1.2.2 Products
Prove the corollary to proposition 2 of ch.1 §2.2: If \( a \) is an element of a multiplicative group, and \( n, m \in \mathbb{N} \), then

a) \( a^n a^m = a^{n+m} \)

b) \( (a^n)^m = a^{nm} \)  

(2 points)

1.2.3 The group \( S_3 \)

a) Compile the group table for the symmetric group \( S_3 \). Is \( S_3 \) abelian?

b) Find all subgroups of \( S_3 \). Which of these are abelian?  

(6 points)

1.2.4 Abelian groups
Let \((G, \lor)\) be a group with neutral element \( e \). Let \( a \in G \) be a fixed element, and define a mapping \( \varphi : G \to G \) by \( \varphi(x) = a \lor x \lor a^{-1} \forall x \in G \).

a) Show that \( \varphi \) defines an automorphism on \( G \), called an inner automorphism.

b) Show that abelian groups have no inner automorphisms except for the identity mapping \( \varphi(x) = x \).

c) Let \( g \lor g = e \forall g \in G \). Prove that \( G \) is abelian.  

(6 points)

1.3.1 Fields

a) Show that the set of rational numbers \( \mathbb{Q} \) forms a commutative field under the ordinary addition and multiplication of numbers.

b) Consider a set \( F \) with two elements, \( F = \{ \theta, e \} \). On \( F \), define an operation “plus” (+), about which we assume nothing but the defining properties

\[ \theta + \theta = \theta \quad , \quad \theta + e = e + \theta = e \quad , \quad e + e = \theta \]

Further, define a second operation “times” (\( \cdot \)), about which we assume nothing but the defining properties

\[ \theta \cdot \theta = e \cdot \theta = \theta \cdot e = \theta \quad , \quad e \cdot e = e \]

Show that with these definitions (and \textbf{no} additional assumptions), \( F \) is a field.  

(7 points)