9. **Particle in homogeneous \( E \) and \( B \) fields**

Consider a point particle (mass \( m \), charge \( e \)) in homogeneous fields \( B = (0, 0, B) \) and \( E = (0, E_y, E_z) \). Treat the motion of the particle nonrelativistically.

a) Show that the motion in \( z \)-direction decouples from the motion in the \( x-y \) plane, and find \( z(t) \).

b) Consider \( \xi := x + iy \). Find the equation of motion for \( \xi \), and its most general solution.

*hint:* Define the **cyclotron frequency** \( \omega = eB/mc \), and remember how to solve inhomogeneous ODEs.

c) Show that the time-averaged velocity perpendicular to the plane defined by \( B \) and \( E \) is given by the **drift velocity**

\[
\langle v \rangle = c E \times B / B^2
\]

Show that \( E_y/B \ll 1 \) is necessary and sufficient for the non relativistic approximation to be valid.

d) Show that the path projected onto the \( x-y \) plane can have three qualitatively different shapes, and plot a representative example for each.

(6 points)

10. **Harmonic oscillator coupled to a magnetic field**

Consider a charged 3-d classical harmonic oscillator (oscillator frequency \( \omega_0 \), charge \( e \)). Put the oscillator in a homogeneous time-independent magnetic field \( B = (0, 0, B) \). Show that the motion remains oscillatory, and find the oscillation frequencies in the directions parallel and perpendicular, respectively, to \( B \).

(4 points)

11. **Relativistic motion in parallel electric and magnetic fields**

Consider a relativistic charged particle (mass \( m \), charge \( e \)) in parallel homogeneous electric and magnetic fields \( E = (0, 0, E) \), \( B = (0, 0, B) \).

a) Show that the equation of motion for the \( z \)-component of the momentum \( p_z \) decouples from \( p_x \) and \( p_y \), and that the momentum perpendicular to the \( z \)-axis is a constant of motion: \( p_x^2 + p_y^2 \equiv p_{\perp}^2 = \text{const} \).

b) Choose the zero of time such that \( p_z(t = 0) = 0 \), and show that with a suitable chosen origin the \( z \)-component of the particle’s position can be written

\[
z(t) = \frac{1}{cE} \sqrt{T_0^2 + c^2e^2E^2t^2}
\]

where \( T_0 \) is the kinetic energy (i.e., the energy of the particle without the potential energy due to the fields) at time \( t = 0 \).

*hint:* If you have trouble, recall Einstein’s law of falling bodies from PHYS 611. You can find my version at http://pages.uoregon.edu/dbelitz/teaching/2013_14/PHYS_611-4/, Assignment # 5, Problem 21.

.../over
c) Introduce a parameter $\varphi$ via $d\varphi/dt = ceB/T(t)$, with $T(t)$ the time-dependent kinetic energy. Show that the orbit of the particle can be represented in the parametric form

$$
x = \frac{cp_\perp}{eB} \sin \varphi , \quad y = \frac{cp_\perp}{eB} \cos \varphi , \quad z = \frac{T_0}{eE} \cosh\left(\frac{E\varphi}{B}\right)
$$

and explicitly find the relation between $\varphi$ and $t$.

*hint:* Consider $\pi := p_x + ip_y$ and note that $|\pi| = p_\perp = \text{const.}$ by the result of part a).

d) Describe and visualize the orbit, and discuss the motion in the limits of large and small times.

(14 points)
9. a) Eq. of motion: 
\[ m \dddot{\mathbf{r}} = e \mathbf{E} + \frac{e}{c} \mathbf{v} \times \mathbf{B} \]

\[ \mathbf{u} + \mathbf{v} = (0,0,\mathbf{u}) \quad \mathbf{E} = (0,E_y,E_z) \]

\[ \begin{align*}
& m \dddot{x} = \frac{e}{c} \mathbf{u} \cdot \mathbf{B} \\
& m \dddot{y} = cE_y - \frac{e}{c} \mathbf{v} \times \mathbf{B} \\
& m \dddot{z} = cE_z
\end{align*} \]  

(1)

(2)

(3)

(4)

\[ x(t) = x_0 + v_0 t + \frac{eE_z}{mc} t^2 \]

b) Define \[ s := x + i y \]

(1) + i \cdot (2) \Rightarrow 

\[ m \dddot{s} = i cE_y - i \frac{e}{c} \mathbf{v} \times \mathbf{B} \]

Define \[ w := \frac{c}{mc} \text{ y-dynamic energy} \]

\[ \dddot{s} + iw \dot{s} = i \frac{e}{mc} E_y \]  

(5)

Speed while of inhomogeneous eq. \[ \dddot{s} = \frac{eE_y}{mc} \]

General while of homogeneous eq. \[ \dddot{s} = a e^{-i \omega t} \text{ (c.c.)} \]

\[ s(t) = a e^{-i \omega t} + \frac{eE_y}{mc} \]  

is the most general while of (k).

c) With \[ e = be^{ix}, \ b, x \in \mathbb{R} \]

\[ \dddot{s} = b e^{-i(\omega - x) t + \frac{eE_y}{mc}} \]

\[ \dddot{s} \] just shifts the two of him \[ \Rightarrow x = 0 \text{ w.l.o.g.} \]
\[ x + iy = bu + \tau + ie_\phi \]
\[
\begin{align*}
\dot{x} &= bu + cE_\phi \mu \\
\dot{y} &= -b \mu u \\
\end{align*}
\]

\[ \langle \dot{y} \rangle = 0, \quad \langle \dot{x} \rangle = cE_\phi \mu = \frac{cE_\phi \mu}{\mu^2} \]

\[ \text{time-averaged velocity} \]

\[ \frac{cE_\phi \mu}{\mu^2} = \frac{c(E\vec{a})}{\mu^2} \]

\[ \text{bulk velocity} \]

\[ \langle \vec{v} \rangle = \frac{c}{\mu^2} E \times \vec{a} \]

\[ \text{drift velocity} \]

condition for \( \mu \ll c : \quad \frac{E_\phi}{\mu} \ll 1 \]

\[ \text{moneyed dshift velocity for non-}\]

\[ \text{relativistic approximation} \]

\[ x(t=0) = 0 = y(t=0) \quad \text{for} \]
\[
\begin{align*}
\dot{x}(t) &= \frac{b}{\mu} u + \frac{cE_\phi}{\mu} t \\
\dot{y}(t) &= \frac{b}{\mu} (u - \mu t - 1) \\
\end{align*}
\]

\( \) The path is a trochoid.

To visualize it, put \( \mu = 1 \) and define \( C = \frac{cE_\phi}{\mu} \).

\[ x(t) = b u + Ct \\
\dot{y}(t) = b (u - \mu t - 1) \]

This is the projection of the path onto the \( x-y \) plane.
For \(C < b\) the trochoid has loops:

```math
\begin{align*}
& b = 1; \\
& c = 0.5; \\
& x[t_] := b \sin[t] + c t \\
& y[t_] := b (\cos[t] - 1) \\
\text{ParametricPlot}\{x[t], y[t]\}, \{t, 0, 4 \pi\}
\end{align*}
```

For \(C > b\) it does not:

```math
\begin{align*}
& b = 1; \\
& c = 2; \\
& x[t_] := b \sin[t] + c t \\
& y[t_] := b (\cos[t] - 1) \\
\text{ParametricPlot}\{x[t], y[t]\}, \{t, 0, 4 \pi\}, \text{AspectRatio} \to 0.3
\end{align*}
```

And for \(C = b\) it degenerates into a cycloid:

```math
\begin{align*}
& b = 1; \\
& c = 1; \\
& x[t_] := b \sin[t] + c t \\
& y[t_] := b (\cos[t] - 1) \\
\text{ParametricPlot}\{x[t], y[t]\}, \{t, 0, 4 \pi\}, \text{AspectRatio} \to 0.3
\end{align*}
```
10) In addition to the restoring force $-m\omega_0^2 x$, the particle is subject to a force from
\[ e \frac{d^2 x}{dt^2} = e \left( \begin{array}{c} 0 \\ -x \\ 0 \end{array} \right) \]

The components are
\[
\begin{align*}
\ddot{x} + \omega_0^2 x &= 0 \\
\ddot{y} + \omega_0^2 y &= -R \dot{y} \\
\ddot{z} + \omega_0^2 z &= 0
\end{align*}
\]

Define $\tilde{f} := x + iy$ and write
\[ (1) + i \cdot (2) \Rightarrow \tilde{f}'' + \omega_0^2 \tilde{f} = -i R \tilde{f}' \\

\Rightarrow \tilde{f}(t) = \tilde{f}_0 e^{i \omega t} \\
\Rightarrow -\omega^2 + \omega_0^2 = i R \omega \\
\Rightarrow \omega^2 + R \omega - \omega_0^2 = 0 \\
\Rightarrow \omega = \frac{1}{2} \left( -R \pm \sqrt{R^2 + 4 \omega_0^2} \right) = \pm \sqrt{\omega_0^2 + \frac{R^2}{4} - \frac{R^2}{4}} \\
\Rightarrow \text{The motion in the } x-y \text{ plane is oscillatory, i.e., the trajectory is on} \\
\Rightarrow \omega = \sqrt{\omega_0^2 + \frac{R^2}{4}} \pm \frac{R}{2} \]
(a) The logarithm is

\[ L = L_0 + cE \tau + \frac{c}{\gamma} \frac{\gamma}{\gamma - 1} \hat{a} \]

which gives

\[ L_0 = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} \]

and

\[ \hat{a} = \frac{1}{c} \left( \begin{array}{c} -\frac{\beta_x}{\gamma} \\ \frac{\beta_y}{\gamma} \end{array} \right) \Rightarrow \nabla \times \hat{a} = \left( \frac{\partial \gamma}{\partial \beta_x} \right) \]

\[ \Rightarrow \frac{\partial L}{\partial \tau} = cE \text{ along } \partial \Omega \]

\[ \Rightarrow \hat{p} = cE \text{ depends only on } p_x, p_y \]

\[ \Rightarrow \hat{p}_x = cE \text{ is constant along } p_x, p_y \]

The force is \( \mathbf{F} = \mathbf{J} - \mathbf{E} \cdot \mathbf{d} \) direction is

\[ F_x = \frac{c}{\gamma} (\hat{a} \times \hat{b})_x = \frac{\gamma}{c} v_y, \quad F_y = \frac{c}{\gamma} (\hat{a} \times \hat{b})_y = -\frac{\gamma}{c} v_x \]

\[ \Rightarrow \frac{\partial \gamma}{\partial \beta_x} = \frac{\gamma v_y}{c} + \frac{\gamma v_x}{c} \]

When

\[ \hat{p} = (p_x, p_y, p_z) = \frac{\partial L}{\partial \tau} = \frac{mv}{\sqrt{1 - v^2/c^2}} \text{ is the moment} \]

\[ \Rightarrow \frac{d}{dt} \left( p_x^L p_y^L \right) = 2(p_x^L p_x^L + p_y^L p_y^L) = \frac{2mc^2}{1 - \frac{v^2}{c^2}} \left( v_x v_y - v_y v_x \right) = \]

\[ \Rightarrow p_x^L + p_y^L = p_z^L = \omega \tau \]

(b) The kinetic energy is

\[ T = \sqrt{\frac{\partial L_0}{\partial \tau} + L_0} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \sqrt{m^2 c^4 + \gamma^2 p^2} \]

\[ = \sqrt{\frac{m^2 c^4 + c^4 p^2 + c^2 p^2 + c^2 p^2}{1 - \frac{v^2}{c^2}}} = \frac{1}{\gamma} \sqrt{p^2 - c^2} \]

When

\[ \frac{p}{v_0} = \frac{mc^4 + c^2 p^2}{v_0} = \gamma(t, 0) \]
\[ t = \frac{ct}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{ct}{\sqrt{1 - \frac{v^2}{c^2}}} \]  

\[ t(\tau) = t_0 + \frac{c^2 E t}{\sqrt{1 - c^2 E^2 c^{-2}}} = t_0 + \frac{c^2 E x}{c^2 E^2 x} \int_0^{\tau} \frac{dx}{\sqrt{1 - c^2 E^2 x^{-2}}} = t_0 + \frac{c^2 E x}{c^2 E^2 x} \int_0^{\tau} \frac{dx}{1 - x^2} \]

\[ = t_0 + \frac{1}{c^2 E} \int_0^{\tau} \frac{dx}{1 + x^2} = t_0 + \frac{1}{c^2 E} \left( \frac{\pi}{2} \right) - \frac{1}{c^2 E} \]

\[ = \frac{1}{c^2 E} \left( \frac{\pi}{2} - \frac{1}{c^2 E} \right) \]

\[ \text{valid if } t_0 = \frac{1}{c^2 E} \]

\[ c) \text{ Define } \hat{e} = \vec{p} + i \vec{p}_y \]

\[ \hat{e} = \frac{c \vec{u}}{c} (-i) (v_x + i v_y) = \frac{c \vec{u}}{c} (-i) \frac{c}{c} \vec{\omega} = -i \frac{c \vec{u}}{c} \vec{\omega} \]

\[ \text{Define } \varphi \text{ by } \frac{du}{dt} = d\varphi \]

\[ \Rightarrow \frac{d\varphi}{d\varphi} = -i \varphi \Rightarrow \hat{\varphi} = p_x e^{-i \varphi} \]

\[ \hat{e} = p_x e^{-i \varphi} \]

\[ \frac{d\varphi}{d\varphi} = \frac{i}{c} \left( v_x + i v_y \right) = i \frac{d}{dt} \left( x + i y \right) = \frac{c}{c} \frac{d}{d\varphi} \left( x + i y \right) = \frac{i}{c} \frac{d}{d\varphi} \left( x + i y \right) \]

\[ \Rightarrow \frac{dx}{d\varphi} = \frac{c p_x}{c} \omega \varphi, \quad \frac{dy}{d\varphi} = -\frac{c p_y}{c} \omega \varphi \]

\[ \Rightarrow x = \frac{c p_x}{c} \omega \varphi, \quad y = \frac{c p_y}{c} \omega \varphi \]

\[ \text{valid if } \omega \text{ is real} \]

\[ \text{don'neg} \]
\[ \frac{d\phi}{dt} = \frac{C_1 \phi \dot{\phi}}{T_0^2 + C_1^2 \dot{\phi}^2} \]

\[ \Rightarrow \frac{d\phi}{dt} = \frac{C_1 \phi \dot{\phi}}{T_0} \frac{1}{1 + \frac{C_1^2 \dot{\phi}^2}{T_0^2}} = \frac{C_1 \phi \dot{\phi}}{T_0^2 + C_1^2 \dot{\phi}^2} \]

\[ \Rightarrow \omega \frac{d\phi}{d\phi} = \omega \arccos \left( \frac{C_1 \dot{\phi}}{T_0} \right) = \frac{1}{1 + \frac{C_1^2 \dot{\phi}^2}{T_0^2}} = \frac{1}{1 + \frac{\omega^2}{T_0^2}} \]

while \( b) \Rightarrow t = \frac{1}{\omega} \)

\[ \Rightarrow t = \frac{T_0^2}{\omega \phi} \frac{\omega}{cT_0} \]

Now we have the orbit in a more explicit form:

\[ x = \frac{C_1 \phi}{cT_0} \sin \phi, \quad y = \frac{C_1 \phi}{cT_0} \cos \phi, \quad t = \frac{T_0}{cT_0} \omega \phi \]

when \( \phi \) is related to \( t \) via

\[ \phi = \frac{\omega}{cT_0} \arccos \left( \frac{C_1 \dot{\phi}}{T_0} \right) \]

or

\[ t = \frac{T_0}{cT_0} \frac{\omega}{cT_0} \phi \]

d) The orbit is a helix when pitch remains constant with inclination angle. Using the unit of light and light

\[ r_i = \frac{cT_0}{cT_0} = 1 \text{ in } \text{L} \]

\[ x = \bar{v}_i \phi, \quad y = \bar{v}_i \phi, \quad t = t_0 \omega \left( \frac{\phi}{1T_0} \right) \]

\[ \text{NN is a graph for } t_0 = \frac{T_0}{cT_0}, \quad E/Q = 0.1 \]
\[ n(198) = z_0 = 1 \]
\[ EB = 0.1 \]
\[ x[\phi_] := \sin(\phi) \]
\[ y[\phi_] := \cos(\phi) \]
\[ z[\phi_] := z_0 \cosh(EB \phi) \]
\[ \text{ParametricPlot3D}[[x[\phi], y[\phi], z[\phi]], \{\phi, 0, 8 \pi\}] \]
\[ \text{Out}(198) = 1 \]
\[ \text{Out}(199) = 0.1 \]

For \( t \gg \frac{T_0}{cE} \) or long \( T \to cE t \)

\[
\Rightarrow \quad \phi = \frac{cE \tau}{t} \Rightarrow \frac{cE \tau}{t} \frac{1}{cE} t = \frac{\pi}{E} \frac{1}{t} \to 0 \quad \text{the angular velocity} \to 0 \text{ as } t \to \infty.
\]

\( t(t) \to ct \to \dot{t} \to c \) the velocity is time-linear opposed.

For \( t \ll \frac{T_0}{cE} \) or have \( \phi = c \frac{E \tau}{T_0} + o(t) \) with angular velocity

\[ c = \frac{c^2 E}{10} \]

\[ \tau(t) = \tau_0 + \frac{1}{2} \frac{c^2 E}{10} t^2 + o(t^3) \] Galilean rel. t.