilon) - F(u-i\varepsilon) \right] \, du = \frac{1}{2\varepsilon} \int g(u) \left( \frac{1}{x-u+i\varepsilon} - \frac{1}{x-u-i\varepsilon} \right) \, du
\]

\[
= \frac{1}{2\varepsilon} \int g(u) \frac{dx}{(x-u)^2 + \varepsilon^2} \left. \right|_{-\infty}^{\infty} = \int g(u) \frac{dx}{x-u} \delta(x-u)
\]

\[
= \int du \int_{-\infty}^{\infty} \frac{f(x) \delta(x-u)}{x-u} \, dx
\]

Theorem 2: Let \( F: \mathbb{C} \to \mathbb{C} \) be a function with the properties:

1. \( F(t) \) is analytic for \( |t| < \varepsilon \)
2. \( F(t) \to 0 \) for \( |t| \to \infty \)
3. \( \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \left[ F(u+i\varepsilon) - F(u-i\varepsilon) \right] \) exists and defines a
   \[ \text{generalized} \] \[ \text{limit} \] \[ f(u) \] .

Then \( F(t) \) can be written as \( \mathcal{H} \) plus, i.e.,

\( F(t) = \mathcal{H} \frac{f(u)}{u-t} \) \[ \text{and} \] \[ F(t) \] \[ \text{is the unique} \] \[ \mathcal{H} \] \[ \text{plus of} \] \[ f(u) \] .

\[ \text{Proof: Books} \]

Proposition 2: Let \( F(t) \) be the \( \mathcal{H} \) plus of a real generalized \( f \) function. \( f(u) \neq 0 \). Then

\[ f(u) > 0 \iff \left( \text{in} \ F(t) \geq 0 \text{ for } |t| \geq 0 \right) \]
\[ F(x+iy) = \int_{x}^{\infty} \frac{f(u)}{u-(x+iy)} \, du = \int_{x}^{\infty} \frac{(u-x-iy)f(u)}{(u-x)^2 + y^2} \, du \]

\[ = \int_{x}^{\infty} \frac{(u-x)f(u)}{(u-x)^2 + y^2} \, du + iy \int_{x}^{\infty} \frac{f(u)}{(u-x)^2 + y^2} \, du \]

\[ = e^{-y} \int_{x}^{\infty} \frac{f(u)}{(u-x)^2 + y^2} \, du \quad \text{for } t > 0 \]

Define \( f(u) \) such that \( \lim_{t \to 0} F(x+iy) > 0 \) if \( f(t) \neq 0 \) and \( \lim_{t \to 0} F(x+iy) > 0 \) if \( f(t) > 0 \) for \( u \in \mathbb{R} \).

Now let \( \lim_{t \to 0} F(x+iy) > 0 \) for \( u \in \mathbb{R} \).

Define \( f(u) = \frac{1}{i} \left[ F(u+i0)-F(u-i0) \right] = \frac{1}{i} \left[ \lim_{t \to 0} F(u+i0) - \lim_{t \to 0} F(u-i0) \right] \geq 0 \) for \( u \in \mathbb{R} \).

Remark: (1) Positivity of this \( \nu \) is equivalent to \( \lim_{t \to 0} F(x) > 0 \) for \( u \in \mathbb{R} \).

2.4. Higher examples of Hilbert-Stephan's graphs

(1) Higher function \( F(x) \) for \( u > 0 \)

\[ F(x) = \frac{-1}{x+iy} \quad \text{is analytic for } u > 0 \]

\[ \text{and } \lim_{t \to \infty} \frac{1}{x} \left[ F(u+i0)-F(u-i0) \right] = \frac{1}{i} \left( \frac{-1}{u+iy} + \frac{1}{u-iy} \right) = \frac{1}{u^2+y^2} \quad \text{is finite} \]

\[ \Rightarrow \lim_{t \to \infty} \int \frac{1}{x} \left[ F(u+i0)-F(u-i0) \right] \, dx \quad \text{is finite} \]
\[ \int \frac{1}{u-t} \frac{1}{u^*} \frac{1}{u^*} = \int \frac{1}{\delta} \frac{1}{(u-u^*)^2} (u-u^*) \]

Remark: (1) The sphere is the region part of \( F(t) \) in

\[ F^+(u) = \frac{1}{u^*} \]

\[ F^-(u) = \frac{1}{u} \frac{1}{u^*} \]

2.5 Spectral representation and Kremer-Brouij notions

Using the results of the previous paragraphs, we have the following known: A causal \( F(t) \) can be written in terms of its joint \( F^+(u) \)

\[ \int \frac{1}{u-t} \frac{F^+(u)}{u^*} \]

Remark: (1) (4) is called spectral representation or Helmholz expansion of \( F(t) \).

(2) For \( t = u^* + i\epsilon \) the denominator takes the form

\[ \frac{1}{u-u^*} = \frac{u-u^* + i\epsilon}{(u-u^*)^2 + \epsilon^2} = \frac{u-u^*}{(u-u^*)^2 + \epsilon^2} + \frac{i\epsilon}{(u-u^*)^2 + \epsilon^2} \]

(3) We know that \( \lim_{\epsilon \to 0} \frac{\epsilon}{(u-u^*)^2 + \epsilon^2} = 0 \) if \( u \neq u^* \) is a polynomial.
The function \( \lim_{\varepsilon \to 0} \frac{x}{x^2 + \varepsilon^2} \) is called the principal-value generalized \( \frac{d}{dx} f(x) \) in the limit as \( x \to \infty \).

Remark: (1) One can show that this result holds for large values of \( \varepsilon \). In particular, \( F(x) \) and \( f(x) \) are related by \( \frac{d}{dx} F(x) = f(x) \).

Wolfey: The purpose of the preceding part of the course is to learn the \textit{Kramers-Kronig} relations:

\[
F'(u) = \int \frac{dx}{\varepsilon} F''(x), \quad F''(u) = -\int \frac{dx}{\varepsilon} \frac{F'(x)}{x-u}
\]

\[
proof: \quad F'(u) = \frac{1}{\varepsilon} \left[ F(u+i\varepsilon) + F(u-i\varepsilon) \right] = \frac{1}{\varepsilon} \int \frac{dx}{\varepsilon} F''(x) \left( \frac{1}{x-u+i\varepsilon} + \frac{1}{x-u-i\varepsilon} \right)
\]

\[
= \frac{1}{\varepsilon} \int \frac{dx}{\varepsilon} F''(x) \frac{(x-u)}{(x-u)^2 + \varepsilon^2} = \frac{1}{\varepsilon} \int \frac{dx}{\varepsilon} \frac{F''(x)}{x-u}
\]

Now write \( \hat{F}(t) = \exp(\text{int}) F(t) \). The premises of [2.1] theorem 2 are fulfilled \( \Rightarrow \hat{F}(t) \) can be written in the form \( \hat{F}(0) \). \( \hat{F}''(0) = \frac{i\pi}{\varepsilon} \hat{F}'(0) \).

\[
\Rightarrow \hat{F}'(0) = \frac{1}{\varepsilon} \left[ \hat{F}(u+i\varepsilon) + \hat{F}(u-i\varepsilon) \right] = \frac{1}{\varepsilon} \hat{F}'(0) \]

\[
\hat{F}'(0) = \frac{1}{\varepsilon} \left[ \hat{F}(u+i\varepsilon) + \hat{F}(u-i\varepsilon) \right] = \frac{i\pi}{\varepsilon} \hat{F}'(0) \]

\[
\Rightarrow \hat{F}''(0) = -\frac{i}{\varepsilon} \hat{F}'(0) = -\frac{i}{\varepsilon} \int \frac{dx}{\varepsilon} \frac{F''(x)}{x-u} = -\int \frac{dx}{\varepsilon} \frac{F''(x)}{x-u}
\]
2.6 Applications: The dielectric function

The linear relation between the potential $\Phi$ and the electric field $\vec{E}$ is given by a function $\chi(t)$, as the result of $\psi$ applying Copley rule:

$$\vec{P}(t) = \chi(t) \vec{E}(t)$$

with $\chi(t)$ a complex function.

$$\chi(\omega + i\epsilon) = \chi'(\omega) + i \chi''(\omega)$$

where $\chi', \chi''$ denote Kramers-Kronig.

$$\chi(\omega) = 1 + 4\pi \chi(t)$$

$$\chi'(\omega) = 1 + \frac{4\pi}{\omega} \chi'(\omega), \quad \chi''(\omega) = \frac{4\pi}{\omega} \chi''(\omega)$$

$$\chi'(\omega) = 1 + \frac{4\pi}{\omega} \chi'(\omega) = 1 + \frac{4\pi}{\omega} \int \frac{dx}{x-\omega} \frac{\chi'(x)}{x} = 1 + \frac{4\pi}{\omega} \int \frac{dx}{x-\omega} \frac{\chi'(x)}{x}$$

$$\chi''(\omega) = \frac{4\pi}{\omega} \chi''(\omega) = -\frac{4\pi}{\omega} \int \frac{dx}{x-\omega} \frac{\chi'(x)}{x}$$

Kramers 1926

Example: Resonant $\chi(\omega)$ of SiN films

E. This is confusing!
Infrared dielectric properties of low-stress silicon nitride

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Fig. 3. (Color online) Real and imaginary parts (solid red curves) of the dielectric function as extracted from the data