5. **Equivalence relations**

Consider a relation $\sim$ on a set $X$ as in ch. 1 §1.3 def. 1, but with the properties

i) $x \sim x \ \forall x \in X$ (reflexivity)

ii) $x \sim y \Rightarrow y \sim x \ \forall x, y \in X$ (symmetry)

iii) $(x \sim y \land y \sim z) \Rightarrow x \sim z$ (transitivity)

Such a relation is called an **equivalence relation**. Which of the following are equivalence relations?

a) $n$ divides $m$ on $\mathbb{N}$.

b) $x \leq y$ on $\mathbb{R}$.

c) $g$ is perpendicular to $h$ on the set of straight lines $\{g, h, \ldots\}$ in the cartesian plane.

d) $a$ equals $b$ modulo $n$ on $\mathbb{Z}$, with $n \in \mathbb{N}$ fixed.

*hint:* “$a$ equals $b$ modulo $n$”, or $a = b \mod(n)$, with $a, b \in \mathbb{Z}$, $n \in \mathbb{N}$, is defined to be true if $a - b$ is divisible on $\mathbb{Z}$ by $n$; i.e., if $(a - b)/n \in \mathbb{Z}$.

(3 points)

6. **Bounds for $n!$**

Prove by mathematical induction that

$$n^n/3^n < n! < n^n/2^n \ \forall n \geq 6$$

*hint:* $(1 + 1/n)^n$ is a monotonically increasing function of $n$ that approaches Euler’s number $e$ for $n \to \infty$.

(4 points)

7. **All ducks are the same color**

Find the flaw in the “proof” of the following

*proposition:* All ducks are the same color.

*proof:* $n = 1$: There is only one duck, so there is only one color.

$n = m$: The set of ducks is one-to-one correspondent to $\{1, 2, \ldots, m\}$, and we assume that all $m$ ducks are the same color.

$n = m+1$: Now we have $\{1, 2, \ldots, m, m+1\}$. Consider the subsets $\{1, 2, \ldots, m\}$ and $\{2, \ldots, m, m+1\}$. Each of these represent sets of $m$ ducks, which are all the same color by the induction assumption. But this means that ducks #2 through $m$ are all the same color, and ducks #1 and $m+1$ are the same color as, e.g., duck #2, and hence all ducks are the same color.

*remark:* This demonstration of the pitfalls of inductive reasoning is due to George Pólya (1888 - 1985), who used horses instead of ducks.

(2 points)

... /over
8. **Products**

Prove the corollary to proposition 2 of ch.1 §2.2: If $a$ is an element of a multiplicative group, and $n, m \in \mathbb{N}$, then

a) $a^n a^m = a^{n+m}$

b) $(a^n)^m = a^{nm}$

(2 points)
5) a) No, bin it is not symmetric. E.g., 2<4 but 4 ≠ 2.
b) No, since it is not symmetric. E.g., 2<4 but 4 ≠ 2.
c) No, since it is not reflexive. No bin is perpendicular to itself.
d) Yes.

Proof: (i) \( a-a = 0 \) is divisible by \( n \) \( \Rightarrow \ a = a \mod(n) \)

(ii) \( \exists k \in \mathbb{Z} : a-b = kn \)
\( \Rightarrow b-a = (-k)n \) \( \Rightarrow b-a \) is divisible by \( n \)
\( \Rightarrow b = a \mod(n) \)

(iii) \( \exists k, l \in \mathbb{Z} : a-b = kn \) and \( b-c = ln \)
\( \Rightarrow a-c = (a-b)+(b-c) = kn+ln = (k+l)n \)
\( \Rightarrow a-c = a \mod(n) \)

\( \Rightarrow a=b \mod(n) \) is an equivalence relation on \( \mathbb{Z} \).
6. First prove \( n^6/2^6 < n! \) \( \forall n \geq 6 \)

\[ n = 6: \quad 6^6/2^6 = 64 < 720 = 6! \quad \checkmark \]

Assume \( m^m/2^m < m! \)

Then \( \frac{(m+1)^{m+1}}{2^{m+1}} = \frac{m^m}{2^m} \left(1+\frac{1}{m}\right)^m (m+1) < \frac{m^m}{2^m} \frac{e}{2} (m+1) \)

\[ < \frac{m^m}{2^m} (m+1) < m! (m+1) = (m+1)! \]

Now prove \( n^6/2^6 > n! \) \( \forall n \geq 6 \)

\[ n = 6: \quad 6^6/2^6 = 1728 = 720 = 6! \quad \checkmark \]

Assume \( m^m/2^m > m! \)

Then \( \frac{(m+1)^{m+1}}{2^{m+1}} = \frac{m^m}{2^m} \left(1+\frac{1}{m}\right)^m (m+1) \geq \frac{m^m}{2^m} \frac{e}{2} (m+1) \geq m! (m+1) \)

\[ = (m+1)! \]

\[ \therefore \quad \frac{n^6}{2^6} < n! < \frac{5^5}{2^5} \quad \forall n \geq 6 \]
7.) The problem here will be 2.

The inductive step from $n = m$ to $n = m+1$ relies on the fact that the sets $\{1, 2, \ldots, m\}$ and $\{1, 2, \ldots, m+1\}$ have common elements. But for $n = 2$, we have $m = 1$, and hence the two sets are $\{1\}$ and $\{1, 2\}$, which have no common elements!

In order for the proof to be valid, one must prove that any two sets have the same color, which is not possible.
2.) a) We want to show that \( a^n \cdot a^m = a^{n+m} \).

   Let \( m = 1 \):
   \[
   a^1 \cdot a^1 = (\frac{n}{m} a) a = \frac{n+1}{m+1} a = a^{n+1}.
   \]
   By the recursion definition,

   \[
   n \to n+1: \quad a^n \cdot a^{n+1} = a^n \cdot a^{n+1} = a^{n+(n+1)}
   \]

   The statement holds for \( m = 1 \) by induction.

b) We want to show that \( (a^n)^m = a^{nm} \).

   Let \( m = 1 \):
   \[
   (a^n)^1 = a^n = a^{nm}.
   \]

   \[
   m \to m+1: \quad (a^n)^m = (a^n)^m a^n = a^{nm} a^n = a^{nm+1}
   \]

   The statement holds for \( m = 1 \) by induction.