1. **Russell’s Paradox (B. Russell, 1901)**

a) Consider the set $M$ defined as the set of all sets that do not contain themselves as an element: $M = \{ x; x \notin x \}$. Discuss why this is a problematic definition.

b) A less abstract version of Russell’s paradox is known as the barber’s paradox: Consider a town where all men either shave themselves, or let the barber shave them and don’t shave themselves. Now consider the statement

*The barber is a man in town who shaves all men who do not shave themselves, and only those.*

Discuss why this definition of the barber is problematic (assuming there actually is a barber in town).

*hint: Ask “Does the barber shave himself?”*

c) Suppose the definition of the barber is modified to read

*The barber shaves all men in town who do not shave themselves, and only those.*

Discuss what this modification does to the paradox.

(3 points)

2. **Distributive property of the union and intersection relations**

Convince yourself graphically of the distributive property of the relations $\cup$ and $\cap$, ch.1 §1.1 remark (9)(iii). That is, for any three sets $A, B, C$,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(2 points)

3. **Mappings**

Are the following $f : X \to Y$ true mappings? If so, are they surjective, or injective, or both?

a) $X = Y = \mathbb{Z}$, \hspace{1cm} $f(m) = m^2 + 1$.

b) $X = Y = \mathbb{N}$, \hspace{1cm} $f(n) = n + 1$.

c) $X = \mathbb{Z}$, $Y = \mathbb{R}$, \hspace{1cm} $f(x) = \log x$.

d) $X = Y = \mathbb{R}$, \hspace{1cm} $f(x) = e^x$.

(2 points)

4. **Parabolic Mapping**

Consider $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = an^2 + bn + c$, with $a, b, c \in \mathbb{Z}$.

a) For which triplets $(a, b, c)$ is $f$ surjective?

b) For which $(a, b, c)$ is $f$ injective?

(4 points)
1) a) happen to contain itself as a whole. But, by its definition, it does not.

b) happen to contain itself as a whole. But, again, by its definition, it does.

There is no third possibility, so the definition is logically self-contradictory.

This is like the barber, this means that the

b) happen the barber shaves himself. But he is shaven by the barber and by definition does not shave himself.

c) happen the barber does not shave himself. But he is shaven by the barber, and hence he does shave himself.

This is the same logical problem as in part a).

c) By dropping the requirement that the barber "is a man in town" the problem goes away: The barber can be a woman, or a man from a distant town.

Note: A logically possible conclusion from c) is also that the town has no barber.
2) \[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]

\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]

\[ (A \cup B) \cap (A \cup C) = \]
1. e) If \( f \) is a mapping, it is neither injective (\( f(n) \neq f(m) \forall n \neq m \)) nor surjective (\( f(n) = f(m) \forall n \neq m \)).

b) If \( f \) is a mapping, it is not injective (1 \in \mathbb{N} has no pre-image). It is surjective, but \( f \) is monotonically increasing.

c) If \( f \) is not a mapping, \( x \leq 0 \) has no image.

d) If \( f \) is a mapping, \( \exp(x) \) is defined \( \forall x \in \mathbb{R} \). It is not injective, \( \exp(f(x)) > 0 \forall x \in \mathbb{R} \). It is injective, \( \exp(x) \) is monotonic.
4. a) \( f(n) \) has an absolute minimum if \( a \neq 0 \)

\[ \Rightarrow \] \( c = 0 \) is necessary for \( f \) to be negative.

Now consider \( f(n) = bn + c \)

\[ \text{If } b = 0, \text{ then } f(n) = c \text{ and hence not negative.} \]

\[ \text{If } b \geq 2 \text{ or } b \leq -2, \text{ then } f(n) \text{ never equals } c + 1, \text{ and hence } f \text{ is not negative.} \]

\[ \text{If } b = \pm 1 \text{ for any } c, \text{ then } f(n) \text{ covers all of } \mathbb{Z}. \]

\[ \Rightarrow f \text{ is negative for } (0, 1, c) = (0, \pm 1, c \in \mathbb{Z}) \]

b) For \( f \) to be injective, \( f(n) = f(m) \) must imply \( n = m \). Let \( n = m + x \), will \( x \in \mathbb{Z} \).

Then \( f(n) = f(m) \) becomes the form

\[ cx^2 + bx + c = x^2 + 2ex + e \]

\[ \Rightarrow ax^2 + (2e - b)x + c - e = 0 \quad (x) \]

\( x = 0 \) is always a solution, which implies \( n = m \).

For \( x \neq 0 \), the only solution of \( (x) \) is

\[ x = -2e - b/a \]

As long as this solution is \( \in \mathbb{Z} \), \( f \) is injective.

\[ \Rightarrow \text{For } f \text{ to be injective, } b \text{ must not be divisible by any } e \in \mathbb{Z}. \]

\[ \Rightarrow f \text{ is injective for } (c \in \mathbb{Z}, b \in \mathbb{Z} \setminus a \mathbb{Z}, c \in \mathbb{Z}) \]