47. **Synchrotron radiation** (continued)

c) Use the integral representation of the Bessel function \( J_{2m} \) from Problem 46 to show that \( P_m \), the power radiated into the \( m \)-th harmonic, can be expressed in terms of Bessel functions as

\[
P_m = \frac{e^2}{R} \frac{m \omega_0}{\omega} \left[ 2\beta^2 J'_{2m}(2m\beta) - (1 - \beta^2) \int_0^{2m\beta} dx J_{2m}(x) \right]
\]

where \( \beta = v/c \) and \( J'_{2m} \) is the derivative of \( J_{2m} \) with respect to its argument.

*note:* One can also obtain this by integrating the final result from ch. 4 §6.2 over the angles, but that’s harder.

d) Show that for \( \beta \approx 1 \) the power peaks at \( m_{\text{max}} \propto \gamma^3 \), where \( \gamma = 1/\sqrt{1 - \beta^2} \).

*hint:* Analyze the \( J' \) contribution in detail and do what you can on the second term.

e) Estimate the peak frequency of the power spectrum and the corresponding wavelength for the Advanced Light Source (1.9 GeV electrons in a circular orbit with radius \( R \approx 20 \text{m} \)), and for a typical radio galaxy (5 GeV electrons in a field \( B \approx 5 \mu \text{G} \)).

(7 points)

f) **This part is optional.** If you run out of stuff to do, work through Jackson ch. 14.6 to derive his result (14.84) for the angular distribution of the synchrotron radiation. Start with the expression for the power spectrum in the parallel-polarization state in § 6.3 and integrate over \( T \) to undo the Wigner structure. Then follow Jackson’s logic and approximations, taking into account the difference between his coordinate system and ours, and also the different zeros of time. Repeat this for the perpendicular polarization, then add up the two and integrate over the frequency to obtain the angular distribution.

*note:* Note that the approximations in the calculation are valid only for large \( \gamma \) and small angles about the orbital plane, and also involve a large-frequency approximation. Coming up with a *complete* expression for the angular distribution for all angles that captures both the ultrarelativistic and nonrelativistic limits is remarkably difficult.

48. **Scattering by a dielectric sphere**

a) Argue on general grounds that the dipole moment of a dielectric sphere (radius \( a \), dielectric constant \( \epsilon \)) subject to an external electric field \( E_{\text{ext}} \) is given by

\[
d = f(\epsilon) a^3 E_{\text{ext}} ,
\]

where the function \( f \) has the properties \( f(\epsilon \to 1) = 0, f(\epsilon \to \infty) = \text{const.} \). (We will determine \( f(\epsilon) \) explicitly next week, see Problem 49.)

b) Find the scattering cross section for radiation with wavelength \( \lambda \gg a \) scattered by the sphere.

(6 points)
c) Problem 4.6 \rightarrow
\begin{align*}
\tau \omega^2 (y) &= \frac{d^2}{dy^2} \omega_2 (mx) \omega_2 (y - \frac{x}{2}) \\
\Rightarrow \tau \omega^2 (y) &= -\frac{d}{dy} \omega_2 (mx) \left[ \frac{1}{m} \frac{x}{2} \right] \omega_2 (y - \frac{x}{2}) \\
\therefore \int_0^1 \tau \omega^2 (x) &= \frac{d}{dy} \left[ \omega_2 (mx) \left( \frac{1}{m} \frac{x}{2} \right) \right] _0^1 (y - \frac{x}{2})
\end{align*}
-> \( P_n = \frac{e^{i\omega_0}}{R} \left\{ -2A^2 \int_0^1 \frac{dx}{x} \sum_{m=0}^n c_m (\omega A) \left( \frac{2m}{A} \right) \right\} \)

1. \( R \approx \frac{\lambda A^2}{\epsilon} \left[ 2A^2 \sum_{m=0}^n c_m (\omega A) - (1-A^2) \int_0^1 \frac{dx}{x} f_m (x) \right] \)

\[ \text{d) Problem 46} \quad \text{For } m \ll \frac{1}{\lambda A}, \lambda A \ll \frac{1}{\lambda} \]

\[ w_n \sum_{m=0}^n c_m (\omega A) \propto \begin{cases} \frac{m^{1/2}}{\lambda} & \text{for } m \ll \lambda^2 \\ \frac{m e^{-2m/\lambda^2}}{\lambda^2} & \text{for } m \gg \lambda^2 \end{cases} \]

This contribution to \( P_n \) gets end \( m = \frac{1}{\lambda A} = \lambda^2 \)

Now consider the second contribution. Define \( f_n (\lambda) = \int_0^1 dx \sum_{m=0}^n c_m (\omega A) \)

\[ \frac{d}{d\lambda} f_n (\lambda) = 2m f_m (\omega A) \]

\[ \text{Prove that} \propto \begin{cases} \frac{m^{1/2}}{\lambda} & \text{for } m \ll \lambda^2 \\ \frac{m e^{-2m/\lambda^2}}{\lambda^2} & \text{for } m \gg \lambda^2 \end{cases} \]

It must to integrate this over \( \lambda \) to obtain \( f_n (\lambda) \), but this cannot change the exponential dropoff for \( m \gg \lambda^2 \)

-> It expect this weak contribution to \( P_n \) to be negligible similar to the first one, but negligible number become of 1+ \( (1-A^2)^m \) in practice which is unmet :\]
Check this numerically:

\[ J_p[m, x] := \frac{(\text{BesselJ}[m-1, x] - \text{BesselJ}[m+1, x])}{2} \]
\[ J[m, x] := \text{BesselJ}[m, x] \]

\[ \beta = 0.99 \]
\[ \gamma = \frac{1}{\sqrt{1 - \beta^2}} \]
\[ \gamma^3 \]
\[ 0.99 \]
\[ 7.08881 \]
\[ 356.222 \]

\[ \beta = 0.99 \]
\[ \gamma = \frac{1}{\sqrt{1 - \beta^2}} \]
\[ \gamma^3 \]
\[ P[m] := m (2 \beta^2 J_p[2 m, 2 m \beta] - (1 - \beta^2) \text{NIntegrate}[J[2 m, x], \{x, 0, 2 m \beta\}]) \]
\[ \text{DiscretePlot}[P[m], \{m, 1, 500\}] \]
\[ 0.99 \]
\[ 7.08881 \]
\[ 356.222 \]

$\text{Aborted}$

Here is the first contribution to the spectrum:

\[ P_1[m] := 2 m \beta^2 J_p[2 m, 2 m \beta] \]
\[ \text{DiscretePlot}[P_1[m], \{m, 1, 1000\}] \]

It peaks for \( m \) between 200 and 250, whereas \( m_{\text{max}} = 356 \), so that's consistent. Now calculate the second contribution:
DiscretePlot[m (1 - \(\beta^2\)) NIntegrate[J[2 m, x], {x, 0, 2 m \beta}], {m, 1, 500}]

So this peaks a bit later than the first contribution, but it's always smaller:

Do[
  Print[P1[n 100]],
  {n, 5, 10}]

1.95462
1.75526
1.55763
1.37012
1.197
1.04009

Do[
  Print[n 100 (1 - \(\beta^2\)) NIntegrate[J[2 n 100, x], {x, 0, 2 n 100 \beta}]],
  {n, 5, 10}]

0.657352
0.617884
0.56815
0.514239
0.459964
0.407651
e) \[ E = \gamma mc^2 = \gamma \times 0.51 \text{ keV} = 1.9 \times 10^2 \text{ GeV} \]

\[ \gamma = 2.725 \quad (\Rightarrow \frac{\nu}{c} = 0.999999996 \ldots) \]

\[ \omega_0 = \frac{\frac{\omega}{c^2}}{R} = \frac{\gamma}{c^2} \sqrt{1 - \frac{1}{\gamma^2}} = 1.5 \times 10^{-7} \text{ Hz} \]

\[ \omega_{\text{max}} = \omega_0 \gamma^2 = 8 \times 10^{17} \text{ Hz} \]

\[ \lambda_{\text{max}} = \frac{2 \pi}{\omega_{\text{max}}} = \frac{2 \pi c}{\omega_{\text{max}}} = 2.4 \times 10^{-7} \text{ m} = 24 \text{ Å} \]

\[ \text{Galaxy:} \quad \gamma = \frac{5 \times 10^3}{0.51} = 10^4 \]

\[ \omega_0 = \frac{\frac{\omega}{c^2}}{m c^2} = \frac{4.8 \times 10^{-10} \times 5 \times 10^{-6}}{9.1 \times 10^{-28} \times 3 \times 10^{10} \times 10^{-4}} \text{ Hz} = 10^{-2} \text{ Hz} \]

\[ \omega_{\text{max}} = \omega_0 \gamma^2 = 10^{-2} \times 10^{12} \text{ Hz} = 10^{10} \text{ Hz} \]

\[ \lambda_{\text{max}} = \frac{2 \pi c}{\omega_{\text{max}}} = 19 \text{ Å} \quad \text{radio waves} \]
48. a) The sphere is homogeneous and isotropic, so the polarization vector is proportional to the external field:
\[ \vec{P} \propto \vec{E}_{\text{ext}} \]

The dipole moment is the volume time \( \vec{P} \rightarrow \vec{d} \times a^3 \vec{E}_{\text{ext}} \)

The proportionality factor will depend on the geometry and on the material properties, i.e., on \( \varepsilon \):
\[ \vec{d} = f(\varepsilon) a^3 \vec{E}_{\text{ext}} \]

Now, for \( \varepsilon \rightarrow 0 \) the susceptibility \( \chi \) and hence \( \vec{P} \) \& \( \vec{d} \) vanish, so we may have
\[ f(\varepsilon \rightarrow 0) = 0 \]

For \( \varepsilon \rightarrow \infty \) the susceptibility diverges
\[ \text{Hence, there is no field inside the sphere} \]
\[ \vec{P} = \vec{E}_{\text{ext}} \times (\text{a pure geometric factor}) \]
\[ f(\varepsilon \rightarrow \infty) = \infty \]

with the electric charge determined by the geometry.

b) If the conductor \( \rightarrow \) for electric dipole, reduce, the total reduced power is
\[ P_{\text{ed}} = \frac{1}{3} \varepsilon_0 \left( \frac{\vec{d}}{c} \right)^2 = \frac{2\omega_0}{3c^2} \vec{d}^2 \]

\( \vec{d} \) from part a) \( \rightarrow \) here
\[ \vec{d} = f(\varepsilon) a^3 \vec{E}_{\text{ext}} \]
For the electric dipole to give the dominant contribution, \( \frac{\omega}{c} \) must vary slowly over the spin, i.e.,

\[ \frac{\omega}{c} \ll \frac{\omega}{\lambda c} \]

or

\[ \lambda = \frac{\omega}{c} \gg \ell \]

In this limit, we have

\[ P_{\text{scatt}} = P_{\text{ed}} = \frac{\omega^4}{2c^3} a^2 \cdot \frac{\ell}{c^2} \left( f(E) \alpha^2 \right)^2 \mathcal{E}_x^2 \]

and the Poynting vector is

\[ |\mathbf{P}| = \frac{\mathcal{E}_x^2}{c} \]

ed the scattering cross section is

\[ \sigma = \frac{P_{\text{scatt}}}{|\mathbf{P}|} = \frac{\omega^4}{2c^4} \alpha^6 \left( f(E) \right)^2 \]

\[ = \frac{\sigma}{\lambda^4} (2c)^4 \left( f(E) \right)^2 \mathcal{E}_x^2 \]

vill \( \lambda = \omega/c = \ell/c \) be the width of the radiative