4.1.2. Polaritons

As a model for a dielectric, consider a polarization field \( \mathbf{P}(\mathbf{x},t) \) that determines the sources of the electromagnetic fields according to

\[
\mathbf{j} = \partial_t \mathbf{P}, \quad \rho = -\mathbf{\nabla} \cdot \mathbf{P}.
\]

In addition to Maxwell’s equations, the dynamics of the system are governed by an equation of motion for \( \mathbf{P} \),

\[
(\partial_t^2 + \omega_0^2) \mathbf{P}(\mathbf{x},t) = a^2 \mathbf{E}(\mathbf{x},t) \quad (\ast),
\]

where \( \omega_0 \) is a characteristic frequency and \( a \) is a real parameter (which dimensionally also is a frequency). This models the dielectric as a harmonic oscillator that is driven by the electric field.

a) Show that Maxwell’s equations plus (\ast) have solutions given by both longitudinal (\( \mathbf{k} \parallel \mathbf{E}, \mathbf{P} \)) and transverse (\( \mathbf{k} \perp \mathbf{E}, \mathbf{P} \)) monochromatic plane waves, and find the frequency-wavenumber relations for the various solutions.

b) Show that the transverse waves in the long-wavelength limit are photon-like, viz.,

\[
\omega_T(\mathbf{k} \to 0) = (c/n)|\mathbf{k}|,
\]

and determine the index of refraction \( n \).

c) Show that no homogeneous wave propagation is possible in a frequency band \( \omega_- < \omega < \omega_+ \), and find \( \omega_\pm \).

Derive the Lyddane-Sachs-Teller relation

\[
\frac{\omega_+^2}{\omega_-^2} = \epsilon(\omega = 0)
\]

where \( \epsilon(\omega) = 1 + 4\pi a^2/(\omega_0^2 - \omega^2) \) is the dielectric function of the dielectric.

d) Discuss the frequency-wavenumber relation for all possible waves explicitly, especially in the limits \( k \to 0 \) and \( k \to \infty \), and plot the result.

(14 points)
4.2.1. Liénard-Wiechert potentials
Consider a point charge $e$ that moves on a given trajectory $X(t)$ with velocity $v(t) = \dot{X}(t)$ which results in charge and current densities

$$\rho(x,t) = e \delta(x - X(t)) \quad , \quad j(x,t) = e v(t) \delta(x - X(t))$$

Show that the resulting retarded potentials have the form

$$\varphi(x,t) = e \frac{|x - X(t_\perp)| - v(t_\perp) \cdot (x - X(t_\perp))}{c}$$

$$A(x,t) = \frac{1}{c} v(t_\perp) \varphi(x,t)$$

where $t_\perp$ is the solution of

$$t_\perp = t - \frac{1}{c} |x - X(t_\perp)| \quad (*)$$

These are known as Liénard-Wiechert potentials after Alfred-M Marie Liénard and Emil Wiechert, who derived them in 1898 and 1900, respectively.

**hint:** Show that the equation (*) for $t_\perp$ has one and only one solution.

(6 points)

4.2.2. Potential of a uniformly moving charge
Consider a charge $e$ moving uniformly along the $x$-axis with velocity $v$: $X(t) = (vt, 0, 0)$. Determine the Liénard-Wiechert potentials explicitly, and show that the result is that same as the one obtained in ch. 2 §2.4 by means of a Lorentz transformation.

(6 points)
4.2.1.) ch 5 §3 =>

\[ \varphi(x',t') = \int_{-\infty}^{\infty} dt' \frac{\delta(t-t' - \frac{1}{c} |x-x'|)}{|x-x'|} \phi(x',t') \]

\[ = \int_{-\infty}^{\infty} dt' \frac{1}{|x-x'|} \delta(t-t' - \frac{1}{c} |x-x'|) \delta(x' - \vec{x}(t')) \]

\[ = e^{\int dt' \frac{1}{|x-x'(t')|} \delta(t-t' + \frac{1}{c} |x-x'(t')|)} \]

\[ \text{when } f(t') = t' - t + \frac{1}{c} |x-x'(t')| \]

\[ \Rightarrow \text{New un know of } f(t') \]

\[ \text{let } (ct, \vec{x}) = (0,0) \text{ un } \Delta \]

\[ \Rightarrow ct' = |\vec{x}(t')| \]

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\[ \delta(f(t')) = \frac{1}{1\delta(t')} \delta(t'-t_-) \]

\[ \text{when } f(t_-) = 0 \Rightarrow t_- = t - \frac{1}{c} |x-x'(t_-)| \]

\[ \text{dimension above } \Rightarrow (x) \text{ has a unique solution } \]

\[ \frac{d}{dt} f(t') = 1 + \frac{1}{c} \frac{|x-x'(t')|}{|x-x'(t')|} (-) \vec{v}(t') \cdot (x-x'(t')) \]

\[ = 1 - \frac{1}{c} \vec{v}(t') \cdot (x-x'(t')) \frac{1}{|x-x'(t')|} \]

\[ > 0 \text{ min } \left| \frac{\vec{v}(t')}{c} \right| < 2 \]
\[ \varphi(x, t) = e^{\frac{i}{\hbar} \int_{t'} \frac{1}{x-x'(t')} \delta(t-t') \cdot \frac{1}{x-x'(t')} \frac{1}{1 - \frac{i}{\hbar} \varphi(t) \cdot (x-x'(t))}} \]

For \( \xi \neq \Delta \),

\[ \bar{k}(\xi, t) = \frac{1}{\hbar} \int d\xi' dt' \frac{1}{x-x'} \delta(t-t' - \frac{i}{\hbar} |x-x'|) \varphi(x', t') \]

\[ = \frac{1}{\hbar} \int d\xi' dt' \frac{1}{x-x'} \delta(t-t') \varphi(t') \delta(x-x'(t')) \]

\[ = \frac{1}{\hbar} \int dt' \frac{1}{x-x'(t')} \varphi(t') \delta(x-x'(t')) \]

\[ = \frac{1}{\hbar} \varphi(t) \int d\xi' \frac{1}{x-x'(t')} \frac{1}{x-x'(t')} \delta(t-t') \]

\[ = \frac{1}{\hbar} \varphi(t) \varphi(x, t) \]
4.2.2. \( \text{W} \)\( \text{h} \)\( \text{i} \)\( \text{n} \)\( \text{n} \)\( \text{e} \)\( \text{r} \)\( \text{e} \)\( \text{s} \)\( \text{B} \)\( \text{r} \)\( \text{K} \) for the special case
\[
\vec{x}(t) = (vt, 0, 0), \quad \vec{v}(t) = (v, 0, 0)
\]

The eq. for \( t_- \) needs
\[
t_- = t - \frac{1}{c} \sqrt{\left(x - vt_\right)^2 + y^2 + z^2}
\]

\[\cdots\]

\[
x - vt_- = x - vt - \frac{v}{c} \sqrt{\left(x - vt_\right)^2 + y^2 + z^2}
\]

\[
\left(x - vt_-\right)^2 = 2(x - vt)(x - vt_-) + \left(x - vt\right)^2 = \frac{v^2}{c^2} \left(x - vt_\right)^2 + \frac{v^2}{c^2} \left(y^2 + z^2\right)
\]

\[
\left(x - vt_-\right)^2 - 2(x - vt)(x - vt_-) + \left(x - vt\right)^2 - \frac{v^2}{c^2} \left(y^2 + z^2\right) = 0
\]

\[
\frac{x - vt_-}{1 - \frac{v^2}{c^2}} = \frac{1}{c} \sqrt{2(x - vt) - \sqrt{4(x - vt)^2 - 4(y^2 + z^2) + \frac{4v^2}{c^2}(y^2 + z^2)}}
\]

\[
= \frac{1}{c} \sqrt{x - vt \pm \sqrt{\frac{v^2}{c^2} \left(x - vt\right)^2 + \frac{v^2}{c^2} \left(y^2 + z^2\right)}}
\]

The physical (retarded) value yields the smaller value for \( t_- \) \( \Rightarrow \) The physical value has \( t_+ \) the above eq.

Define \( R^\circ(x, t) := \sqrt{\left(x - vt\right)^2 + y^2 + z^2} \) as in \( \text{p} \) \( 2.5 \)

\[
x - vt_- = \frac{1}{c} \sqrt{x - vt + \frac{v}{c} R^\circ(x, t)}
\]

This is the explicit solution for \( t_- \).

\[\text{Problem 35} \Rightarrow \]

\[
\frac{e}{\varphi(x, t)} = \sqrt{\left(x - vt\right)^2 + y^2 + z^2} - \frac{v}{c} \left(x - vt_-\right)
\]

\( \text{U} \) \( \text{J} \) \( \text{f} \) \( \text{2.5} \) \( \Rightarrow \) \( \text{t} \) \( \text{m} \) \( \text{d} \) \( \text{t} \) \( \text{n} \) \( \text{o} \) \( \text{L} \) \( \text{n} \) \( \text{b} \) \( \text{u} \) \( \text{h} \) \( \text{L} \) \( \text{s} \) \( \text{e} \) \( \text{p} \) \( \text{h} \) \( \text{l} \) \( \text{d} \) \( \text{e} \) \( \text{p} \) \( \text{h} \) \( \text{l} \) \( \text{s} \) \( \text{e} \) \( \text{p} \) \( \text{h} \) \( \text{t} \) \( \text{p} \) \( \text{h} \) \( \text{s} \) \( \text{e} \) \( \text{p} \)
\[ (x-vt)^2 - \frac{y^2}{c^2} = R^+ \]

\[ (x-vt)^2 + \frac{y^2}{c^2} = \left( R^+ + \frac{y}{c} (x-vt) \right)^2 \]

\[ = \frac{y^2}{c^2} (x-vt)^2 + 2 \frac{y}{c} (x-vt) R^+ + R^+ \]

\[ (x-vt)^2 + \frac{y^2}{c^2} = \left( R^+ + \frac{y}{c} (x-vt) \right) R^+ \]

\[ f^2 \left[ (x-vt) + \frac{y}{c} R^+ \right] + \frac{y^2}{c^2} = \left( R^+ \right)^2 + 2 \frac{y}{c} \left[ (x-vt) + \frac{y}{c} R^+ \right] R^+ \]

\[ f^2 (x-vt)^2 + 2 \frac{y}{c} (x-vt) R^+ \left( R^+ \right)^2 + \frac{y^2}{c^2} = \left( R^+ \right)^2 + 2 \frac{y}{c} (x-vt) R^+ \]

\[ \left( R^+ \right)^2 (1 - \frac{y^2}{c^2}) = \left( R^+ \right)^2 (1 - \frac{u^2}{c^2}) \]

\[ = \left( R^+ \right)^2 \frac{1}{1 - \frac{u^2}{c^2}} \]

\[ \left( R^+ \right)^2 (x-vt)^2 + \frac{y^2}{c^2} = \left( R^+ \right)^2 \frac{1}{1 - \frac{u^2}{c^2}} f^2 \]

\[ \text{Now, let} \]

\[ e = \sqrt{(x-vt)^2 + \frac{y^2}{c^2} - \frac{y}{c} (x-vt)} = R^+(x_1,v) \]

\[ \Rightarrow \phi(x_1,v) = \frac{e}{R^+(x_1,v)} \]

\[ \text{and from Problem 4.2.1} \]

\[ A(x_1,v) = \frac{\dot{v}}{c} \phi(x_1,v) = \frac{\dot{v}}{c R^+(x_1,v)} \]

\[ \text{Thus, we have some results as x \neq u \neq 2} \]

\[ \text{Note: solve via Computer, as x \neq u \neq 2.5, c would come!} \]