1.2.1. Energy-momentum tensor

Consider the electromagnetic field in the absence of matter.

a) Show that the tensor field

\[ H_\mu^\nu(x) = (\partial_\mu A_\alpha(x)) \frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\alpha(x))} - \delta_\mu^\nu \mathcal{L} \]

obeys the continuity equation

\[ \partial_\nu H_\mu^\nu(x) = 0 \quad (*) \]

*note*: Notice that \( H_\mu^\nu(x) \) is a generalization of Jacobi’s integral in Classical Mechanics.

b) Show that (*) also holds for

\[ \hat{T}_\mu^\nu = H_\mu^\nu + \partial_\alpha \psi_\mu^{\alpha} \]

where \( \psi_\mu^{\alpha} \) is any tensor field that is antisymmetric in the second and third indices, \( \psi_\mu^{\alpha}(x) = -\psi_\mu^{\alpha'}(x) \).

c) Show that \( \psi_\mu^{\alpha} \) can be chosen such that \( \hat{T}_\mu^\nu(x) = T_\mu^\nu(x) \), which provides an alternative proof that \( T_\mu^\nu(x) \) obeys (*).

(5 points)

1.2.2. Energy-momentum conservation in the presence of matter

Prove the corollary of ch. 1 §2.3: In the presence of matter, the energy-momentum tensor obeys the continuity equation

\[ \partial_\nu T_\mu^\nu(x) = \frac{-1}{c} J_\nu(x) \]

(2 points)

1.2.3. Energy-momentum tensor for a massive scalar field

Consider the massive scalar field from ch. 0 §2.5:

\[ \mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{m^2}{2} \varphi^2 \]

and the tensor field \( H_\mu^\nu \) defined analogously to Problem 1.2.1:

\[ H_\mu^\nu = (\partial_\mu \varphi) \frac{\partial \mathcal{L}}{\partial (\partial_\nu \varphi)} - \delta_\mu^\nu \mathcal{L} \]

Determine \( H_\mu^\nu \) explicitly and show that

\[ \partial_\nu H_\mu^\nu = 0 \]

*hint*: Use the Euler-Lagrange equation determined in ch. 0 §2.5.

(2 points)
1.2.4. Coulomb gauge

Consider the 4-vector potential $A^\mu(x) = (\varphi(x), A(x))$. Show that one can always find a gauge transformation such that

$$ \nabla \cdot A(x) = 0 $$

This choice is called *Coulomb gauge*. (2 points)
1.2.1) a) \[ \partial_v \delta \frac{\partial x}{\partial \partial A_k} = \partial_x \delta \frac{\partial x}{\partial \partial A_k} = \frac{\partial x}{\partial \partial A_k} \partial_x \frac{\partial x}{\partial \partial A_k} + \frac{\partial x}{\partial \partial A_k} \partial_x \frac{\partial x}{\partial \partial A_k} \]

\[ \Rightarrow 0 = \partial_x \left( \frac{\partial x}{\partial \partial A_k} \partial_x \frac{\partial x}{\partial \partial A_k} - \delta \frac{\partial x}{\partial \partial A_k} \right) = \partial_v \delta \frac{\partial x}{\partial \partial A_k} \]

b) \[ \partial_v \partial_x \frac{\partial x}{\partial \partial A_k} = - \partial_v \partial_x \frac{\partial x}{\partial \partial A_k} = - \partial_x \partial_v \frac{\partial x}{\partial \partial A_k} = - \partial_v \partial_x \frac{\partial x}{\partial \partial A_k} \]

\[ \Rightarrow \partial_v \partial_x \frac{\partial x}{\partial \partial A_k} = 0 \Rightarrow \partial_v \tilde{\nu} = 0 \]

c) \[ \text{Define } \frac{\partial x}{\partial \partial A_k} = \frac{1}{\xi_0} A^T F_{\partial A_k} = - \frac{1}{\xi_0} A^T F_{A^T} = - \frac{\partial x}{\partial \partial A_k} \]

\[ \Rightarrow \partial_v \tilde{\nu} = 0 \Rightarrow \partial_v \nu = 0 \]

\[ \tilde{\nu} = A^T \nu + \partial_x \frac{\partial x}{\partial \partial A_k} = (\partial^2 A_k) \frac{\partial x}{\partial \partial A_k} - \frac{\partial x}{\partial \partial A_k} \frac{\partial x}{\partial \partial A_k} + \frac{1}{\xi_0} \partial_x \frac{\partial x}{\partial \partial A_k} \]

\[ = (\partial^2 A_k) \frac{\partial x}{\partial \partial A_k} + \frac{1}{\xi_0} \frac{\partial x}{\partial \partial A_k} F_{\partial A_k} + \frac{1}{\xi_0} \partial_x \frac{\partial x}{\partial \partial A_k} \]

\[ = \frac{1}{\xi_0} \left( \partial^2 A_k - \partial_x \partial^2 A_k \right) F_{\partial A_k} + \frac{1}{\xi_0} \frac{\partial x}{\partial \partial A_k} F_{\partial A_k} \]

\[ = \frac{1}{\xi_0} \tilde{F}_{\partial A_k} - \tilde{F}_{\partial A_k} \frac{\partial x}{\partial \partial A_k} = \tilde{F}_{\partial A_k} \]

\[ \Rightarrow \partial_v \tilde{\nu} = 0 \]
1.22. Consider the proof of the proposition in § 2.2.

The only difference is that now the \( E \) of nodes

\[
\partial \nu F^\nu x = \frac{\partial}{\partial x}
\]

\[
\Rightarrow \partial \nu F^\nu x = \frac{1}{4\pi} \left[ -\left( \partial \nu F^\nu x \right) F^\nu x - F^\nu x \partial \nu F^\nu x + \frac{1}{2} \partial F F^\nu \partial^\nu F^\nu x \right]
\]

\[
= -\frac{1}{2} F^\nu \partial^\nu F^\nu
\]

\[
= 0 \quad \text{by § 2.2}
\]
\[ u_\mu = (\partial_\mu \varphi) \frac{\partial^2 \varphi}{\partial (\partial_\nu \varphi)} - \delta^\mu_\nu \varphi \]

\[ = (\partial_\mu \varphi)(\partial^\nu \varphi) - \delta^\mu_\nu (\partial_\nu \varphi)(\partial^\nu \varphi) + \delta^\mu_\nu \frac{\partial}{\partial \varphi} \varphi \]

\[ \Rightarrow \partial_\nu u_\mu = (\partial_\nu \partial_\mu \varphi)(\partial^\nu \varphi) + (\partial_\nu \varphi)(\partial_\mu \partial^\nu \varphi) - (\partial_\mu \varphi)(\partial_\nu \partial^\nu \varphi) + \frac{\partial^2 \varphi}{\partial \varphi} \partial_\nu \varphi \]

\[ = (\partial_\nu \partial_\mu \varphi)(\partial^\nu \varphi) - (\partial_\mu \partial_\nu \varphi)(\partial^\nu \varphi) + (\partial_\nu \varphi)(\partial_\mu \partial^\nu \varphi + \frac{\partial \varphi}{\partial \varphi} \partial_\mu \varphi) \]

\[ = (\partial_\nu \varphi)(\partial_\mu \partial^\nu \varphi + \frac{\partial \varphi}{\partial \varphi} \partial_\mu \varphi) = 0 \]

by the boundary condition

\[ (\partial_\nu \varphi)(\partial_\mu \partial^\nu \varphi + \frac{\partial \varphi}{\partial \varphi} \partial_\mu \varphi) = 0 \]
1.2.1.) Gauge loop: 
\[
\begin{align*}
A^f & \rightarrow A^f - \partial^f x \\
\rightarrow A & \rightarrow A - \hat{\nabla} X \\
\rightarrow \hat{\nabla} A & \rightarrow \hat{\nabla} A - \hat{\nabla} X
\end{align*}
\]

Now choose \( X \) as any solution of the Abelian eq. 
\[
\hat{\nabla}^2 x(x) = \hat{\nabla} \cdot \hat{\nabla} (x)
\]

The un transformed \( A \) has the property 
\[
\hat{\nabla} \cdot A(x) = \hat{\nabla} \cdot A(x) - \hat{\nabla}^2 X(x) = 0
\]