2.2.1. Planar charge distributions

a) Consider a homogeneously charged infinitesimally thin ring with radius $R$ and total charge $Q$ that is oriented perpendicular to the $z$-axis. Calculate the electric field on the $z$-axis.

b) The same for a homogeneously charged disk with charge density $\sigma$ and radius $R$. Consider the limits $z \to \infty$, $z \to 0$, and $R \to \infty$, and ascertain that they makes sense.

(4 points)

2.2.2. Spherically symmetric charge distributions

Consider a spherically symmetric static charge distribution (in spherical coordinates): $\rho(x) = \rho(r)$.

a) Express the electric field in terms of a one-dimensional integral over $\rho(r)$, and the electrostatic potential by a one-dimensional integral over the field.

Hint: Make an ansatz for a purely radial field, $E(x) = E(r) \hat{e}_r$, and integrate Gauss’s law over a spherical volume.

Explicitly calculate and plot the field $E(x)$ and the potential $\varphi(x)$ for

b) a homogeneously charged sphere

$$\rho(x) = \begin{cases} \rho_0 & \text{if } r \leq r_0 \\ 0 & \text{if } r > r_0 \end{cases}$$

c) a homogeneously charged spherical shell

$$\rho(x) = \sigma_0 \delta(r - r_0) .$$

(8 points)

2.2.3. Electrostatics in $d$ dimensions (to be continued later)

Consider the third Maxwell equation in $d$ dimensions:

$$\nabla \cdot E(x) = S_d \rho(x)$$

with the electric field $E$ a $d$-vector, and $S_d$ the area of the $(d - 1)$-sphere: $S_{2n} = 2\pi^n/(n-1)!$ and $S_{2n+1} = 2^{n+1}n!\pi^n/(2n)!$ for even and odd dimensions, respectively. Define a scalar potential $\varphi(x)$ in analogy to the $3 - d$ case, such that

$$E(x) = -\nabla \varphi(x)$$

and consider Poisson’s equation

$$\nabla^2 \varphi(x) = -S_d \rho(x)$$

Note: Here we consider a generalization of electrostatics to $d$-dimensional space, NOT a $d$-dimensional charge distribution embedded in 3-dimensional space.
a) Show that the Green function $G_d(x)$ function for Poisson’s equation, i.e., the solution of
\[ \nabla^2 G_d(x) = -S_d \delta(x) \]
is given by
\[ G_d(x) = \frac{1}{d-2} \frac{1}{|x|^{d-2}} \]
for all $d \neq 2$, and by
\[ G_2(x) = \ln(1/|x|) \]
for $d = 2$.

*hint:* For $d = 1$, differentiate directly, using $d \text{sgn} x/dx = 2 \delta(x)$. For $d \geq 2$, show that $G_d(x)$ is a harmonic function for all $x \neq 0$, then integrate $\nabla^2 G_d$ over a hypersphere around the origin and use Gauss’s law.

(4 points)