0.2.4. Functional derivative

Let $F[\varphi]$ be a functional of a real-valued function $\varphi(x)$. For simplicity, let $x \in \mathbb{R}$; the generalization to more than one dimension is straightforward. We can (sloppily) define the functional derivative of $F$ as

$$\frac{\delta F}{\delta \varphi(x)} := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( F[\varphi(y) + \epsilon \delta(y-x)] - F[\varphi(y)] \right)$$

a) Calculate $\delta F/\delta \varphi(x)$ for the following functionals:

i) $F = \int dx \varphi(x)$

ii) $F = \int dx \varphi^2(x)$

iii) $F = \int dx (\varphi'(x))^2$ where $\varphi'(x) = d\varphi/dx$

\textit{hint:} Integrate by parts and assume that the boundary terms vanish.

iv) $F = \int dx V(\varphi'(x))$ where $V$ is some given function.

\textit{remark:} Blindly ignore terms that formally vanish as $\epsilon \to 0$ unless you want to find out why the above definition is very problematic. It does work for operational purposes, though.

b) Consider a Lagrangian density $L(\varphi(x), \partial_\mu \varphi(x))$ and an action $S = \int d^4x L$. Show that extremizing $S$ by requiring $\delta S/\delta \varphi(x) \equiv 0$ with the above definition of the functional derivative leads to the Euler-Lagrange equations

$$\partial_\mu \frac{\partial L}{\partial (\partial_\mu \varphi)} = \frac{\partial L}{\partial \varphi}$$

(3 points)

0.2.5. Massive scalar field

Consider the Lagrangian density for a massive scalar field from the example in ch. 0 §2.5.

a) Generalize this Lagrangian density to a complex field $\phi(x) \in \mathbb{C}$:

$$L = \frac{1}{2} (\partial_\mu \phi(x)) (\partial^\mu \phi^*(x)) - \frac{m^2}{2} |\phi(x)|^2$$

with $\phi^*$ the complex conjugate of $\phi$. What are the Euler-Lagrange equations now?

b) Consider a local gauge transformation, $\phi(x) \to \phi(x) e^{i\Lambda(x)}$, with $\Lambda(x)$ a real field that characterizes the transformation. Is the Lagrangian from part b) invariant under such a transformation?

(2 points)
0.2.6. Particle in homogeneous $E$ and $B$ fields

Consider a point particle (mass $m$, charge $e$) in homogeneous fields $B = (0, 0, B)$ and $E = (0, E_y, E_z)$. Treat the motion of the particle nonrelativistically.

a) Show that the motion in $z$-direction decouples from the motion in the $x$-$y$ plane, and find $z(t)$.

b) Consider $\xi := x + iy$. Find the equation of motion for $\xi$, and its most general solution.

*hint:* Define the cyclotron frequency $\omega = eB/mc$, and remember how to solve inhomogeneous ODEs.

c) Show that the time-averaged velocity perpendicular to the plane defined by $B$ and $E$ is given by the drift velocity

$$\langle v \rangle = c \frac{E \times B}{B^2}$$

Show that $E_y/B \ll 1$ is necessary and sufficient for the non relativistic approximation to be valid.

d) Show that the path projected onto the $x$-$y$ plane can have three qualitatively different shapes, and plot a representative example for each.

(6 points)

0.2.7. Harmonic oscillator coupled to a magnetic field

Consider a charged 3-d classical harmonic oscillator (oscillator frequency $\omega_0$, charge $e$). Put the oscillator in a homogeneous time-independent magnetic field $B = (0, 0, B)$. Show that the motion remains oscillatory, and find the oscillation frequencies in the directions parallel and perpendicular, respectively, to $B$.

(4 points)
b) \( 0 = \frac{\delta}{\delta \phi(x)} \int d^3 \xi \; \mathcal{L}(\phi_1, \partial_1 \phi_1) \)

\[ = \frac{\lambda}{\epsilon \eta^3} \int d^3 \xi \; \mathcal{L}(\phi_1 + \epsilon \delta(y-\xi), \partial_y \phi_1 + \epsilon \partial_y \delta(y-\xi)) - \frac{\partial \mathcal{L}}{\partial \phi_1} \]

\[ = \frac{\partial \mathcal{L}}{\partial \phi_1} - \frac{\partial \mathcal{L}}{\partial (\partial_y \phi_1)} \]
0.2.5

1. Let \( \phi(x) \) and \( \phi^*(x) \) be independent fields.
   
   Minimizing with respect to \( \phi^* \) yields
   
   \[
   \left( \partial_x^2 + m^2 \right) \phi(x) = 0
   \]

   and minimizing with respect to \( \phi \) just yields the case
   
   \[
   \left( \partial_x^2 + m^2 \right) \phi^*(x) = 0
   \]

2. Under \( \phi(x) \rightarrow \phi(x) e^{i \Delta(x)} \) or \( \phi(x) \rightarrow -\phi(x) e^{-i \Delta(x)} \)

   \[
   |\phi(x)|^2 \rightarrow |\phi(x)|^2
   \]

   and

   \[
   \partial_x \phi(x) \rightarrow (\partial_x \phi(x)) e^{i \Delta(x)} + i (\partial_x \Delta(x)) \phi(x) e^{i \Delta(x)}
   \]

   \[
   \partial_x \phi^*(x) \rightarrow (\partial_x \phi^*(x)) e^{-i \Delta(x)} - i (\partial_x \Delta(x)) \phi^*(x) e^{-i \Delta(x)}
   \]

   \[
   \Rightarrow \partial_x \phi(x) \partial_x \phi^*(x) \rightarrow \partial_x \phi(x) \partial_x \phi^*(x) - i (\partial_x \Delta(x)) (\partial_x \phi(x)) \phi^*(x)
   + i (\partial_x \Delta(x)) (\partial_x \phi^*(x)) \phi(x)
   + (\partial_x \Delta(x)) (\partial_x \Delta(x)) |\phi(x)|^2
   \]

   \[
   + \partial_x \phi \partial_x \phi^*
   \]

   \[
   \Rightarrow \text{The Lagrange is not invariant}
   \]
a) Eq. of motion:
\[ m \ddot{x} = e E_x + \frac{e}{c} \dot{x} \times B, \]
\[ m \ddot{y} = e E_y - \frac{e}{c} \dot{y} \times B, \]
\[ m \ddot{z} = e E_z. \]

\[ \Rightarrow \dot{z}(t) = \dot{z}_0 + v_0 t + \frac{e E_z}{mc} t^2. \]

b) Define \( \bar{J} = x + iy \)
\[ (1) + i \cdot (2) \Rightarrow m \ddot{\bar{J}} = i e E_y - i \frac{e}{c} \dot{\bar{J}}. \]

Define \( \omega = \frac{e}{mc} \) gyrodynes.

\[ \Rightarrow \ddot{\bar{J}} + i \omega \dot{\bar{J}} = i \frac{e E_y}{mc}. \]

Spiral motion of electron: \( \bar{J} \).
Counter-rotating motion of beam: \( \dot{\bar{J}} = a e^{-i \omega t} \) (c.e.c.)

\[ \Rightarrow \ddot{\bar{J}}(t) = a e^{-i \omega t} + \frac{e E_y}{mc} i. \]

c) With \( c = b e^{ix}, b, x \in \mathbb{R} \)
\[ \Rightarrow \dot{z} = b e^{-i(x-\omega t)} + c E_y / mc. \]

\[ \Rightarrow x \text{ just shifts the two of him} \Rightarrow \dot{z} = 0 \text{ m.m.g.} \]
\[ x + i y = b \omega u t - i b \omega u t + eE_y/u \]

\[ \begin{align*}
  x &= b \omega u t + eE_y/u \\
  y &= -b \omega u t
\end{align*} \] (**) 

\[ <y> = 0, \quad <x> = -eE_y/u = \frac{CE_y}{\Omega} \quad \text{kinetic angular velocity} \]

\[ \frac{eE_y}{\Omega^2} = \frac{C (E \times \bar{q})_x}{\Omega^2} \]

\[ \begin{align*}
  \langle \vec{v} \rangle &= \frac{e}{\Omega} \vec{E} \times \bar{q} \quad \text{drift velocity} \\
  \end{align*} \]

valid for \( v \ll c : \quad \frac{E_y}{\Omega} \ll 1 \]

d) \[ \text{When} \quad x(t=0) = 0 = y(t=0) \]

\[ \begin{align*}
  x(t) &= \frac{b}{\Omega} \omega u t + \frac{eE_y}{\Omega} t \\
  y(t) &= \frac{b}{\Omega} (\omega u t - 1)
\end{align*} \]

\[ \text{The path is a trochoid.} \]

To vidalize it, put \( w = 1 \) and define \( C = \frac{CE_y}{\Omega} \).

\[ \begin{align*}
  x(t) &= b \omega u t + Ct \\
  y(t) &= b (\omega u t - 1)
\end{align*} \]

This is the projection of the path onto the \( x-y \) plane.
For $C < b$ the trochoid has loops:

```math
b[21] := b = 1;
c = 0.5;
x[t_] := b \sin[t] + c t
y[t_] := b (\cos[t] - 1)
ParametricPlot[{x[t], y[t]}, {t, 0, 4 \pi}]
```

For $C > b$ it does not:

```math
b[60] := b = 1;
c = 2;
x[t_] := b \sin[t] + c t
y[t_] := b (\cos[t] - 1)
ParametricPlot[{x[t], y[t]}, {t, 0, 4 \pi}, AspectRatio -> 0.3]
```

And for $C = b$ it degenerates into a cycloid:

```math
b[91] := b = 1;
c = 1;
x[t_] := b \sin[t] + c t
y[t_] := b (\cos[t] - 1)
ParametricPlot[{x[t], y[t]}, {t, 0, 4 \pi}, AspectRatio -> 0.3]
```
In addition to the restoring force \(-m_0 x\), the potential is negligible. The point from

\[
\frac{\ddot{r}}{c} = \frac{\varepsilon}{c} (\frac{x_0}{\sqrt{1-X_0^2}}, -X_0, 0)
\]

\text{The top of motion on}

\begin{align*}
\dot{x} + u_0 \dot{x} &= \frac{R}{\sqrt{1-X_0^2}} \\
\dot{\dot{x}} + u_0 \dot{x} &= -R x \\
\ddot{x} + u_0^2 x &= 0
\end{align*}

\text{will \( R = \varepsilon \sqrt{1-X_0^2} \) be the angular frequency.}

\text{For oscillations in the \( x-y \) plane, the frequency } \omega = u_0 \text{ is unchanged.}

\text{Define } \dot{y} := x + i y \text{ and write}

\begin{align*}
(1) + i \cdot (2) &\Rightarrow \dot{\dot{y}} + u_0 \dot{y} = -i Re \dot{y}
\end{align*}

\text{Existence } \dot{y}(t) = \dot{y}_0 e^{i \omega t}

\Rightarrow -u_0^2 + \omega_0^2 = \omega R

\Rightarrow \omega = \sqrt{-u_0^2 + \omega_0^2} = \sqrt{\omega_0^2 + R^2/4} - R/2

\text{The motion in the } x-y \text{ plane is oscillatory, i.e., an}

\text{wave propagates on}

\[
\omega = \sqrt{\omega_0^2 + R^2/4} \pm R/2
\]