17. **R as a metric space**

Consider the reals $\mathbb{R}$ with $\rho : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $\rho(x, y) = |x - y|$. Show that this definition makes $\mathbb{R}$ a metric space.

(3 points)

18. **Limits of sequences**

a) Show that a sequence in a metric space has at most one limit.

*hint:* Assume there are two limits, and use the triangle inequality to show that they must be the same.

b) Show that every sequence with a limit is a Cauchy sequence.

(3 points)

19. **Banach space**

Let $B$ be a $K$-vector space ($K = \mathbb{R}$ or $\mathbb{C}$) with null vector $\theta$. Let $||\ldots|| : B \rightarrow \mathbb{R}$ be a mapping such that

(i) $||x|| \geq 0 \forall x \in B$, and $||x|| = 0$ iff $x = \theta$.

(ii) $||x + y|| \leq ||x|| + ||y|| \forall x, y \in B$.

(iii) $||\lambda x|| = |\lambda| \cdot ||x|| \forall x \in B, \lambda \in K$.

Then we call $||\ldots||$ a **norm** on $B$, and $||x||$ the **norm** of $x$.

Further define a mapping $d : B \times B \rightarrow \mathbb{R}$ by

$d(x, y) := ||x - y|| \forall x, y \in B$

Then we call $d(x, y)$ the **distance** between $x$ and $y$.

a) Show that $d$ is a metric in the sense of §4.5, i.e., that every linear space with a norm is in particular a metric space.

If the normed linear space $B$ with distance/metric $d$ is complete, then we call $B$ a **Banach space** or **B-space**.

b) Show that $\mathbb{R}$ and $\mathbb{C}$, with suitably defined norms, are B-spaces. (For the completeness of $\mathbb{R}$ you can refer to §4.5 example (3), and you don’t have to prove the completeness of $\mathbb{C}$ unless you insist.)

Now let $B^*$ be the dual space of $B$, i.e., the space of linear functionals $\ell$ on $B$, and define a norm of $\ell$ by

$||\ell|| := \sup_{||x||=1} \{ |\ell(x)| \}$

c) Show that the such defined norm on $B^*$ is a norm in the sense of the norm defined on $B$ above.

(In case you’re wondering: $B^*$ is complete, and hence a B-space, but the proof of completeness is difficult.)

(5 points)

20. **Hilbert space**

a) Show that the norm on a Hilbert space defined by §4.7 def. 1 is a norm in the sense of §4.6 def. 1.

*hint:* Use the Cauchy-Schwarz inequality (§4.7 lemma).

b) Show that the mappings $\ell$ defined in §4.7 def. 4 are linear forms in the sense of §4.3 def. 1(a).

(3 points)
Position definiteness of $y$ given in obvious.

From the triangle inequality:

By definition of $|x|$, we have $x, y \in \mathbb{R}$

$\Rightarrow 0 \leq 2(x-y)(y-z) + 2|x-y| |y-z|

\Rightarrow (x-z)^2 = x^2 - 2xz + z^2 \leq x^2 - 2xz + z^2 + 2(x-y)(y-z) + 2|x-y| |y-z|

\Rightarrow x^2 - 2xz + z^2 + 2(y-z)^2 - 2(y-z)^2 - 2|x-y| |y-z|

\Rightarrow x^2 - 2xz + z^2 + 2y^2 - 2y^2 + 2|x-y| |y-z|

\Rightarrow (x-y)^2 + (y-z)^2 + 2|x-y| |y-z|

\Rightarrow (|x-y| + |y-z|)^2

\Rightarrow (x-z)^2 \geq 0 \Rightarrow

|x-z| \leq |x-y| + |y-z| \quad \text{triangle inequality}
12. a) Let \( x_n \) be a sequence, \( y_n \). Suppose \( x_n \to x^* \) and \( y_n \to y^* \).

\[
\Rightarrow s(x^*, y^*) \leq s(x^*, x_n) + s(y^*, y_n) + k x_n \quad \forall x_n \text{ by the triangle inequality.}
\]

Thus \( \lim_{n \to \infty} s(x^*, x_n) = \lim_{n \to \infty} s(y^*, y_n) = 0 \)

\[
\Rightarrow s(x^*, y^*) = 0 \quad \Rightarrow x^* = y^*
\]

b) Let \( x_n \) be a sequence, \( x^* \): \( x_n \to x^* \)

\[
\Rightarrow s(x_n, x_m) \leq s(x_n, x^*) + s(x_m, x^*)
\]

Let \( \delta > 0 \). Then \( \exists N \in \mathbb{N} : s(x_n, x^*) < \delta \quad \forall n > N \)

Now let \( \epsilon > 0 \) and \( \delta = \epsilon/2 \). Then \( \exists N > 0 : \)

\[
s(x_n, x_m) \leq s(x_n, x^*) + s(x_m, x^*) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

provided \( n, m > N \).
1. a) \[ d(x,y) = ||x-y|| > 0 \] for all \( x, y \in \mathbb{R} \) by property (i) of \( \mathbb{R} \).

\[ d(x,y) = 0 \iff x = y \iff x = y \]

\[ \Rightarrow \text{positive definiteness} \checkmark \]

\[ d(y-x) = ||y-x|| = ||-(x-y)|| = ||x-y|| \] by property (iii)

\[ = d(x,y) \]

\[ \Rightarrow \text{symmetry} \checkmark \]

\[ d(x+t) = ||x+t|| = ||x-y+y-t|| \leq ||x-y|| + ||y-t|| \] by property (c)

\[ = d(x,y) + d(y,t) \]

\[ \Rightarrow \text{triangle inequality} \checkmark \]

b) Consider \( \mathbb{R} \) as an \( \mathbb{R} \)-vector space and define

\[ ||x|| = |x| \neq 0 \quad \forall x \in \mathbb{R} \]

Then \( ||\cdot|| : \mathbb{R} \rightarrow \mathbb{R} \) has all the properties required of a norm. Furthermore, if \( \mathbb{R}^\mathbb{R} (\mathbb{R}) \rightarrow \) any linear space has a limit \( \Rightarrow \mathbb{R} \) is complete and hence a \( \mathbb{R} \)-space.

Define the vector norm \( \mathbb{C} \) in \( \mathbb{C} \)-space defined by

\[ ||z|| = |z| = \sqrt{z\bar{z}} \]

This makes \( \mathbb{C} \) a \( \mathbb{R} \)-space (among completeness).
1. Let the norm be defined for a norm $\| \cdot \|_1$.

(i) \[ \| x \| = \text{np} \left( \frac{|x(x)|}{\|x\|} \right) \Rightarrow \| x \|_{\infty} \leq \| x \|_1 \]

The unit vector in $\mathbb{R}^n$ is the null functional $\phi$, defined by $\phi(x) = 0 \neq x \in \mathbb{R}$.

\[ \Rightarrow \| x \| = 0 \]

ii. Let $\| x \| = 0$. Then the set of $x \in \mathbb{R}$ with $\| x \| = 1$ forms $\mathbb{Q}$, i.e., the set of $x$.

\[ \Rightarrow \| x \| = 0 \text{ if } x \neq 0 \]

(iii) \[ \| x \| = \text{np} \left( \frac{|x(x)|}{\|x\|} \right) \leq \text{np} \left( \frac{|x(x)|}{\|x\|} + |x(x)| \right) \]

\[ \Rightarrow \| x \| = \| x \|_{\infty} + \| x \|_1 \]

Let $i$ be such that the triangle inequality for $C$.

\[ \| x \| = \text{np} \left( \frac{|x(x)|}{\|x\|} \right) + \text{np} \left( |x(x)| - |x(x)| \right) \]

\[ \Rightarrow \| x \|_{\infty} + \| x \|_1 \neq 0 \text{ for } x \in \mathbb{R}, x \neq 0 \]
20. a) \( (x,x) = (x,x)^* \in \mathbb{R} \) \( \forall (x,x) \geq 0 \Rightarrow \|x\| = (x,x)^{1/2} \geq 0 \forall x \in \mathbb{R} \)

\( \alpha \) \( (x,x) = 0 \) \( \iff \) \( x = 0 \)

\( \|x+y\|^2 = (x+y,x+y) = (x,x) + (y,y) + (x,y) + (y,x) \)

\( = (x,x) + (y,y) + (x,y) + (y,x) \)

\( = \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \)

\( \Rightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\|^2 + \|y\|^2) \)

\( \Rightarrow \|x+y\| \leq \|x\| + \|y\| \)

\( \Rightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\|^2 + \|y\|^2) \)

\( \Rightarrow \|x+y\| \leq \|x\| + \|y\| \)

\( \Rightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\|^2 + \|y\|^2) \)

\( \Rightarrow \|x+y\| \leq \|x\| + \|y\| \)

\( \Rightarrow \|ax\|^2 = (ax,ax) = a^2(x,x) = |a|^2 \|x\|^2 \)

\( \Rightarrow \|ax\| = |a|\|x\| \forall a \in \mathbb{C}, x \in \mathbb{H} \Rightarrow \|\cdot\| \text{ is a norm} \)

b) The definition \( \ell(x) = (y,x) \) is linear:

(1) \( \ell(x+z) = (y,x+z) = (y,x) + (y,z) = \ell(x) + \ell(z) \forall x,z \in \mathbb{H} \)

(2) \( \ell(\lambda x) = (y,\lambda x) = \lambda (y,x) = \lambda \ell(x) \forall x \in \mathbb{H}, \lambda \in \mathbb{C} \)

\( \Rightarrow \ell \) is a linear form