

Problem Assignment # 6

11/06/2019
due 11/13/201921. Lorentz transformations in M_2

Consider the 2-dimensional Minkowski space M_2 with metric $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and 2×2 matrix representations of the pseudo-orthogonal group $O(1, 1)$ that leaves g invariant.

a) Let $\sigma, \tau = \pm 1$, and $\phi \in \mathbb{R}$. Show that any element of $O(1, 1)$ can be written in the form

$$D_{\sigma, \tau}(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$$

To study $O(1, 1)$ it thus suffices to study the matrices $D(\phi) := D_{+1, +1} = \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix}$.

b) Show explicitly that the set $\{D(\phi)\}$ forms a group under matrix multiplication (which is a subgroup of $O(1, 1)$ that is sometimes denoted by $SO^+(1, 1)$), and that the mapping $\phi \rightarrow D(\phi)$ defines an isomorphism between this group and the group of real numbers under addition.

c) Show that there exists a matrix J (called the *generator* of the subgroup) such that every $D(\phi)$ can be written in the form

$$D(\phi) = e^{J\phi}$$

and determine J explicitly.

(6 points)

22. Time-like and space-like intervals

Consider two points (ct_x, x^1, x^2, x^3) and (ct_y, y^1, y^2, y^3) in Minkowski space. The interval between the two points is called *time-like* if

$$c^2(t_x - t_y)^2 > (x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2 \quad ,$$

and *space-like* if

$$c^2(t_x - t_y)^2 < (x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2 \quad .$$

Show that an interval that is time-like or space-like in some inertial frame is also time-like or space-like in any other inertial frame. (This reflects the invariance of the speed of light.)

(2 points)

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23. Special Lorentz transformations in M_4

Consider the Minkowski space M_4 .

a) Show that the following transformations are Lorentz transformations:

$$\text{i) } D^\mu_\nu = \begin{pmatrix} 1 & 0 \\ 0 & R^i_j \end{pmatrix} \equiv R^\mu_\nu \quad (\text{rotations})$$

where R^i_j is any Euclidian orthogonal transformation.

$$\text{ii) } D^\mu_\nu = \begin{pmatrix} \cosh \alpha & \sinh \alpha & 0 & 0 \\ \sinh \alpha & \cosh \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv B^\mu_\nu \quad (\text{Lorentz boost along the } x\text{-direction})$$

with $\alpha \in \mathbb{R}$.

$$\text{iii) } D^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv P^\mu_\nu \quad (\text{parity})$$

$$\text{iv) } D^\mu_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv T^\mu_\nu \quad (\text{time reversal})$$

b) Let L be the group of all Lorentz transformations. Show that the rotations defined in part a) i) are a subgroup of L , and so are the Lorentz boosts defined in part a) ii).

c) Let $I^\mu_\nu = \delta^\mu_\nu$ be the identity transformation. Show that the sets $\{I, P\}$, $\{I, T\}$, and $\{I, P, T, PT\}$ are subgroups of L .

(4 points)

24. General Lorentz transformations in M_4

Let D be a general Lorentz transformation in M_4 .

a) Show that $|D^0_0| \geq 1$, and that $(D^0_1)^2 + (D^0_2)^2 + (D^0_3)^2 = (D^1_0)^2 + (D^2_0)^2 + (D^3_0)^2$.

b) Let $L_{++} = \{D \in L; \det D > 0, D^0_0 > 0\}$. (This is called the set of proper orthochronous Lorentz transformations, and one can show that it is a subgroup of L .) Show that any Lorentz transformation can be written as an element of L_{++} followed by either P , or T , or PT . It thus suffices to study L_{++} .

c) Show that any element of L_{++} can be written as a spatial rotation $R(\Phi, \Theta, \Psi)$ followed by a Lorentz boost $B(\alpha)$ followed by a rotation about the 3-axis followed by a rotation about the 2-axis. In a symbolic notation:

$$D = \begin{pmatrix} 1 & 0 \\ 0 & R_2(\phi)R_3(\theta) \end{pmatrix} B(\alpha) \begin{pmatrix} 1 & 0 \\ 0 & R(\Phi, \Theta, \Psi) \end{pmatrix}$$

L_{++} is thus characterized by six parameters: 3 Euler angles Φ, Θ, Ψ , the boost parameter α , and two additional rotation angles ϕ and θ .

(7 points)