13. Function space
Consider the set $C$ of continuous functions $f : [0,1] \to \mathbb{R}$. Show that by suitably defining an addition on $C$, and a multiplication with real numbers, one can make $C$ an additive vector space over $\mathbb{R}$.

(2 points)

14. The space of rank-2 tensors
a) Prove the theorem of ch.1 §4.3: Let $V$ be a vector space $V$ of dimension $n$ over $K$. Then the space of rank-2 tensors, defined via bilinear forms $f : V \times V \to K$, forms a vector space of dimension $n^2$.

b) Consider the space of bilinear forms $f$ on $V$ that is equivalent to the space of rank-2 tensors, and construct a basis of that space.

hint: On the space of tensors, define a suitable addition and multiplication with scalars, and construct a basis of the resulting vector space.

(5 points)

15. Cross product of 3-vectors
Let $x, y \in \mathbb{R}^3$ be vectors, and let $\epsilon_{ijk}$ be the Levi-Civita symbol. Show that the (covariant) components of the cross product $x \times y$ are given by

$$(x \times y)_i = \epsilon_{ijk} x^j y^k$$

(1 points)

16. Symmetric tensors
Let $V$ be an $n$-dimensional vector space over $K$ with some basis, let $f : V \times V \to K$ be a bilinear form, and let $t$ be the rank-2 tensor defined by $f$. Show that $f$ is symmetric, i.e. $f(x, y) = f(y, x)$ $\forall x, y \in V$, if and only if the components of the tensor with respect to the given basis are symmetric, i.e., $t_{ij} = t_{ji}$.

(2 points)
12) On $C$, define $(f+g)(x) := f(x) + g(x)$

If $f$ and $g$ are continuous, then so is the sum defined $(f+g)$

Furthermore, let $f(x) \in R$, $C$ inherits all of the other sum properties from $(\mathbb{R}, +)$

$\Rightarrow C$ is an additive group

Now define multiplication: let scalars $\lambda \in R$ by $(\lambda f)(x) := \lambda f(x)$

If $f$ is continuous, then so is the product $(\lambda f)$.

Furthermore, let $\lambda \in R$ and $f(x) \in R$, this multiplication will

satisfy the bilinear and associative, so it inherits these properties from $R$ under ordinary addition and multiplication of matrices.

Thus, $(1f)(x) = 1f(x) = f(x) \quad \forall x \in [0,1] \Rightarrow 1f = f$

$\Rightarrow C$ is a $R$-vector space
He knows that the rank-2 forms on one-to-one correspond to bilinear forms \( f(x, y) \). On the set of bilinear forms, define an addition by

\[
(f + g)(x, y) = f(x, y) + g(x, y)
\]

This makes the set of forms an additive group. Define a multiplication with scalars by

\[
(\lambda f)(x, y) = \lambda f(x, y), \quad \lambda \in k
\]

This makes the space of forms a k-vector space.

On the space of rank-2 forms \( T, u, v \), this corresponds to defining the forms \( t + u \) as the linear combination

\[
(t + u)_{ij} = t_{ij} + u_{ij}
\]

And the linear \( \lambda t \) as the scalar multiplication

\[
(\lambda t)_{ij} = \lambda t_{ij}
\]

The space of forms is now a k-vector space.

Consider a basis \( \{ e_i \} \) of \( V \), and construct \( n^2 \) forms

\[
E_{ij} := e_i \otimes e_j
\]

with \( (\text{w.r.t. } v) \) coordinates

\[
(E_{ij})^{k\ell} = \delta_i^k \delta_j^\ell
\]

Define a linear \( t \) as a linear combination of the \( E_{ij} \).
\[ t = \sum_{j} E_{ij} \text{ will vanish at } \chi_{i} \text{ for } i \neq k \]

This has no coordinates:

\[ t_{i} = \sum_{j} \delta_{ij} E_{i} \]

\[ \Rightarrow \text{ Any real-2 vector can be written as a linear combination of the } E_{ij}, \text{ with the coordinates } t_{ij} \text{ of } t \text{ on the coordinates } E_{ij}. \]

\[ t = \sum_{j} t_{ij} E_{ij} \]

\[ \Rightarrow \text{ The } E_{ij} \text{ span the plane } \]

In order for a vector to be in the null plane, all of its coordinates must be zero, so \[ t = 0 \Leftrightarrow t_{ij} = 0 \text{ for } i, j \]

\[ \Rightarrow \text{ The } E_{ij} \text{ are linearly independent} \]

\[ \Rightarrow \text{ The } \mathbb{R}^{2} \text{ real-2 vector } E_{ij} \text{ from a basis of the span of real-2 planes, } \text{ed plane the span has dimension } 2. \]

b) Let \( f_{ij} \) be the bilinear form that corresponds to the basis \( E_{ij} \). Then

\[ f_{ij}(e_{i}, e_{j}) = (E_{ij})_{kl} \delta_{ik} \delta_{lj} \]

For arbitrary \( x_{ij} \in \mathbb{V} \) we have

\[ f_{ij}(x_{ij}) = x_{ij}^{k} f(e_{i}, e_{j}) = x_{ij}^{k} \delta_{ik} \delta_{lj} = x_{ij}^{l} \]

\[ \Rightarrow \text{ The set of } \mathbb{R}^{2} \text{ bilinear forms } f_{ij} \text{ defined by } \]

\[ f_{ij}(x_{ij}) = x_{ij}^{l} \]
(15.) Let \( x = (x^1, x^2, x^3) \) and \( y = (y^1, y^2, y^3) \). The cross product is defined by

\[
x \times y = (x^2 y^3 - x^3 y^2, x^3 y^1 - x^1 y^3, x^1 y^2 - x^2 y^1)
\]

On the other hand,

\[
\epsilon_{ijk} x^i \times y^j \times y^k = \begin{cases} 
  x^2 y^3 - x^3 y^2 & \text{if } i = 1 \\
  x^3 y^1 - x^1 y^3 & \text{if } i = 2 \\
  x^1 y^2 - x^2 y^1 & \text{if } i = 3 
\end{cases} = (x \times y)^2
\]
16. Let \( f(x, y) = f(y, x) \) \( \forall x, y \in V \).

The two maps are defined by \( t_{ij} = f(e_i, e_j) \)

\[ t_{ji} = f(e_j, e_i) = f(e_i, e_j) = t_{ij} \]

Now let \( t_{ij} = t_{ji} \)

\( \Rightarrow f(e_i, e_j) = f(e_j, e_i) \) for all basis vectors \( e_i \)

Let for any \( x, y \in V \) in \( V \)

\[ f(x, y) = x^i y^j f(e_i, e_j) = y^j x^i f(e_j, e_i) = y^j x^i f(e_j, e_i) = f(y, x) \]