3.1.1. Electromagnetic waves and gauge invariance

a) Show that the Lorenz gauge, \( \frac{1}{c} \partial_t \varphi + \nabla \cdot \mathbf{A} = 0 \), still does not uniquely determine the potentials of an electromagnetic wave: Let \( f \) be an arbitrary scalar solution of the wave equation, \( \Box f = 0 \). Then the transformation \( \mathbf{A} \to \mathbf{A} + \nabla f, \varphi \to \varphi - \frac{1}{c} \partial_t f \) leaves both the wave equation for the 4-vector potential and the fields unchanged.

b) Show in particular that the gauge of an electromagnetic wave can always be chosen such that \( \varphi = 0, \nabla \cdot \mathbf{A} = 0 \).

(3 points)

3.1.2. Plane waves

Consider the scalar field
\[
\psi(x, t) = \cos(k \cdot x - \omega t),
\]
where \( k \) is a Euclidian vector.

a) What is necessary and sufficient to make \( \psi \) a solution of the wave equation?

b) Perform a Lorentz boost, and show that the transformed wave again has the form
\[
\psi'(x', t') = \cos(k' \cdot x' - \omega' t').
\]

How are \( k' \) and \( \omega' \) related to \( k \) and \( \omega \)?

(3 points)

3.1.3. Spherical waves

Consider the wave equation
\[
\left( \frac{1}{c^2} \partial_t^2 - \nabla^2 \right) f(x, t) = 0
\]

Find and discuss the most general solution that has the form
\[
f(x, t) = u(r, t)/r
\]
where \( r = |x| \).

(3 points)
3.1.4. **Cosmological redshift**

Edwin Hubble observed the following relation between the wavelength of spectral lines in galaxies and the distance of the galaxies from the earth:

\[
\frac{\lambda - \lambda_0}{\lambda} = Hr/c
\]

where \(\lambda\) is the wavelength of a spectral line as observed in the galaxy, \(\lambda_0\) is the wavelength of the same spectral line as measured in the laboratory, \(r\) is the distance of the galaxy, and \(c\) is the speed of light. \(H\) is observed to be roughly \(H \approx 68\) (km/s)/Mpc (1 Mpc = \(3.26 \times 10^6\) light years).

a) Assuming that the observed red shift is due to the nonrelativistic Doppler effect, and that the motion of the galaxy is purely radial, find a relation between the distance of a galaxy and its velocity with respect to the earth.

b) How long did it take a galaxy that’s now at distance \(r\) to get there? Use the result to estimate the age of the universe.

c) Hubble’s original estimate was \(H \approx 530\) (km/s)/Mpc. Why does this value pose a problem?

(3 points)

3.2.1. **General solution of the wave equation**

Consider a one-dimensional wave equation

\[
(\partial_t^2 - c^2 \partial_x^2) f(x, t) = 0
\]

Show that the general solution constructed by Fourier transform in ch. 3 §2.2 has the form of the d’Alembert solution from ch. 3 §1.2, and vice versa.

(2 points)

4.1.1. **Wave equations for the electromagnetic fields**

Show directly from the Maxwell equations, without introducing potentials, that the fields obey the inhomogeneous wave equations

\[
\Box E = -4\pi \left( \nabla \rho + \frac{1}{c^2} \partial_t j \right), \quad \Box B = \frac{4\pi}{c} \nabla \times j.
\]

(2 points)