

Problem Assignments # 9

11/27/2019
due 12/04/2019

31. Applications of the residue theorem

Use complex analysis to evaluate the real integrals

a)

$$\int_{-\infty}^{\infty} dx \frac{\sin x}{x}$$

hint: Write $\sin x = (e^{ix} - e^{-ix})/2i$ and consider the resulting two integrals with complex integrands. Why is this a good strategy?

b)

$$\int_{-\infty}^{\infty} dx \frac{\sin x}{x} \frac{1}{1+x^2}$$

and check your results by means of Wolfram Alpha.

Let $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$. Use the residue theorem to show that

c)

$$\int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\pi/a}$$

Now let $a \in \mathbb{R}$ and consider the integral

d)

$$\int_{-\infty}^{\infty} \frac{dx}{x} \frac{1}{x^2 + a^2}$$

and define its Cauchy principal value by

$$\lim_{R \rightarrow 0} \left[\int_{-\infty}^{-R} dx f(x) + \int_R^{\infty} dx f(x) \right]$$

with $f(x) = 1/x(x^2 + a^2)$. Determine the Cauchy principal value using the residue theorem. Is the result consistent with the expectation for a real symmetric integral over an antisymmetric integrand?

hint: Go around the pole on a semicircle of radius R and let $R \rightarrow 0$.

(14 points)

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34. More 1-d Fourier transforms

Consider a function of time $f(t)$ and define its Fourier transform

$$\hat{f}(\omega) := \int dt e^{i\omega t} f(t)$$

and its Laplace transform $F(z)$ as

$$F(z) = \pm i \int dt e^{izt} f_{\pm}(t) \quad (\pm \text{ for } \text{sgn}(\text{Im } z) = \pm 1)$$

with z a complex frequency and $f_{\pm}(t) = \Theta(\pm t) f(t)$. Further define

$$F''(\omega) = \frac{1}{2i} [F(\omega + i0) - F(\omega - i0)] \quad , \quad F'(\omega) = \frac{1}{2} [F(\omega + i0) + F(\omega - i0)]$$

Calculate $F''(\omega)$ and $F'(\omega)$ for

a) $f(t) = e^{-|t|/\tau}$

b) $f(t) = e^{i\omega_0 t}$

hint: $\lim_{\epsilon \rightarrow 0} \epsilon / (x^2 + \epsilon^2) = \pi \delta(x)$, with $\delta(x)$ the familiar Dirac delta-function, which we will study in detail in Week 10.

Show that in both cases $\int \frac{d\omega}{\pi} \frac{F''(\omega)}{\omega} = F'(\omega = 0)$.

note: These concepts are important for the theory of response functions.

(4 points)

36. Distribution limits (This problem is optional and may show up again in Winter term)

a) Show that the sequences

$$f_n(x) = \frac{1}{\pi x} \sin(nx) \quad (n = 1, 2, \dots)$$

and

$$g_n(x) = \frac{1}{\pi n} \frac{1}{x^2 + 1/n^2} \quad (n = 1, 2, \dots)$$

yield the δ -function as $n \rightarrow \infty$ in a distribution-limit sense:

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} g_n(x) = \delta(x)$$

b) Show that

$$\frac{d}{dx} \text{sgn } x = 2 \delta(x)$$

and

$$\frac{d}{dx} \Theta(x) = \delta(x)$$

with $\Theta(x)$ the step function.

c) Show that

$$\lim_{\epsilon \rightarrow 0} \epsilon |x|^{\epsilon-1} = 2 \delta(x)$$

hint: Start with ch.3 §2.4 example 6 and differentiate.

(6 points)