PHYS 610 Mathematical Methods for Scientists F 2019

Problem Assignment # 3 10/16/2019
due 10/23/2019

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9. **Pauli group**

The Pauli matrices are complex 2×2 matrices defined as

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ,$$

Now consider the set P_1 that consists of the Pauli matrices and their products with the factors -1 and $\pm i$:

$$P_1 = \{\pm\sigma_0, \pm i\sigma_0, \pm\sigma_1, \pm i\sigma_1, \pm\sigma_2, \pm i\sigma_2, \pm\sigma_3, \pm i\sigma_3\}$$

Show that this set of 16 elements forms a (nonabelian) group under matrix multiplication called the Pauli group. It plays an important role in quantum information theory.

(3 points)

10. **The group S_3**

a) Compile the group table for the symmetric group S_3. Is S_3 abelian?

b) Find all subgroups of S_3. Which of these are abelian?

(6 points)

11. **Abelian groups**

Let $(G, \vee)$ be a group with neutral element e. Let $a \in G$ be a fixed element, and define a mapping $\varphi : G \to G$ by $\varphi(x) = a \vee x \vee a^{-1}\ \forall x \in G$.

a) Show that φ defines an automorphism on G, called an *inner automorphism.*

b) Show that abelian groups have no inner automorphisms except for the identity mapping $\varphi(x) = x$.

c) Let $g \vee g = e\ \forall g \in G$. Prove that G is abelian.

(6 points)

12. **Fields**

a) Show that the set of rational numbers $\mathbb{Q}$ forms a commutative field under the ordinary addition and multiplication of numbers.

b) Consider a set F with two elements, $F = \{\theta, e\}$. On F, define an operation "plus" $(+)$, about which we assume nothing but the defining properties

$$\theta + \theta = \theta \quad , \quad \theta + e = e + \theta = e \quad , \quad e + e = \theta$$

Further, define a second operation "times" $(\cdot)$, about which we assume nothing but the defining properties

$$\theta \cdot \theta = e \cdot \theta = \theta \cdot e = \theta \quad , \quad e \cdot e = e$$

Show that with these definitions (and **no** additional assumptions), F is a field.

(7 points)