2.3.2. Legendre polynomials

Consider the ODE

\[(1 - x^2)y'' - 2xy' + \lambda y = 0\]

with \(\lambda\) a constant. Show that a necessary condition for the existence of a polynomial solution is

\[\lambda = n(n + 1)\]

with \(n = 0, 1, \ldots\) What else do you need to require in order to get a condition that is necessary and sufficient? Convince yourself that these considerations correctly produce the first three Legendre polynomials up to an overall normalization factor.

**hint:** Make a power-series ansatz and require that the series terminates.

(4 points)

2.3.3. Associated Legendre functions

**note:** When comparing with the reference book by Abramowitz and Stegun, note that their \(P^m_\ell(x)\) equals \((-)^{3m/2}\) times our \(P^m_\ell(x)\).

Show that

\[\left(\sqrt{1 - x^2} \frac{d}{dx} - m \frac{x}{\sqrt{1 - x^2}}\right) P^m_\ell(x) = (\ell + m)(\ell - m + 1) P^{m-1}_\ell(x)\]

**hint:** First differentiate Legendre’s ODE \(m - 1\) times to show that

\[(1 - x^2) \frac{d^{m+1}}{dx^{m+1}} P_n(x) - 2mx \frac{d^m}{dx^m} P_n(x) + (n + m)(n - m + 1) \frac{d^{m-1}}{dx^{m-1}} P_n(x) = 0\]

Then use this in evaluating \(\sqrt{1 - x^2} dP^m_\ell(x)/dx\).

(3 points)
2.3.4. **Spherical harmonics**

Prove that the spherical harmonics have the following properties:

\[ Y_{\ell}^m(\Omega)^* = (-)^m Y_{\ell}^{-m}(\Omega) \]  
(1)

\[ \cos \theta Y_{\ell}^m(\Omega) = \left( \frac{(\ell + 1 - m)(\ell + 1 + m)}{(2\ell + 1)(2\ell + 3)} \right)^{1/2} Y_{\ell+1}^m(\Omega) + \left( \frac{(\ell - m)(\ell + m)}{(2\ell - 1)(2\ell + 1)} \right)^{1/2} Y_{\ell-1}^m(\Omega) \]  
(2)

\[ \sin \theta e^{\pm i\varphi} Y_{\ell}^m(\Omega) = \pm \left( \frac{(\ell \mp m - 1)(\ell \pm m)}{(2\ell - 1)(2\ell + 1)} \right)^{1/2} Y_{\ell-1}^{m\pm 1}(\Omega) \mp \left( \frac{(\ell \pm m + 1)(\ell \pm m + 2)}{(2\ell + 1)(2\ell + 3)} \right)^{1/2} Y_{\ell+1}^{m\pm 1}(\Omega) \]  
(3)

\[ \hat{L}_{\pm} Y_{\ell}^m(\Omega) = \pm \left( \ell \pm m \right) \frac{1}{2} Y_{\ell}^{m\mp 1}(\Omega) \]  
(4)

where

\[ \hat{L}_{\pm} = e^{\mp i\varphi} \left[ \mp \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \varphi} \right] \]

*hint:* Use the properties of the associated Legendre functions we quoted in ch.3 §3.2, as well as Problem 2.3.3.

(9 points)

2.3.5. **Field due to distant charges**

Consider the electric field generated by a charge density \( \rho(y) \) that vanishes inside a sphere with radius \( r_0 \): \( \rho(y) = 0 \) for \( |y| \leq r_0 \). Show that

a) If \( \rho \) is invariant under parity operations, \( \rho(-y) = \rho(y) \), then the electric field at the origin vanishes.

b) If \( \rho(y) \) is invariant under rotations about the \( z \)-axis through multiples of an angle \( \alpha \) with \( |\alpha| < \pi \), then the field-gradient tensor at the origin has the form \( \varphi_{ij}(x = 0) = \begin{pmatrix} \varphi & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & -2\varphi \end{pmatrix} \)

c) If \( \rho(y) \) has cubic symmetry, i.e., if \( \rho(y) \) is invariant under rotations through \( \pi/2 \) about any of the three axes \( x \), \( y \), and \( z \), then the field-gradient tensor at the origin vanishes.

(6 points)