1.3.1. Energy density
Show that the argument from ch. 1 §3.6 remark (3) for \( u(x, t) \) being the energy density of the electromagnetic field still holds if the field is coupled to \( N \) relativistic particles rather than one nonrelativistic one.

(3 points)

1.3.2. Addition of velocities
Consider a particle that has a velocity \( v \) in some inertial frame. Find the velocity of the particle in another inertial frame that moves with a velocity \( V \) with respect to the first one. Use the result to show that the velocity in the second frame is less than \( c \), provided it was less than \( c \) in the first one.

(2 points)

1.3.3. Galileo transformations of Maxwell’s equations

a) Show explicitly which of Maxwell’s equations are or are not invariant under Galileo transformations.

\textit{hint:} Consider the transformations of all vectors (the 4-gradient, the fields, and the 4-current) to zeroth order in \( 1/c \), but keep the terms of \( O(1/c) \) in Maxwell’s equations. In other words, note that if you do a Lorentz transformation consistently to a given order in \( 1/c \), then of course all of Maxwell’s equations are invariant.

b) Suppose you had never heard of Lorentz transformations, but were familiar with Galilean mechanics. What are the two logical conclusions you could draw from the result of part a)? (Obviously, one of them by now is of historical interest only.)

(4 points)

1.3.4. Lorentz transformations of fields
Consider static and homogeneous fields \( E \) and \( B \) that are not parallel to one another in some inertial frame.

a) Show that there exists an inertial frame in which \( E \) and \( B \) are parallel, and that the two frames are related by a Lorentz boost whose velocity is given by the solution of the equation

\[
\frac{V}{c} (E^2 + B^2) = (1 + V^2/c^2) E \times B
\]

b) Show explicitly that this equation has one and only one physical solution that obeys \( |V|/c < 1 \), that there always is a physical solution, and that the result in the limit of almost parallel fields in the original reference frame is sensible.

c) Are there other inertial frames in which \( E \) and \( B \) are parallel? If so, how many?

(7 points)
Under the relativity principle, the kinetic energy is

\[ E_{\text{K}} = \frac{1}{2} m \frac{d^2 \vec{r}}{dt^2} - L_o = \frac{m u^2}{1 - v^2/c^2} + m u^2 \frac{v}{c^2} (1 - v^2/c^2) = \frac{m u^2}{1 - v^2/c^2} \]

Now consider

\[ \frac{\partial}{\partial t} \vec{p} = \frac{\partial}{\partial t} \frac{d \vec{r}}{dt} = \frac{d}{dt} \frac{m \vec{v}}{1 - v^2/c^2} = \frac{m \vec{v} \cdot \vec{a}}{(1 - v^2/c^2)^{1/2}} + m u^2 \frac{v}{c^2} \frac{v}{c^2} \frac{1}{(1 - v^2/c^2)^{1/2}} \]

\[ \frac{m \vec{v} \cdot \vec{a}}{(1 - v^2/c^2)^{1/2}} = \frac{d}{dt} E_{\text{K}} \]

But from the eq. of motion we have

\[ \vec{p} = m \vec{V} = m \vec{E} + m \vec{u} \times \vec{E} \]

\[ \Rightarrow \frac{d}{dt} E_{\text{K}} = m \vec{V} \cdot \vec{E} \quad \text{now} \quad m \vec{V} \cdot (\vec{V} \times \vec{E}) \quad \text{0} \]

This is the same expression as \( \frac{d}{dt} (U + F_{\text{EM}}) \),

so Property 4 known also yields

\[ \frac{d}{dt} (U + F_{\text{EM}}) = 0 \quad \text{when} \quad U = \int d^3 x \, u(x, t) \]

\( U \) is the field energy

For \( N \) particles with charge \( Q_i \) (\( i = 1, \ldots , N \)), consider

\[ \int d^3 x \, \vec{j} \cdot \vec{E} = \sum_{i=1}^{N} Q_i \vec{E} \cdot \vec{E} \]

\[ \Rightarrow \frac{d}{dt} E_{\text{K}} = - \sum_{i=1}^{N} Q_i \vec{E} \cdot \vec{E} \]

\( \Rightarrow \text{same result as for} \quad N = 1 \)
\( \text{(1) } x_l \) \( \text{and } t_l \Rightarrow \tilde{x} = x_l + \frac{\nu}{C^2} V_l t, \quad \tilde{t} = t_l + \frac{\nu}{C^2} x_l \)

\[
\begin{align*}
\tilde{\dot{x}} &= \frac{d\tilde{x}}{d\tilde{t}} = \frac{dx_l/dt + \nu}{1 + \frac{\nu}{C^2} dx_l/dt} = \frac{\nu + \nu}{1 + \nu V_l/C^2} \\
\tilde{\dot{t}} &= \frac{d\tilde{t}}{d\tilde{t}} = \frac{d/dt}{1 + \frac{\nu}{C^2} dx_l/dt} = \frac{\nu V_l/C^2}{1 + \nu V_l/C^2}
\end{align*}
\]

When we have ensured that the relative motion of the inertial frames is along the \( x \)-axis.

Now let \( \Delta_x = \nu x/C, \text{ and } \Delta = V/C \).

\[
\begin{align*}
1 - \tilde{\Delta}_x &= 1 - \frac{\Delta_x + \Delta}{1 + \Delta_x/\Delta} = \frac{(1 - \Delta_x)(1 - \Delta)}{1 + \Delta_x/\Delta} \geq 0 \text{ provided } |\Delta_x|, |\Delta| < 1 \\
\end{align*}
\]

\[
\begin{align*}
1 - \tilde{\Delta}_x > 0 \\
\tilde{\varsigma}_x \leq C \quad \text{and} \quad \tilde{\varsigma}_x = C \text{ iff } \nu_x = C \text{ or } V = C
\end{align*}
\]
(1.22) a) Under a Lorentz boost along the line $x - c t = 0$ (12.4.4.1),

\[
\begin{pmatrix}
1 + 0(v/c^2) & 0 & 0 & 0 \\
0 & 1 + 0(v/c^2) & 0 & 0 \\
0 & 0 & 1 + 0(v/c^2) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

leads to the 4-potential:

\[
\begin{pmatrix}
\frac{ct}{c^2} \\
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix}
\frac{ct}{c^2} \\
x \\
y \\
z
\end{pmatrix}
+ \begin{pmatrix}
\frac{ct + o(v/c)}{c^2} \\
x + o(v/c) \\
y \\
z
\end{pmatrix}
\]

To second order in $v/c$,

\[
\bar{t} = t, \quad \bar{x} = x + \bar{V} t
\]

Now transform the derivatives:

\[
\begin{pmatrix}
\frac{\partial}{\partial \bar{t}} \\
\frac{\partial}{\partial \bar{x}} \\
\frac{\partial}{\partial \bar{y}} \\
\frac{\partial}{\partial \bar{z}}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial}{\partial t} \\
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{pmatrix}
+ \begin{pmatrix}
\frac{\partial}{\partial \bar{t}} \\
\frac{\partial}{\partial \bar{x}} \\
\frac{\partial}{\partial \bar{y}} \\
\frac{\partial}{\partial \bar{z}}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial}{\partial t} \\
\frac{\partial}{\partial x} + V \cdot \bar{A} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{pmatrix}
\]

to $o(v/c^2)$

The fields have no time-like component

\[
\bar{E} = E, \quad \bar{B} = B
\]

to $o(v/c^2)$

(ie. also follows explicitly from 1.2.4.4.2).

Finally, the 4-current:

\[
\begin{pmatrix}
\bar{j}_0 \\
\bar{j}_1 \\
\bar{j}_2 \\
\bar{j}_3
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial}{\partial t} \\
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{pmatrix}
+ \begin{pmatrix}
\frac{\partial}{\partial \bar{t}} \\
\frac{\partial}{\partial \bar{x}} \\
\frac{\partial}{\partial \bar{y}} \\
\frac{\partial}{\partial \bar{z}}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial}{\partial t} \\
V \cdot \bar{A} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{pmatrix}
\]

to $o(v/c^2)$
Note under Lorentz's eqs.

(1) \[ \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0 \]  

(2) \[ \frac{1}{2} \frac{\partial \mathbf{E}^2}{\partial t} + \nabla \times \mathbf{E} = \frac{1}{c^2} \frac{\partial \mathbf{B}^2}{\partial t} + \nabla \times \mathbf{E} + \frac{1}{c^2} \left( \nabla \cdot \mathbf{B} \right) \mathbf{E} = \frac{1}{c^4} \left( \nabla \cdot \mathbf{B} \right) \mathbf{E} = 0 \]

(2) \[ \nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} = \frac{4\pi}{\varepsilon_0} \]

(4) \[ -\frac{1}{c^2} \frac{\partial \mathbf{E}^2}{\partial t} + \nabla \times \mathbf{E} = -\frac{1}{c^2} \frac{\partial \mathbf{E}^2}{\partial t} + \nabla \times \mathbf{E} - \frac{1}{c^2} \left( \nabla \cdot \mathbf{B} \right) \mathbf{E} \]

\[ = \frac{4\pi}{\varepsilon_0} - \frac{1}{c^2} \left( \nabla \cdot \mathbf{B} \right) \mathbf{E} \]

\[ = \frac{4\pi}{\varepsilon_0} - \frac{4\pi}{\varepsilon_0} \mathbf{E} - \frac{1}{c^2} \left( \nabla \cdot \mathbf{B} \right) \mathbf{E} + \frac{4\pi}{\varepsilon_0} \]

\[ \rightarrow (1) \text{ and } (2) \text{ are covariant under Galilean boosts, but } (2) \text{ and } (4) \]

\[ \text{on not.} \]

Remark: (1) The eqs. hold when a time derivative on not covariant. The "strong" \[ \nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{E} \] serves their role also the \[ \mathbf{E} \] has \[ \frac{\mathbf{E}}{c} \], this is a velocity \[ \text{em} \]

hies equivalent to a time derivative.

(2) Another way to say it: \[ \text{the eqs. continue to } 0(1/c) \text{ in } (2) \text{ and } (4), \text{ whereas the Galilean boost}

\[ \text{results in the Lorentzian } \mathbf{E} \rightarrow 0(1/c), \text{ so they are not covariant.} \]
(1) Two possibilities:

(1) Maxwell's eye on valid ref to a speed reference frame known as "ether." Michelson-Morley killed this possibility.

(2) Newtonian mechanics is only. This turned out to be the resolution; speed relative fixed the problem.
1.3.4) Let the non-parallel vectors $\vec{e}, \vec{d}$ lie in the $y-z$-plane:

$\vec{e} = (0, e_y, e_z), \quad \vec{d} = (0, d_y, d_z)$

Now consider a boost in the $x$-direction:

$u_2 \xi \eta \zeta \to \vec{e}_x = E_x = 0$ and $\vec{d}_x = D_x = 0$.

$\Rightarrow \vec{e} = (0, \vec{e}_y, \vec{e}_z), \quad \vec{d} = (0, \vec{d}_y, \vec{d}_z)$

$\Rightarrow \vec{e} \times \vec{d} = (\vec{e}_y \vec{d}_z - \vec{e}_z \vec{d}_y, 0, 0)$

$\Rightarrow \vec{e} || \vec{d} \ \Rightarrow \ \vec{e}_y \vec{d}_z - \vec{e}_z \vec{d}_y = 0$

$\Rightarrow \xi \eta \zeta \to$

$0 = (E_y \omega \phi + E_z \omega \phi)(\vec{d}_y \omega \phi + \vec{d}_z \omega \phi) - (E_x \omega (\phi - \phi_0) \vec{d}_y \omega (\phi - \phi_0))$

$= (E_y \vec{d}_z - E_z \vec{d}_y)(\omega \phi + \vec{d}_y \omega \phi) + (E_z \vec{d}_y + E_y \vec{d}_z)(\omega \phi + \vec{d}_z \omega \phi)$

$= \gamma (1 + \gamma^2)(\vec{e} \times \vec{d})_x + \frac{\gamma}{\gamma+1} \frac{\gamma^2}{\gamma+1} (\vec{e} \times \vec{d})_y$

where $\gamma = \frac{1}{\sqrt{1 - u^2}}$

It follows that the distance of the plane defined by $\vec{e} \times \vec{d}$ is arbitrary, then $\vec{V}$ is given by the relative of

$\vec{V} = (\vec{e} \times \vec{d}) = (1 + \gamma V_c \gamma) \vec{e} \times \vec{d}$

b) Now return to the original frame $\xi \eta \zeta$ and denote

$(\vec{e} \times \vec{d})_x = \dot{x}, \quad \vec{e}_y \vec{d}_z = \ddot{y}, \quad V_x = \dot{V}$

$\Rightarrow \dot{x} \gamma^2 - \ddot{y} \gamma + x = 0$
\[ A = \frac{1}{2x} \left( \frac{1 + \sqrt{1 - 4x^2}}{x} \right) \quad (\star) \]

Now consider \( \frac{1}{2} 4x^2 - \left( E^2 \dot{\alpha}^2 \right)^2 - 4E^2 \dot{\alpha}^2 x^2 \quad \text{let} \quad \lambda = \frac{E}{\dot{\alpha}} \)

\[ \left( E^2 \dot{\alpha}^2 \right)^2 - 4E^2 \dot{\alpha}^2 \]

\[ \left( E^2 \dot{\alpha}^2 \right)^2 - 4E^2 \dot{\alpha}^2 \quad \text{may be} \quad 0 \quad (\dagger) \]

\[ \frac{1}{2} \sqrt{\frac{1}{2} x} \quad \text{for any } \quad \lambda \quad \text{then there is at least one solution } \quad (\dagger) \]

\[ \left( \frac{1}{2x} \right)^2 \geq 1 \]

\[ x > 0 \quad \text{w.l.o} \]

\[ \frac{1}{2x} \geq 1 \]

\[ \frac{1}{2x} + \frac{1}{2x} \sqrt{\frac{1}{2} x} \geq 1 \]

Thus the only candidate for a physical solution is

\[ A = \frac{1}{2x} \left( 1 - \sqrt{\frac{1}{2} x} \right) = \frac{1}{2x} - \frac{1}{\sqrt{\left( \frac{1}{2x} \right)^2 - 1}} \]

We shall need to show that this solution obey \( \lambda < 1 \).

When \( x = \frac{1}{2x} > 0 \) w.l.o.

Then we have \( x > 1 \).

Now define \( x - \frac{1}{2x} < 1 \)

\[ \overset{\text{Thus}}{\Rightarrow} x > 1 \]

\[ \Rightarrow \frac{Ax}{\dot{\alpha}} \quad \text{where} \quad \frac{x}{\dot{\alpha}} \quad \text{is the unique physical solution that}

\[ \lambda \quad \text{exists} \]
With almost parallel fields \( \Rightarrow \vec{E} \perp \vec{v} \Rightarrow v \Rightarrow v \to \infty \Rightarrow \Delta \to x - x \sqrt{1 - \frac{v^2}{c^2}} = x - x + \frac{v^2}{2c^2} + O\left(1/v^2\right) = \frac{v}{c^2} + O\left(1/v^2\right) \to 0 \)

which is with:

If the fields are almost parallel to start with, then a small boost velocity will not make them parallel.

c) One \( \vec{E} \parallel \vec{v} \), e.g.: \( \vec{E} = (E_x, 0, 0) \), \( \vec{v} = (v_x, 0, 0) \) =>

the ray limit boost in the wavenumber direction of \( \vec{E} \parallel \vec{v} \) will have \( \vec{E} \parallel \vec{v} \Rightarrow \text{Run on infinite ray not circular frames} \)