2.3.2. Applications of the residue theorem

Use complex analysis to evaluate the real integrals

a) \[ \int_{-\infty}^{\infty} dx \frac{1}{x^4 + 1} \]

b) \[ \int_{-\infty}^{\infty} dx \frac{\sin x}{x} \]

*hint:* Write \( \sin x = (e^{ix} - e^{-ix})/2i \) and consider the resulting two integrals with complex integrands. Why is this a good strategy?

c) \[ \int_{-\infty}^{\infty} dx \frac{\sin x}{x} \frac{1}{1 + x^2} \]

and check your results by means of Wolfram Alpha.

Let \( a \in \mathbb{C} \) with \( \text{Re} a > 0 \). Use the residue theorem to show that

d) \[ \int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}} \]

Now let \( a \in \mathbb{R} \) and consider the integral
e) \[ \int_{-\infty}^{\infty} dx \frac{1}{x^2 + a^2} \]

and define its Cauchy principal value by

\[ \lim_{R \to 0} \left( \int_{-R}^{-\infty} dx f(x) + \int_{R}^{\infty} dx f(x) \right) \]

with \( f(x) = 1/x(x^2 + a^2) \). Determine the Cauchy principal value using the residue theorem. Is the result consistent with the expectation for a real symmetric integral over an antisymmetric integrand?

*hint:* Go around the pole on a semicircle of radius \( R \) and let \( R \to 0 \).

(17 points)
2.3.3. Matsubara frequency sum

Let \( f(z) \) have simple poles at \( z_j \) \((j = 1, 2, \ldots)\), and no other singularities. Let \( f(|z| \to \infty) \) go to zero faster than \( 1/z \). Consider the infinite sum

\[
S = -T \sum_{n=-\infty}^{\infty} f(i\Omega_n)
\]

with \( \omega_n = 2\pi T n \) and \( T > 0 \). Show that

\[
S = \sum_j n(z_j) \text{Res} f(z_j)
\]

where \( n(z) = 1/(e^{z/T} - 1) \) is the Bose distribution function.

**hint:** Show that \( n(z) \) has simple poles at \( z = i\Omega_n \), and integrate \( n(z) f(z) \) over an infinite circle centered on the origin.

**note:** Sums of this form are important in finite-temperature quantum field theory. In this context, \( T \) is the temperature and \( \Omega_n \) is called a “bosonic Matsubara frequency”.

(3 points)