18. **Electrostatics in \( d \) dimensions (continued from Assignment \# 5)**

Consider Problem \#18 as set up in Problem Assignment \#5.

b) Calculate and plot the potential \( \varphi \) and the field \( \mathbf{E} \) for \( d = 2 \) for the case of a homogeneously charged disk, \( \rho(\mathbf{x}) = \rho_0 \Theta(r_0 - |\mathbf{x}|) \).

*hint:* It is easiest to proceed as in the 3-\( d \) case, see Problem 17.

*note:* This problem plays an important role in the theory of the Kosterlitz-Thouless transition, for which part of the 2016 Nobel prize in Physics was awarded.

c) The same for \( d = 1 \) for the case of a uniformly charged rod, \( \rho(x) = \rho_0 \Theta(x_0^2/4 - x^2) \).

*hint:* Integrate Poisson’s formula directly. (8 points)

19. **Helmholtz equation**

Find the most general Fourier transformable solution of the Helmholtz equation

\[
(\kappa^2 - \nabla^2)\varphi(x) = 4\pi \rho(x)
\]

in terms of an integral.

*hint:* The answer is a generalization of Poisson’s formula. (3 points)

20. **Quadrupole moments**

a) Consider a localized charge density as in ch.2 \( \S 3.1 \) and carry the expansion of the potential to \( O(1/r^3) \).

Show that the potential to that order is given by

\[
\varphi(x) = \frac{1}{r} Q + \frac{1}{r^3} \mathbf{x} \cdot \mathbf{d} + \frac{1}{r^5} \sum_{i,j} x_i x_j Q_{ij} + \ldots
\]

with \( Q \) the total charge and \( \mathbf{d} \) the dipole moment, and determine the quadrupole tensor \( Q_{ij} \).

b) Show that the quadrupole tensor is independent of the choice of the origin provided the total charge and the dipole moment vanish.

c) Consider a homogeneously charged ellipsoid \((x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1\) and calculate the quadrupole tensor \( Q_{ij} \) with respect to the ellipsoid’s center. Check to make sure that the result for \( Q_{ij} \) is traceless.

d) Let the charge density be invariant under rotations about the \( z \)-axis through multiples of an angle \( \alpha \), with \( |\alpha| < \pi \). Show that in this case the quadrupole tensor has the form

\[
\begin{pmatrix}
q & 0 & 0 \\
0 & q & 0 \\
0 & 0 & -2q
\end{pmatrix}
\]

Make sure your result from part c) conforms with this for the special case \( a = b \).

e) Consider the homogeneously charged ellipsoid from Problem 20 c), and calculate the quadrupole moments \( Q_{2m} \) as defined in ch.2 \( \S 3.5 \).

(10 points)