1.4.5. \( \mathbb{R} \) as a metric space

Consider the reals \( \mathbb{R} \) with \( \rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) defined by \( \rho(x, y) = |x - y| \). Show that this definition makes \( \mathbb{R} \) a metric space.

(3 points)

1.4.6. Limits of sequences

1. a) Show that a sequence in a metric space has at most one limit.

   *hint*: Assume there are two limits, and use the triangle inequality to show that they must be the same.

2. b) Show that every sequence with a limit is a Cauchy sequence.

(3 points)

1.4.7. Banach space

Let \( B \) be a \( K \)-vector space (\( k = \mathbb{R} \) or \( \mathbb{C} \)) with null vector \( \theta \). Let \( || \ldots || : B \to \mathbb{R} \) be a mapping such that

1. \( ||x|| \geq 0 \) \( \forall \) \( x \in B \), and \( ||x|| = 0 \) iff \( x = \theta \).
2. \( ||x + y|| \leq ||x|| + ||y|| \) \( \forall \) \( x, y \in B \).
3. \( ||\lambda x|| = |\lambda| \cdot ||x|| \) \( \forall \) \( x \in B, \lambda \in K \).

Then we call \( \| \ldots \| \) a **norm** on \( B \), and \( ||x|| \) the **norm** of \( x \).

Further define a mapping \( d : B \times B \to \mathbb{R} \) by

\[
d(x, y) := ||x - y|| \quad \forall \ x, y \in B
\]

Then we call \( d(x, y) \) the **distance** between \( x \) and \( y \).

1. a) Show that \( d \) is a metric in the sense of §4.5, i.e., that every linear space with a norm is in particular a metric space.

   If the normed linear space \( B \) with distance/metric \( d \) is complete, then we call \( B \) a **Banach space** or **B-space**.

2. b) Show that \( \mathbb{R} \) and \( \mathbb{C} \), with suitably defined norms, are B-spaces. (For the completeness of \( \mathbb{R} \) you can refer to §4.5 example (3), and you don’t have to prove the completeness of \( \mathbb{C} \) unless you insist.)

Now let \( B^* \) be the dual space of \( B \), i.e., the space of linear functionals \( \ell \) on \( B \), and define a norm of \( \ell \) by

\[
||\ell|| := \sup_{||x|| = 1} \{ |\ell(x)| \}
\]

1. c) Show that the such defined norm on \( B^* \) is a norm in the sense of the norm defined on \( B \) above.

(In case you’re wondering: \( B^* \) is complete, and hence a B-space, but the proof of completeness is difficult.)

(5 points)

1.4.8. Hilbert space

1. a) Show that the norm on a Hilbert space defined by §4.7 def. 1 is a norm in the sense of the definition in Problem 1.4.7.

   *hint*: Use the Cauchy-Schwarz inequality (§4.7 lemma).

2. b) Show that the mappings \( \ell \) defined in §4.7 def. 4 are linear forms in the sense of §4.3 def. 1(a).

(3 points)