49. **Dipole moment of a dielectric sphere**

Consider a dielectric sphere of radius \( a \) and dielectric constant \( \epsilon \) subject to an external field \( E_{\text{ext}} \).

**a)** Find the electric induction \( D \) inside and outside the sphere as a function of \( E_{\text{ext}} \).

*hint:* Use symmetry arguments to write the potential \( \varphi \) outside the sphere as a superposition of the linear potential that belongs to \( E_{\text{ext}} \) and a dipole potential, and inside the sphere as proportional to the former. This introduces two unknown constants: One for the superposition, and one for the proportionality assumption. Use the boundary condition that the potential must be continuous at the surface of the sphere to eliminate one of those.

**b)** Show that the Maxwell equation \( \nabla \cdot D = 0 \) implies that at the interface between a dielectric and vacuum the normal component of \( D \) must be continuous. This constitutes a second boundary condition.

**c)** Combine the results of parts a) and b) to find the dipole potential of the sphere that is induced by \( E_{\text{ext}} \), i.e., determine the function \( f(\epsilon) \) from Problem 48.

(7 points)

50. **Electromagnetic waves in a dielectric medium**

Coarse grain the dynamical Maxwell equations and assume that the simple relations \( D = \epsilon E \) and \( B = \mu H \) from ch.5 §1, with constant \( \epsilon \) and \( \mu \), remain valid for time-dependent fields. Derive the wave equation for any of the fields (e.g., \( H \)), in analogy to ch.3. What is the speed of light in the medium?

(2 points)

51. **Density response, and the dielectric function**

Consider the coarse-grained charge density \( \bar{\rho}(x) \) from ch. 5 §1.1, and its Fourier transform \( \bar{\rho}(q) \). It is generated by the response of the electrons in the material to an electric potential \( \varphi(q) \). Let \( \delta n \) be the deviation of the (coarse-grained) electron density from its average value, and assume a linear relation between \( \delta n \) and \( -e\varphi \), with \( -e \) the electron charge:

\[
\delta n(q) = -e\chi_{sc}(q)\varphi(q).
\]

Here \( \varphi \) is the total electric potential, which consists of the coarse-grained internal one and the externally applied one, if any, and \( \chi_{sc} \) is called the screened density susceptibility.

**a)** Show that the dielectric function \( \epsilon(q) \), defined as the Fourier transform of the function \( \epsilon(x) \) from ch. 5 §1.1, is related to \( \chi_{sc} \) by

\[
\epsilon(q) = 1 - v_{e}(q)\chi_{sc}(q)
\]

where \( v_{e}(q) = 4\pi e^{2}/q^{2} \) is the Fourier transform of the Coulomb potential.

**b)** Consider the density response of the electrons to the external part of the electric potential \( \varphi_{\text{ext}} \), which determines the dielectric induction via \( D = -\nabla \varphi_{\text{ext}} \):

\[
\delta n(q) = -e\chi_{n}(q)\varphi_{\text{ext}}(q)
\]

with \( \chi_{n} \) the density susceptibility. What is the relation between \( \chi_{n} \) and \( \chi_{sc} \)?

.../over
c) What is the behavior of $\chi_{sc}(q)$ in the long-wavelength limit, $q \to 0$, in a dielectric? What is the behavior of $\chi_n(q)$ in the same limit? Does the result make physical sense?

d) In a metal, as opposed to a dielectric, $\chi_{sc}(q \to 0) \to \chi_0 = \text{const.} < 0$. What does this imply for $\epsilon(q \to 0)$, and for $\chi_n(q \to 0)$? Do these results make physical sense?

notes:

• The negative sign of $\chi_{sc}(q \to 0)$ is conventional (although one might argue that this is an unfortunate convention).

• If you find all of this confusing, you are not alone. Good references include Chapter 3 of the book *Theory of Quantum Liquids* by D. Pines and P. Nozières, and the lecture notes by Peter Young at UC Santa Cruz (https://young.physics.ucsc.edu/232/rpa2.pdf).

(9 points)
49. (a) Spherical symmetry → the potential will depend only on the external field vector \( \vec{E}_{\text{ext}} \), when potential is \( -\vec{E}_{\text{ext}} \cdot \vec{r} \).

1st case: \( r > a \) (outside)

The potential will be that of the external field plus a modification due to the induced dipole moment of the sphere. The latter can also depend on \( \vec{E}_{\text{ext}} \).

\[
\Phi(r > a) = -\vec{E}_{\text{ext}} \cdot \vec{r} + \frac{Q_{\text{ind}}}{r^2} \quad \text{will} \quad C_1 = \text{const.}
\]

\[
\left\{ \begin{array}{l}
\Phi_{\text{ind}}(r) \quad \text{induced potential}
\end{array} \right.
\]

2nd case: \( r < a \) (inside)

The only source of Laplace's eq. Let is finite at \( r = 0 \) and depends only on a finite vector \( \vec{E}_{\text{ext}} \) is

\[
\Phi(r < a) = C_2 \vec{E}_{\text{ext}} \cdot \vec{r} \quad \text{will} \quad C_2 = \text{const.}
\]

\[
\left\{ \begin{array}{l}
\Phi_{\text{ext}} = -C_2 \vec{E}_{\text{ext}}
\end{array} \right.
\]

The field inside the sphere is proportional to the field outside the sphere, which makes sense by symmetry.

Remark (1): One can also argue that this follows directly from the spherical symmetry of the problem.
boundary condition I: \( \psi \) must be continuous at \( r = a \)

\[ - \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{\sigma^2} \frac{\partial}{\partial \phi} \left( \sigma^2 \frac{\partial \psi}{\partial \phi} \right) \]

\[ \Rightarrow \quad \sigma^2 = \sigma^2 \left( 1 + \frac{c_2}{a^2} \right) \]

For the solution inside and outside the sphere or can

\[ \nabla \cdot E_{\text{ext}} = E \frac{\partial E_{\text{ext}}}{\partial r} = E \left( 1 - \frac{c_2}{a^2} \right) \frac{\partial E_{\text{ext}}}{\partial r} \quad (i) \]

\[ \nabla \cdot E_{\text{int}} = - \nabla \cdot \bar{E}_{\text{int}} = E_{\text{int}} - \nabla \cdot \left( \bar{E}_{\text{int}} \cdot \hat{r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial E_{\text{int}}}{\partial r} \right) \]

\[ \Rightarrow \quad E_{\text{int}} = \frac{c_2}{r^2} \left[ \left( \bar{E}_{\text{int}} \cdot \hat{r} \right) \right] \quad (ii) \]

b) Write a \( \mathbf{A} \mathbf{x} \) about unit with normal vector \( \hat{e} \).

\[ \nabla \cdot \mathbf{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z = 0 \]

\[ \Rightarrow \frac{\partial}{\partial z} \quad \text{must be continuous across the surface, that} \]

\[ \frac{\partial}{\partial z} \rightarrow \infty \quad \text{at limit} \quad \nabla \cdot \mathbf{A} = 0 \quad \text{within } \Omega_2 \]

C) Part c) (i) + (ii) plus part b) \[ \Rightarrow \]

\[ E \left( 1 - \frac{c_2}{a^2} \right) = 1 + \frac{c_2}{a^2} \left( 3 - 1 \right) a^2 = 1 + \frac{c_2}{a^2} \]

\[ \Rightarrow \quad E - 1 = (c_2 + 1) \frac{c_2}{a^2} \]
\[ E_{\text{ext}} = \frac{1}{E+2} E_{\text{ext}}^1 \]

\[ d = \frac{4\pi}{3} a^3 \left( \frac{E-1}{E+2} \right) E_{\text{ext}}^1 \]

\[ f(x) = \frac{E-1}{E+2} \]
Coom-jovinid formull eqs:
\[ \nabla \cdot \mathbf{A} = 0 \quad (1) \]
\[ \nabla \cdot \mathbf{J} = 0 \quad (2) \]
\[ \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} + \nabla \times \mathbf{E} = 0 \quad (3) \]
\[ -\frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{H} = 0 \quad (4) \]

\[ \mathbf{D} = \varepsilon \mathbf{E}, \quad \varepsilon = \frac{\mu_0}{\varepsilon_0} \]

(2)
\[ \nabla \times \nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = \frac{\varepsilon_0}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{\varepsilon_0}{c} \frac{\partial}{\partial t} \mathbf{E} \]

\[ \mathbf{D} = \frac{\varepsilon_0}{c} \frac{\partial}{\partial t} \mathbf{E} \]

\[ \nabla \times \nabla \times \mathbf{H} = \frac{1}{c} \mathbf{E} \times \left( \nabla \times \mathbf{H} \right) - \frac{1}{c^2} \mathbf{E} \times \left( \nabla \times \nabla \times \mathbf{H} \right) = 0 \]

\[ \left( \frac{\varepsilon_0}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \mathbf{H} = 0 \quad \text{wave eq.} \]

\[ \text{The speed of light in the medium is } c/\sqrt{\varepsilon_0} \]

\[ \]
$\Sigma(1) \ a) \quad \tilde{E}(\tilde{x}) = -\nabla \phi(\tilde{x}) \Rightarrow \tilde{E}(\tilde{y}) = -i \tilde{\gamma} \phi(\tilde{y})$
\[\Rightarrow \tilde{i} \tilde{\gamma} \tilde{E}(\tilde{y}) = \frac{\tilde{q}}{\tilde{E}} \phi(\tilde{y})\]
\[\Rightarrow \phi(\tilde{y}) = \frac{\tilde{q}}{\tilde{E}} i \tilde{\gamma} \tilde{E}(\tilde{y})\]

Now, $\delta \nu(\tilde{y}) = \chi_{sc}(\tilde{y}) (-e) \phi(\tilde{y})$
\[\Rightarrow -ie \frac{\tilde{q}}{\tilde{E}} \chi_{sc}(\tilde{y}) \tilde{q} \cdot \tilde{E}(\tilde{y})\]

Let $\tilde{e} \cdot \tilde{E}(\tilde{x}) = \frac{\tilde{q}}{\tilde{E}} \chi_{sc}(\tilde{y}) \tilde{q} \cdot \tilde{E}(\tilde{y})$
\[\Rightarrow \delta \nu(\tilde{y}) = \chi_{sc}(\tilde{y}) (-e) \delta \nu(\tilde{y}) + \chi_{sc} \delta \nu(\tilde{y})\]
\[\Rightarrow \delta \nu(\tilde{y}) = \chi_{sc}(\tilde{y}) (-e) \frac{\tilde{q}}{\tilde{E}} \chi_{sc}(\tilde{y}) \tilde{q} \cdot \tilde{E}(\tilde{y}) + \frac{\tilde{q}}{\tilde{E}} \chi_{sc}(\tilde{y}) \tilde{q} \cdot \tilde{E}(\tilde{y})\]
\[\Rightarrow \delta \nu(\tilde{y}) = \chi_{sc}(\tilde{y}) \tilde{q} \cdot \tilde{E}(\tilde{y}) \left[ 1 - \frac{\tilde{q}}{\tilde{E}} \chi_{sc}(\tilde{y}) \right]\]
\[\Rightarrow \delta \nu(\tilde{y}) = \chi_{sc}(\tilde{y}) \tilde{q} \cdot \tilde{E}(\tilde{y})\]

b) $\delta \nu(\tilde{y}) = -ie \frac{\tilde{q}}{\tilde{E}} \chi_{sc}(\tilde{y}) \tilde{q} \cdot \tilde{E}(\tilde{y})$

But we also have

$\delta \nu(\tilde{y}) = -ie \frac{\tilde{q}}{\tilde{E}} \chi_{sc}(\tilde{y}) \tilde{q} \cdot \tilde{E}(\tilde{y})$
\[\Rightarrow -ie \frac{\tilde{q}}{\tilde{E}} \chi_{sc}(\tilde{y}) \tilde{q} \cdot \tilde{E}(\tilde{y})\]
\[ X_{cL}(\bar{q}) = X_{se}(\bar{q})/\varepsilon(\bar{q}) \]

\[ X_{cL}(\bar{q}) = \frac{X_{se}(\bar{q})}{1 - \nu_c(\bar{q})X_{se}(\bar{q})} \]

c) \text{ A dielectric, } \varepsilon(\bar{q} \to 0) = \varepsilon = \text{const.} > 1 \]

\[ X_{se}(\bar{q} \to 0) = -(\varepsilon - 1) \frac{1}{\varepsilon \varepsilon_0 q^2} \]

\[ X_{se}(\bar{q} \to 0) < 0 \text{ and vanishes as } q^4 \]

\[ X_{cL}(\bar{q} \to 0) = -\frac{\varepsilon - 1}{\varepsilon \varepsilon_0} \frac{q^2}{1 + \frac{\varepsilon \varepsilon_0}{\varepsilon_0} (\varepsilon - 1) \frac{q^2}{\varepsilon \varepsilon_0}} \]

\[ \frac{\varepsilon - 1}{\varepsilon \varepsilon_0} \frac{q^2}{1 + \frac{\varepsilon \varepsilon_0}{\varepsilon_0} (\varepsilon - 1) \frac{q^2}{\varepsilon \varepsilon_0}} \]

\[ \text{interpretation}: \] A homogeneous field just displaces all electrons by an equal distance and does not lead to a voltage drop. The difference between \( X_{se} \) and \( X_{cL} \) just reflects the (finite) distance inside the medium, this field is the total field.

\[ \text{d) In metals, } X_{se}(\bar{q} \to 0) = X_0 < 0 \to \]

\[ X(\bar{q} \to 0) = 1 - \frac{\varepsilon \varepsilon_0}{q^2} X_0 \to +\frac{12}{q^2} \to +\infty \text{ when } \bar{q}^2 = 4\varepsilon_0^2 X_0 \]

\[ X_{cL}(\bar{q} \to 0) \to X_0 \frac{q^2}{1 - \frac{\varepsilon \varepsilon_0 X_0}{q^2}} = X_0 \frac{q^2}{12 + q^2} \to 0 \]
Interpretation: In a medium, an infinitesimal field produces an infinitesimal polarization, i.e., 
\[ X \rightarrow 0 \text{ as } \epsilon \rightarrow 0 \text{ in homogeneous limit.} \]

However, in a finite system and homogeneous external field, the limit as \( \epsilon \to 0 \) is a finite displacement of all electrons, i.e., 
\[ X = 0 \text{ for } (\epsilon \to 0). \]